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The Numerical Solution For The Roots Of Equations (Case Of Real Roots)

By

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Contents

CHAPTER 1

The real roots of an equation

This chapter deals with those methods which are applicable to finding the real roots of the equation

$$f(x) = 0 \qquad (1.1)$$

where f(x) is any piecewise continuous function of x having numerical coefficients, whether a polynomial or transendental function.

The finding of the real roots of f(x) can in general be divided into two parts. The first part has as its goal the finding of an approximate value of the root. The second part makes use of this approximate knowledge of the root to obtain the root out to the desired number of significant figures.

1.1 Finding an approximate value of a real root

Sometimes, one may have a good guess as to the approximate value of the root; but in general it will be necessary to use one of the following methods.

a. Graphical Methods

Generally speaking, the best method of finding the approximate value of a root is to plot the function

$$y = f(x) \qquad (1.2)$$

and determine approximately the points at which this plot crosses the x axis. At these points y=0, and hence the corresponding values of x, by Eq.(1.2), satisfy Eq.(1.1) and are therefore real roots of that equation.

In some cases it is preferable to write the equation in the form

$$f_1(x) = f_2(x) \tag{1.3}$$

in which case we plot the two functions

$$y_1 = f_1(x) \quad \text{and} \quad y_2=f_2(x) \tag{1.4}$$

The abscissas of their points of intersection obviously satisfy Eq.(1.3) and hence are the real roots of this equation.

b. Analytical Methods

There are two common analytical methods for finding the approximate value of the root of an equation.

One is to find a simpler equation that has a root approximately equal to the required root of the given equation. This can often be done by neglecting a term known to be small.

The second method makes use of the following theorem.

Theorem: If f(x) is a real function that is continuous between x=a and x=b, where a and b are real numbers, and if f(a) and f(b) are of opposite signs, then there is at least one real root between a and b.

1.2 Method of False Position

Suppose that f(a) and f(b) are of opposite signs, the plot f(x) crosses the x axis between x=a and x=b and that, therefore, a root $x=\bar{x}$ lies between these limits.

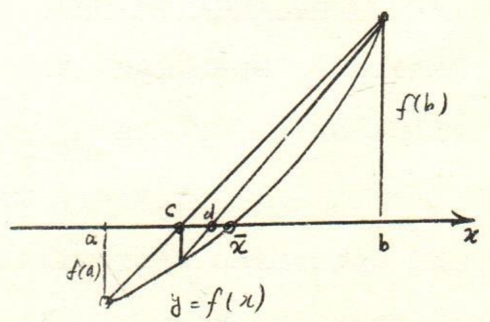

Fig.1.1. Method of Flase Position

The root $x=\bar{x}$ will be approximately given by $x=c$:

$$c = a + \frac{f(a)}{f(a)-f(b)} (b-a) \qquad (1.5)$$

Since f(a) and f(b) are of opposite signs, f(c) must be opposite in sign to one of them; therefore it is possible to apply the approximation again for a still better prediction of the value of $\bar{x}$. Therefore a better approximation

$$d = c + \frac{f(c)}{f(c)-f(b)} (b-c) \qquad (1.6)$$

to the root $x=\bar{x}$ is obtained by replacing a in Eq.(1.5) by c. The method of false position is very simple in principle as it merely replaces the plot of f(x) between any two points a and b by its chord. The simplicity and complete generality of this method make it a very powerful tool for computation.

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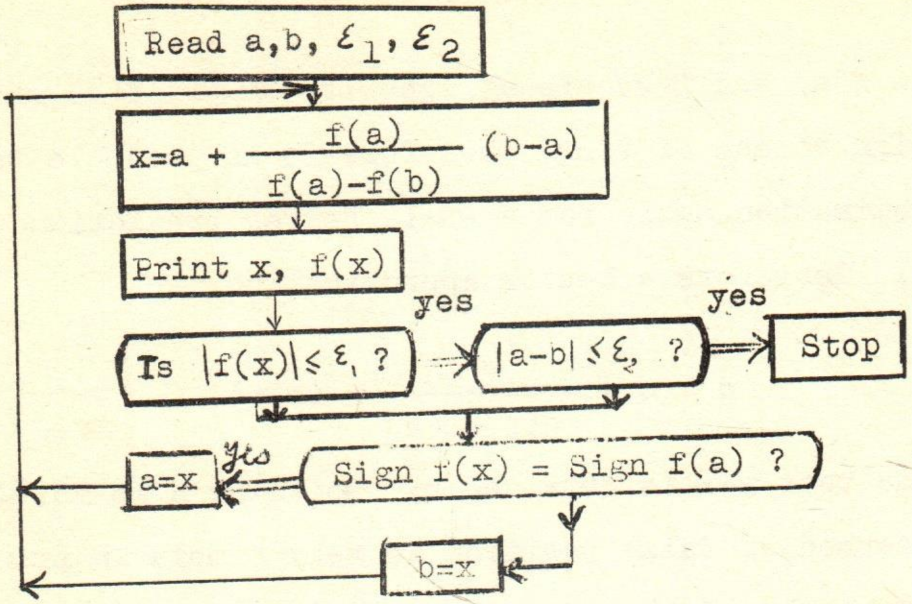
$$f(x) = 0 \quad , \qquad X = a + \frac{f(a)}{f(a)-f(b)}(b-a)$$



