UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

التخطيط القومي

Memo. No. 508

An Optimal Policy for Machine Tool Replacement

by

Dr. Hamdy A. Taha

<u>Operations Research Center.</u>

November 1964.

CAIRO

AN OPTIMAL POLICY FOR MACHINE TOOL REPLACEMENT*

ABSTRACT:

This paper presents a new optimal policy for machine tool replacement. The method used is based on the assumption that, as the tool is used longer, the number of defective items produced by the machine will increases due to the malfunction of the tool. On the other hand, if maintenance or corrective actions are applied to the tool more frequently, the number of defectives produced is expected to decrease. The objective, therefore, is to determine the optimal length of time that should elapse before maintenance or corrective actions are applied to the tool. This would be such as to give a balance between the conflicting costs of maintaining the tool and of reworking and/or scrapping the defective items. Two types of maintenance actions are applied to the tool: (1) replacement, and (2) sharpening or adjustment. The procedure assumes that the application of either of these actions would restore the tool to its original condition. It is also assumed that these maintenance actions are applied at equally spaced intervals of time, as predetermined from optimal results. The type of maintenance to be applied is decided on either a deterministic or a probabilistic basis. This paper also proposes an approximate for solving the integral equation which determines the value of the decision variable of the problem.

INTRODUCTION

During the production process, a machine tool may be subject to two types of maintenance actions: (1) the tool is completely replaced, and (2) the tool is sharpened or adjusted before it is used again.

The purpose of this paper is to answer two questions concerning the application of the maintenance or corrective actions to a machine tool: (1) how long a period of time should elapse before the tool is sharpened or adjusted? and (2) how long a period of time should elapse before the tool is replaced? Before presenting

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* Submitted for publication in the Journal of Industrial Engineering of America

the procedure for answering these questions, a review of the most common method for determing the economic life of a machine tool will be given.

Current method determine the economic life of a machine tool by selecting the length of time which maximizes the amount of metal removed by the tool perunit cost of using the tool during this interval of time.[1]
In this method the costs incurred as a result of using the tool include: (1) machine set-up costs, (2) costs of sharpening and/or adjusting the tool, (3) depreciation expenses on the tool, and (4) overhead burden charged against the tool while it is in operation. The type of maintenance actions to be taken at the end of the economic life of the tool is not specified in this method. It is left up to the maintenace operator to decide whether the tool should be replaced, adjusted or sharpened.

The major drawback of the above method is that it does not take into consideration the effect of producing defective items resulting from the malfunction of the tool, e.g., tool wear. It is conceivable that as the tool is used longer, the percentage of items which do not meet the specifications of the process will increase.

The new method introduced in this paper provides for the above point. It is noted that as the number of maintenance actions applied to the tool is increased, the percentage of

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1 See L. Doyle. Tool Engineering, (New York: Prentice-Hall, 1959), pp. 67-75.

"defective" items that is produced by the machine is decreased.[2] On the other hand, if the number of maintenance actions applied to the tool is decreased, the percentage of defective items produced by the system will increase.

This means that a decrease in the costs of reworking and/or scrapping the defective items occurs at the expense of increasing the costs of applying maintenance actions to the tool, and vice versa. The objective then is to determine the length of the life of the tool which minimizes the sum of these two conflicting costs. This decision problem and its solution are given in the following sections.

DESCRIPTION OF THE DECISION PROBLEM

In this study it will be assumed that the tool is used to process one measurable dimension of the manufactured product, e.g., length, thickness, diameter. Because of factors inherent in the scheme of production, this measured dimension is subject to an inevitable amount of variation from the actual value set for the process. In the terminology of statistical quality control, a system which is subject only to this kind of error is said to be stable or under statistical control. In this case it is expected that a large percentage of the produced items will fall within prespecified control limits.[3] The specification limits of any process are thus set to allow for this inherent variation in the process.

The presence of "assignable causes" in the process, i.e., external causes other than those inherent in the process such as tool malfunction, should result in an increase in the percentage of defective items that are produced by the process.

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2 "Defective" items as defined here are items that do not meet the specifications set for the process.

3- See E. L. Grant. Statistical Quality Control. (New York: McGraw-Hill, 1952).

It will be assumed in this study that any increase in the percentage of defective items over that when the process is stable is caused solely by the poor condition of the tool. This assumption applies more correctly to automatic and semi-automatic machines where the effect of other external factors that may affect the output of the process is not as strong as it is in a manually-operated machine.

In view of the above discussion, the process investigated in this paper can be described mathematically as follows. Let x represent the value of the measured variable (i.e., the variable under control) and let $f(x;\mu,\sigma)$ be the continuous probability density function which represents the variation in x around the process mean, where μ and σ are the process-mean and standard deviation respectively under stable conditions. It is assumed that the variation in f due to the poor condition of the tool will only occur through variations in its mean and/or standard deviation without affecting the type of the distribution function. This means that the distribution function $f(x)$ is also a function of time in so far as its mean and standard deviation are concerned. The notation $f(x;\mu(t),\sigma(t))$ will thus be used to represent the distribution of x at any time during the life of the tool. It should be noted that for the purpose of this analysis we do not think of $\mu(t)$ and $\sigma(t)$ as random variables, but rather as time variables whose variation can be specified, in advance, by certain trends.

Another important assumption should also be made here. The statistical behavior of the system in the period following a maintenance or corrective action will be the same as its statistical behavior during the period when the tool was first used, i.e., the trends of and during any period will always remain the same.

As mentioned in the introduction, we are interested in two major decisions: (1) the length of time that elapses before the tool is sharpened or adjusted, and (2) the length of time that elapses before the tool is completely replaced. This is decided in two different ways: (1) the tool is replaced after it is sharpened or readjusted m times, where m is a fixed integer which is determined depending on the number of times that a tool can be sharpened (or adjusted) before it is scrapped, and (2) at the end of the economic life of the tool the decision as to whether the tool should be sharpened (or adjusted) or replaced with probabilities p and $1-p$ respectively $(0 \leq p \leq 1)$. It is clear that in both cases the decision problem reduces to the determination of a single parameter T, which is the length of time that the tool is used before a maintenance action is applied to it. It should be noted that the interval T does not include the time spent in applying the maintenance actions to the tool, nor does it include the time that is lost because of