Test: $f(x) = xe^x - 2 = 0$

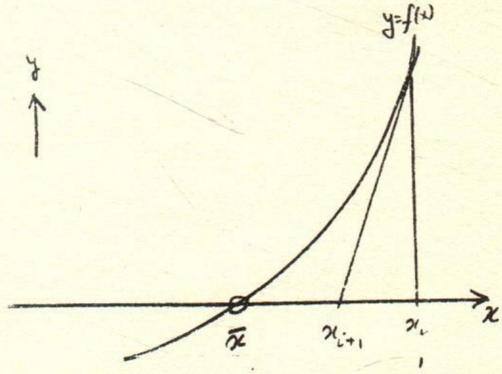
a=0, b=1

Ans. x= 0.8526055

1.3 The Newton-Raphson Method

In Fig.1.2, x_{i+1} is determined by drawing a tangent to the curve at x_i and extending it until it intersects the x axis at the point $(x_{i+1}, 0)$. The slope of the tangent is given by

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$



Newton-Raphson Method

Fig. 1.2.

Solving for x_{i+1}, we obtain the Newton-Raphson formula:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1.7)$$

To determine the rate of convergence of the iterates in the Newton-Raphson method, we expand f(x) about one of the iterates x_i using Taylor's series with a remainder

$$f(x) = f(x_i)+(x-x_i)f'(x_i)+ \frac{1}{2}(x-x_i)^2 f''(\xi)$$

where ξ lies between x and x_i. Substituting $x=\bar{x}$ and solving for $\bar{x}$ in the second term on the right-hand side, we have

$$\bar{x} = x_i - \frac{f(x_i)}{f'(x_i)} - (\bar{x}-x_i)^2 \frac{f''(\xi)}{2f'(x_i)}$$

therefore, by Eq. (1.7)

$$x_{i+1}-\bar{x} = (x_i-\bar{x})^2 \frac{f''(\xi)}{2f'(x_i)} \qquad (1.8)$$

Since $x_j-\bar{x}$ is the error in the jth iterate to the root, Eq.(1.8) states that each iteration squares the error and then multiplies it by the factor $f''(\xi)/2f'(x_i)$, where ξ lies somewhere between x_i and the root $\bar{x}$. It is clear from the above that the convergence will be poor it $f''/2f'$ is large in the neighborhood of the root. This will usually happen if f(x) does not cross the x axis at a sufficiently steep angle.

Iteration-Method of higher Order

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \left(1 + \frac{1}{2} \frac{f''(x_i)f(x_i)}{f'^2(x_i)}\right) \qquad (1.9)$$

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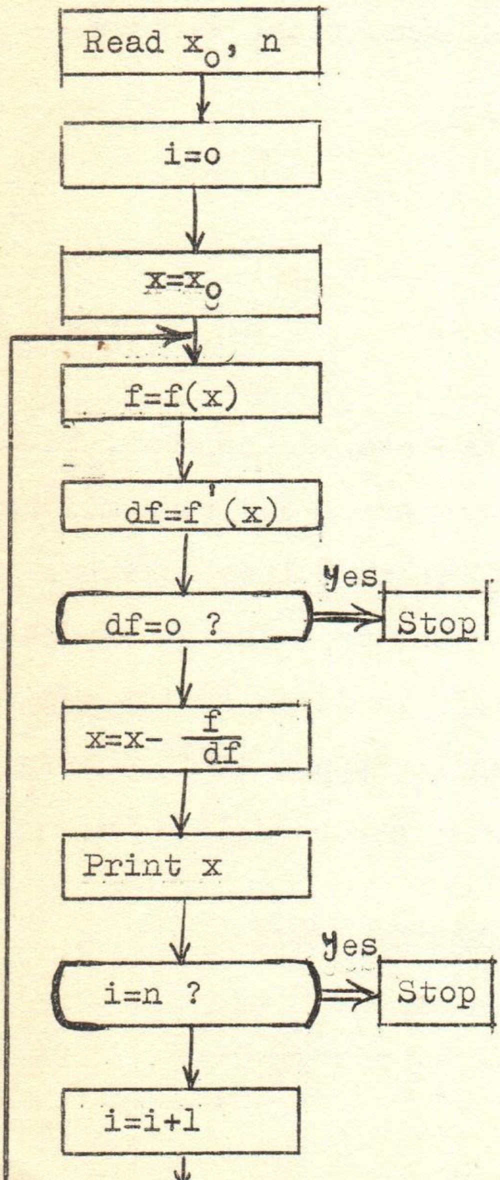
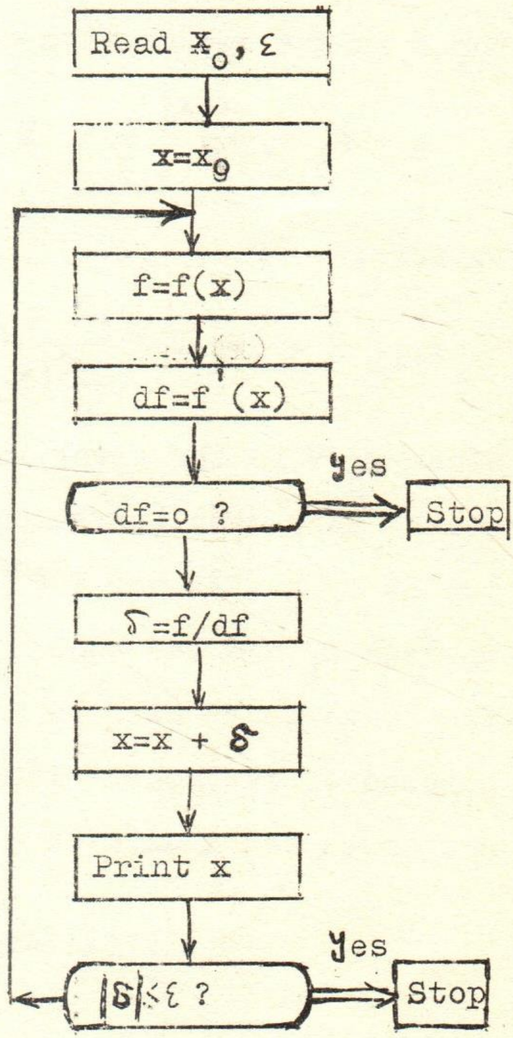
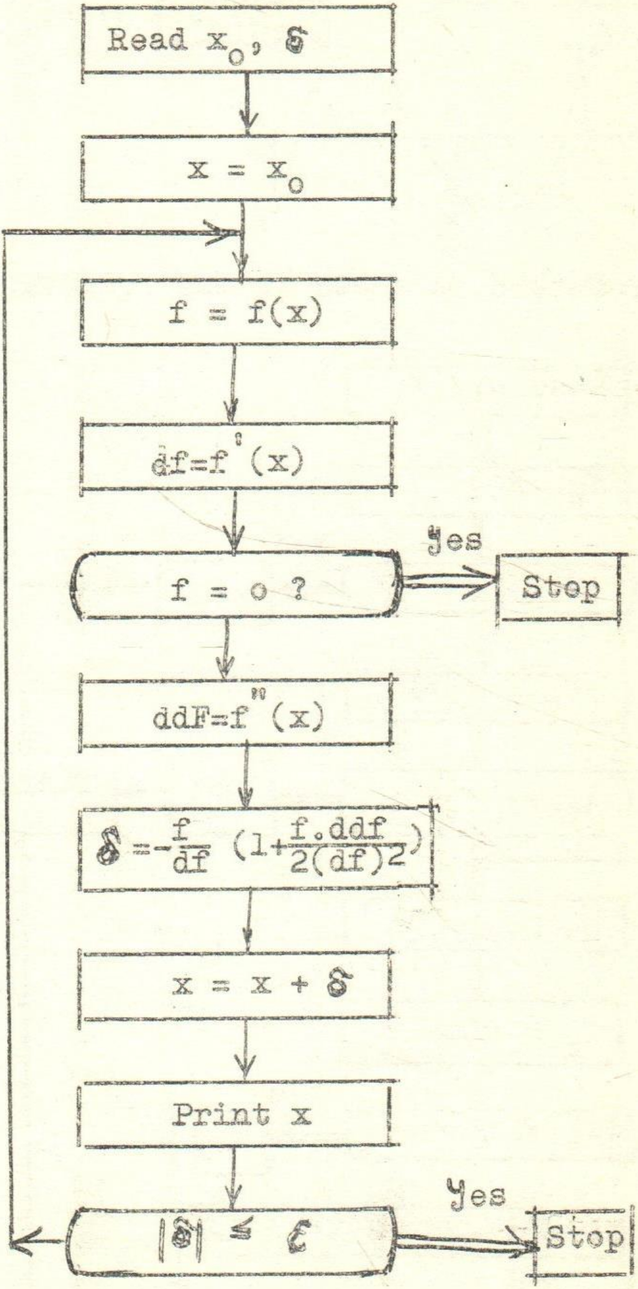
$$x_{i+1} = x_i - \underbrace{\frac{f(x_i)}{f'(x_i)}}_{\delta}$$



Another method



$$x_{i+1} = x_i \underbrace{- \frac{f(x_i)}{f'(x_i)} \left(1 + \frac{1}{2} \frac{f''(x_i) f(x_i)}{f'^2(x_i)}\right)}_{\delta}$$



1.4 The Half - Method

If f(x) is a real function that is continuous between x=u and x=v, where u and v are real numbers, and if f(u) and f(v) are of opposite signs, then there is at least one real root between u and v.

$$m = \frac{1}{2}(u+v) \qquad (1.10)$$

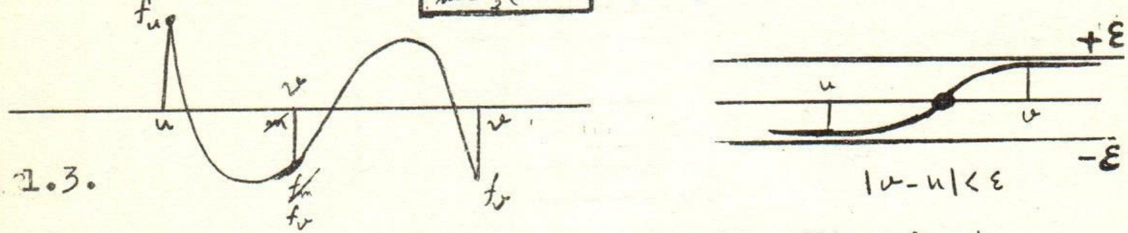
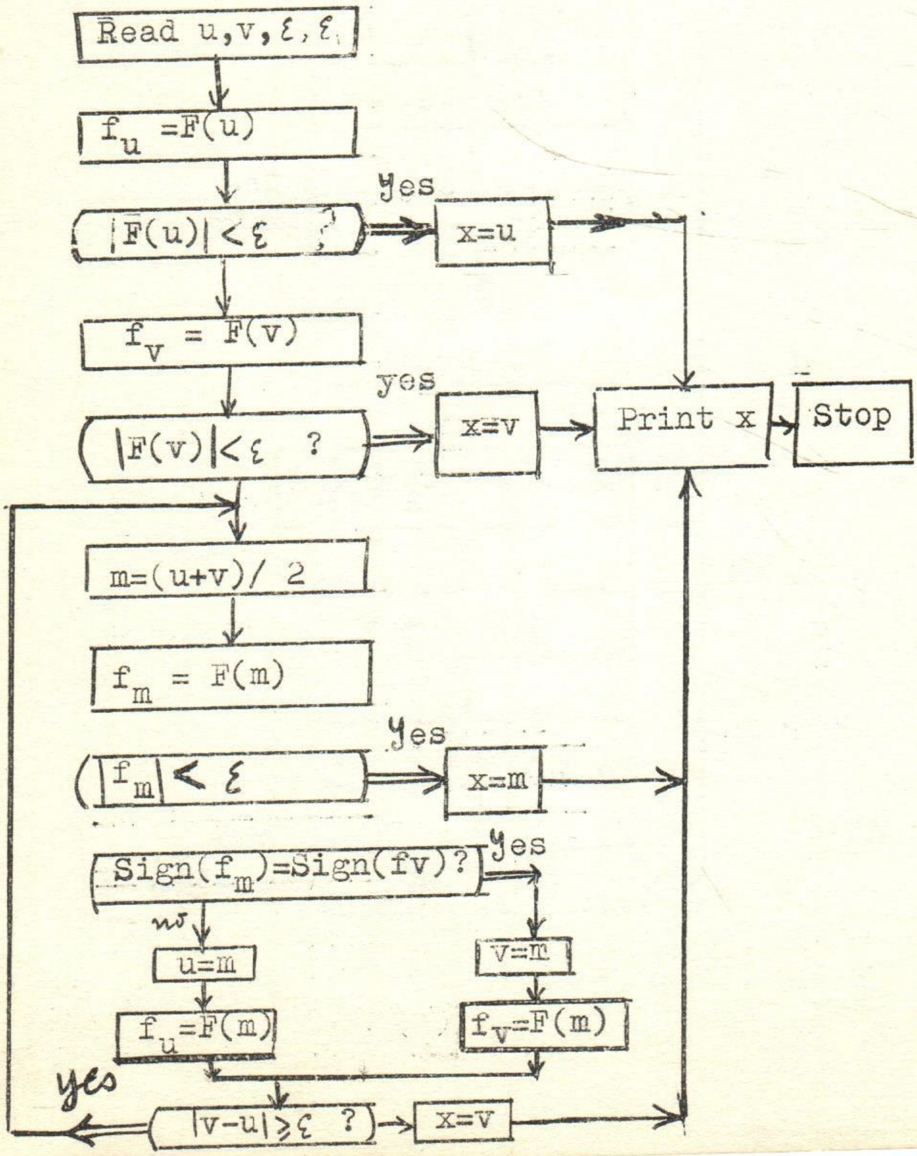


Fig. 1.3.

The Half-Method is shown in the following flow-chart



1.5 Example

Cubic root $\sqrt[3]{a}$

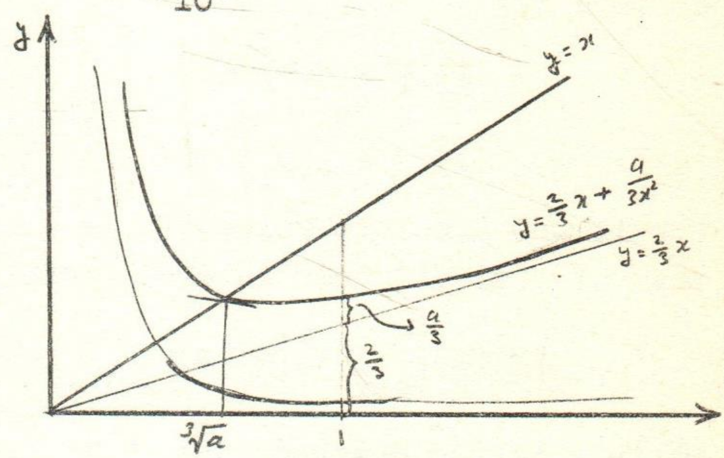
$$f(x) = x^3 - a$$
$$f'(x) = 3x^2$$
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$
$$x_{i+1} = x_i - \frac{x_i^3 - a}{3x_i^2}$$

$$\boxed{x_{i+1} = \left(2x_i + \frac{a}{x_i^2}\right)/3} \qquad (1.11)$$

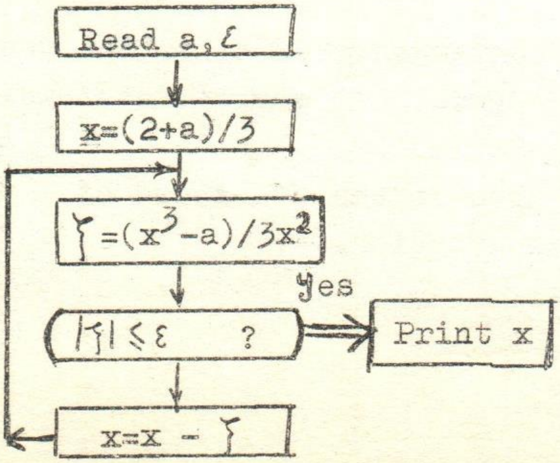
1. Case : $0 < a \leq 1 \curvearrowright x_0 = (2 + a)/3$

2. Case : $a > 1 \curvearrowright \tilde{a} = \frac{a}{10^m}$, m positive integer number

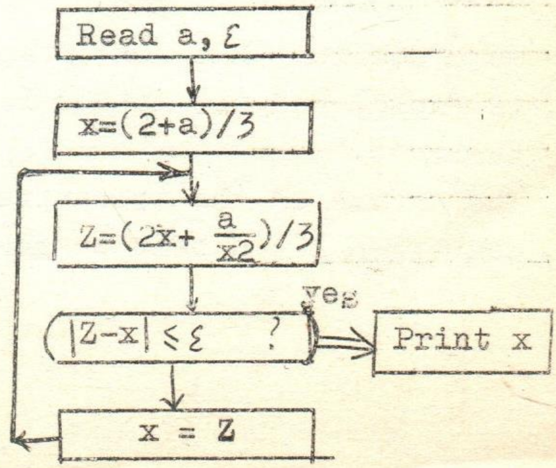
Fig.1.4.



Flow-chart



Other method



1.6 General Method of Iteration

The equation $f(x) = 0$ can be written in the from

$$g(x) = h(x) \qquad (1.12)$$

It is convenient to replace Eq. (1.12) by the set of Simultaneous equations

$$\left.\begin{aligned} y &= g(x) \\ y &= h(x) \end{aligned}\right\} \quad (1.13)$$

If one can solve explicity for x in the second equation, these can be written

$$\left.\begin{aligned} y &= g(x) \\ x &= H(y) \end{aligned}\right\} \quad (1.14)$$

Suppose x_o is any initial guess at the root of $f(x)$; then the iterates $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ may be defined by

$$\left.\begin{aligned} y_{i-1} &= g(x_{i-1}) \\ x_i &= H(y_i) \end{aligned}\right\} \quad i=1,2,3,\ldots$$

Now if the absolute value of the slope of $g(x)$ is less than that of $h(x)$ at their intersection, i.e. if

$$\left|g'(x)\right|_{x=\bar{x}} < \left|h'(x)\right|_{x=\bar{x}} \qquad (1.15)$$

where $\bar{x}$ is the desired root, then, for a sufficiently close guess x_o, $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. If the slope of $g(x)$ is much smaller than $h(x)$, the convergence of x_n to the root is rapid, and the method is a practical way of determining the root.

The nature of the iterative process and its speed of convergence can best be shown graphically.

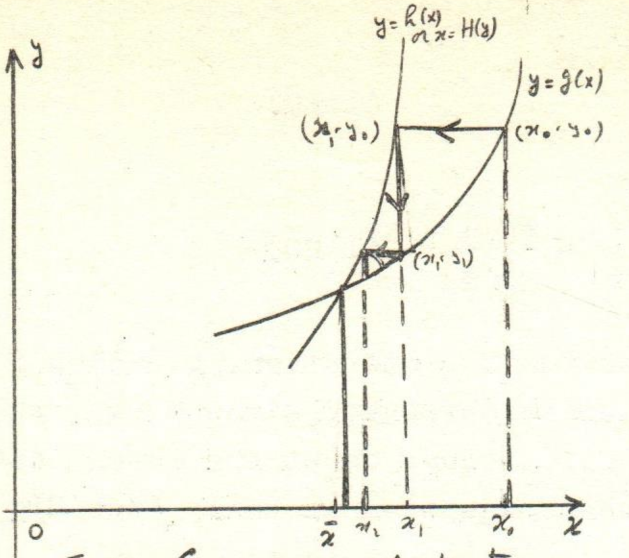


Fig 1.5 Convergence of the iterates $x_0, x_1, \ldots$ to the root $\bar{x}$ when $g(x)$ and $h(x)$ both have positive slopes.

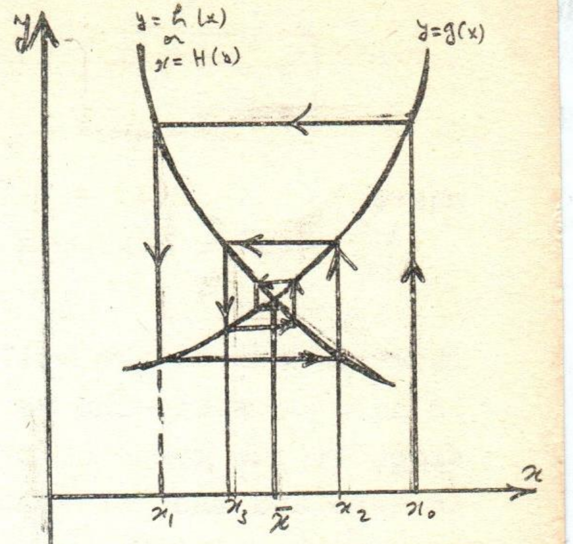


Fig 1.6 Convergence of the iterates $x_0, x_1, \ldots$ to the root $\bar{x}$ when $g(x)$ and $h(x)$ have opposite slopes.

1.7 Special Iterative Forms

The weakness of the iterative method is the difficulty of being able to write $f(x) = 0$ in the form

$$\begin{aligned} y &= g(x) \\ y &= h(x) \end{aligned} \qquad (1.16)$$

and at the same time to make sure that the first equation has a much smaller slope at the intersection and that the second equation is readily solvable for x.

One can overcome these difficulties by establishing standard methods of forming the iterative equation (1.16)

In place of $f(x) = 0$, write

$$F(x) = -\frac{f(x)}{f'(x)} = 0 \qquad (1.17)$$

This function has the same root $\bar{x}$ as $f(x) = 0$, provided $f'(\bar{x}) \neq 0$, and moreover

$$F'(\bar{x}) = -1 + \frac{f(\bar{x})\, f''(\bar{x})}{\left[f'(\bar{x})\right]^2} = -1 \qquad (1.18)$$

Therefore if one deals with $F(x)$ instead of $f(x)$, one obtains the iteration equations

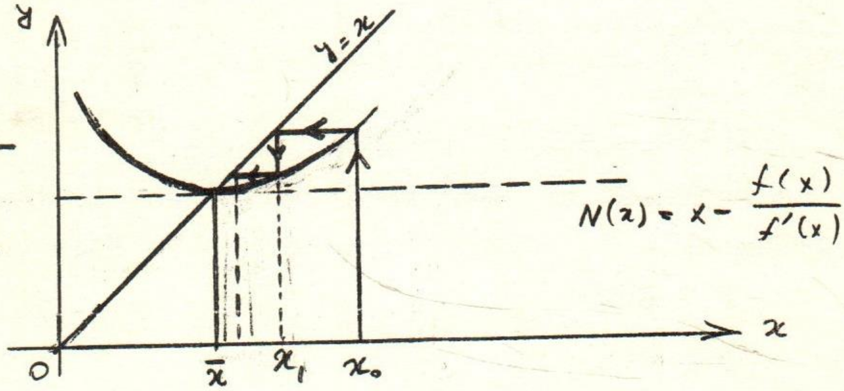
$$\boxed{\begin{array}{l} y = N(x) \\ x = y \end{array}} \qquad (1.19)$$

where

$$N(x) = F(x) + \bar{x} = \bar{x} - \frac{f(x)}{f'(x)} \qquad (1.20)$$

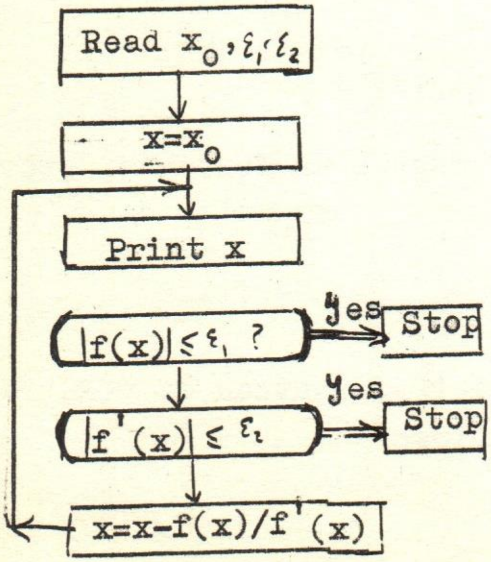
These equations are well suited for iteration, provided $f'(\bar{x})$ is not too small and is readily obtained, since $N'(\bar{x})$, the slope of the graph of the first equation at the intersection, is zero and that of the second equation is unity (see Fig 1.5)

Fig. 1.7. The iterative equivalent of the Newton-Raphson Formula.



Flow-chart

$$y = x - \frac{f(x)}{f'(x)}$$
$$x = y$$



1.8 Birge-Vieta Method

In the Newton-Raphson method any close approximation x_i to a root of a polynomial

$$P_n(x) = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \dots + c_{n-1}x + c_n = 0 \quad (1.21$$

is improved by the formula

$$\boxed{x_{i+1} = x_i - \frac{P_n(x_i)}{P_n'(x_i)}} \qquad i=0,1,2,\dots$$

This same procedure is followed in the Birge-Vieta method but the values of $P_n(x_i)$ and $P_n'(x_i)$ are calculated by synthetic division, as shown in the following diagram:

	c_0	c_1	c_2	c_3	...	c_{n-1}	c_n
x_i		x_iP_0	x_iP_1	x_iP_2	...	x_iP_{n-2}	x_iP_{n-1}
	P_0	P_1	P_2	P_3	...	P_{n-1}	$\underline{P_n(x_i)}$
x		xP_1'	xP_2'	xP_3'	...	xP_{n-1}'	
	P_1'	P_2'	P_3'	P_4'	...	$\underline{P_n'(x_i)}$	

Diagram (1.8)

One can easily prove this diagram as follows:

$$P_n(x) = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \dots + c_{n-1}x + c_n$$

$$\equiv (\dots(((\underbrace{\underbrace{\underbrace{c_0}_{P_0(x)}x+c_1)}_{P_1(x)}x+c_2)}_{P_2(x)}x+c_3)x+\dots + c_n$$

For a given set of coefficients $c_0, c_1, c_2, \dots, c_n$ there can be defined the following Polynomials

$$
\begin{aligned}
P_0(x) &= c_0 \\
P_1(x) &= xP_0(x) + c_1 \\
P_2(x) &= xP_1(x) + c_2 \\
&\dots\dots\dots\dots\dots\dots \\
P_{n-1}(x) &= xP_{n-2}(x) + c_{n-1} \\
P_n(x) &= xP_{n-1}(x) + c_n
\end{aligned}
\qquad (1.22)
$$

Multiplying the first Eq. by x^n, the second by x^{n-1}, the third by x^{n-2}, etc., and adding, we obtain at once Eq. (1.21). Therefore $P_n(x)$ of Eqs. (1.22) is identical with $P_n(x)$ of Eq. (1.21)

Suppose we wish to evaluate $P_n(x_i)$. It is generally more convenient to calculate in order, from Eqs. (1.22) the values of $P_0(x_i)$, $P_1(x_i)$, $P_2(x_i)$, ... , $P_n(x_i)$ then to find all n powers of x_i and substitute in Eq. (1.21)

$$
\begin{array}{c|cccccc}
 & c_0 & c_1 & c_2 & \cdots & c_{n-1} & c_n \\
x_i & & x_iP_0 & x_iP_1 & \cdots & x_iP_{n-2} & x_iP_{n-1} \\
\hline
 & P_0 & P_1 & P_2 & \cdots & P_{n-1} & P_n(x_i)
\end{array}
$$

Diagram (1.9)

If the Eqs. (1.22) are differentiated with respect to x, obtain the equations

$$P_i'(x) = xP_{i-1}'(x) + P_{i-1}(x) \quad , i = 1,2, \ldots , n$$

$$\text{where } P_o'(x) = 0.$$

Comparing these equations with Eqs. (1.22) we see that $P_1'(x)$ $P_2'(x), \ldots , P_n'(x)$ are obtained from $P_o(x), P_1(x), \ldots , P_{n-1}(x)$ in exactly the same way that $P_o(x), P_1(x), \ldots , P_{n-1}(x)$ are obtained from $\mathfrak{C}_o, \mathfrak{C}_1, \ldots , \mathfrak{C}_{n-1}$. Hence diagram (1.9) can be extended to give the diagram (1.8) to obtain the derivative of each of the $P_i(x)$.

Also, if

$$P_n(x) = \mathfrak{C}_o x^n + \mathfrak{C}_1 x^{n-1} + \ldots + \mathfrak{C}_n \equiv \sum_{i=o}^{n} \mathfrak{C}_i x^{n-i}$$

then

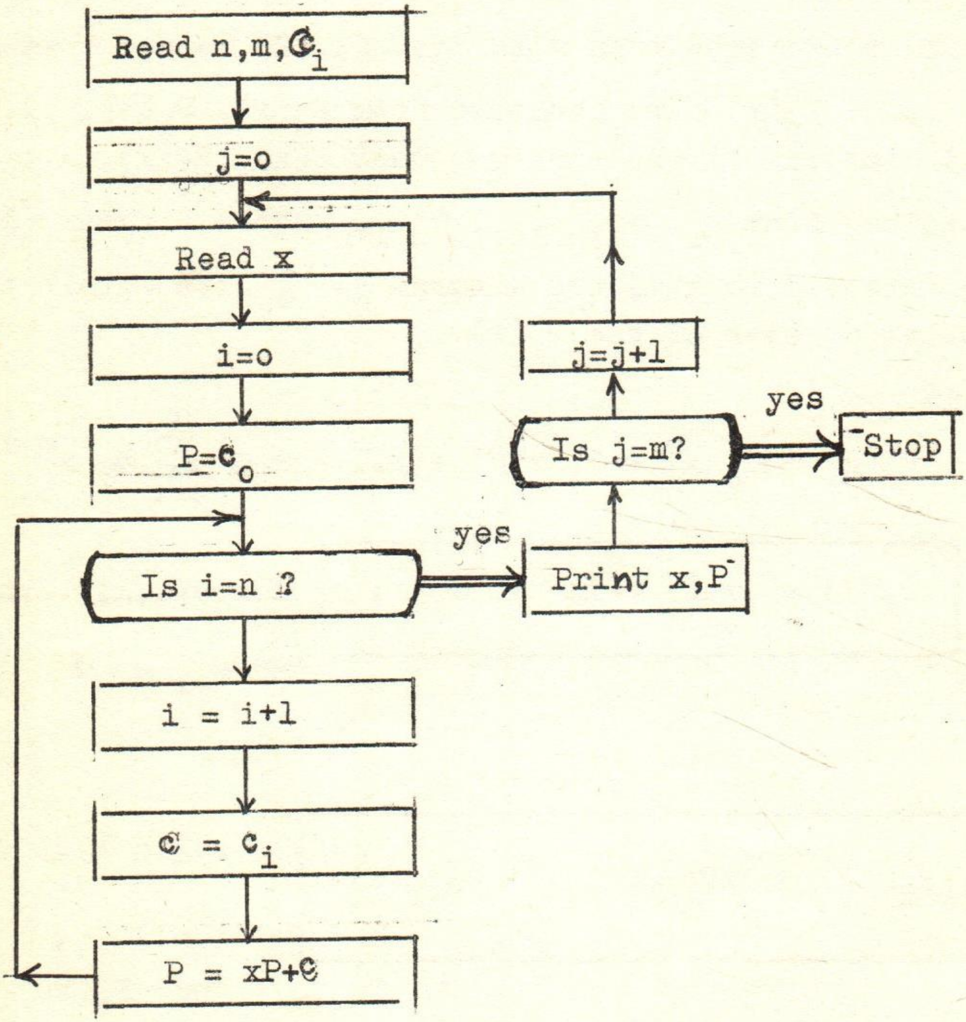
$$\boxed{P_i(x) = x P_{i-1}(x) + \mathfrak{C}_i ,} \quad i = o,1,2,\ldots,n \text{ and } P_{-1} = 0$$

and

$$\boxed{P_i'(x) = xP_{i-1}'(x) + P_{i-1}(x)} \quad , i = 1,2, \ldots , n \text{ and } P_o(x) = 0$$

Flow chart for $P_n(x_j) = \sum_{i=o}^{n} C_i x_j^{n-i}$, j=o,1,..,m

$P_i(x_j) = x_j P_{i-1}(x_j) + c_i$, $i=o,1,2,\ldots,n$ and $P_{-1}=o$
$j=o,1,\ldots,m.$

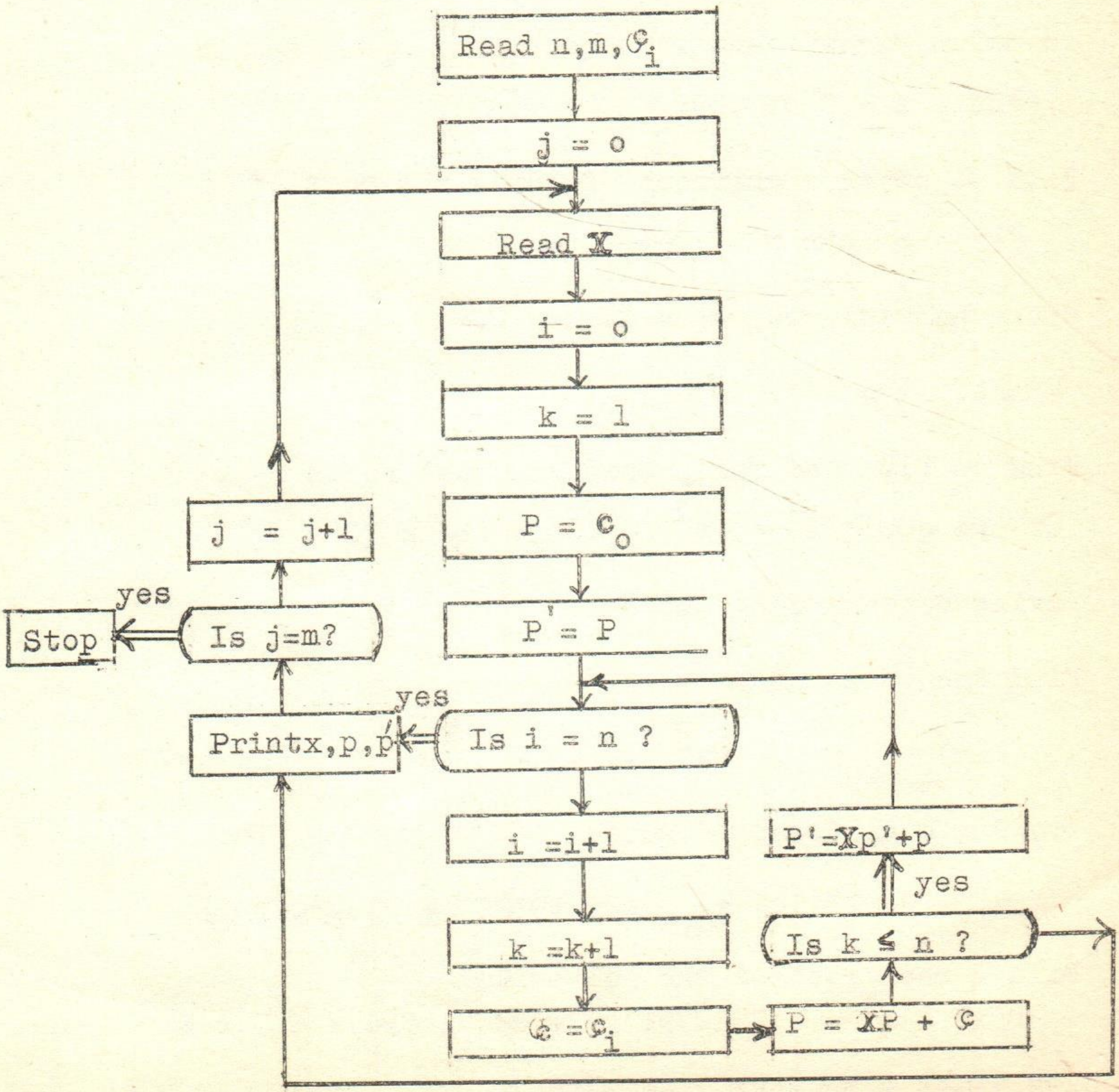


Flow chart for $P_n(x_j) = \sum_{i=o}^{n} c_i x_j^{n-i}$, j=o,1,...,m and $P_n'(x)\Big|_{x=x_j}$ in the same time of computation

$$P_i(x_j) = x_j P_{i-1}(x_j) + c_i \ , \quad i=o,1,\ldots,n; \ P_{-1}=o$$

$$P_k'(x)\Big|_{x=x_j} = x_j P_{k-1}'(x)\Big|_{x=x_j} + P_{k-1}(x_j) \ , \quad k=1,2,\ldots,n; P_o'(x)=$$

j=o,1,2,...,m



1.9 **Problems**

1. Find an approximate root of

$$e^{-x} \sin x + 25x - 1 = 0$$

2. Find the approximate value of a real root of

$$f(x) = x^2 - \log x - 10 = 0$$

3. Determine the smallest positive real root

$$f(x) = x e^x - 2 = 0$$

to seven significant figures.

(Ans. $x = 0.8526055$)

4. Find to seven significant figures the root of

$$f(x) = \sin x - \frac{x+1}{x-1} = 0$$

Whose approximate value is -0.4

(Ans. $x = -0.4203625$)

5. Find to five decimal places the real roots of the cubic $t^3 - 4t^2 - 6t + 4 = 0$ lying between zero and one.

6. Find the roots of

$$P(x) = x^3 - 3x^2 + 4x - 5 = 0$$

to six significant figures

($x_1 = 2.21341$, $x_{2,3} = 0.393295 \pm 1.45061\, i$)

7. Find a real root of the polynomial

$$P(x) = x^5 - 6.2842731\, x^4 + 23.714994\, x + 3 = 0$$

to eight significant figures

(Ans. $x = 1.7799319$)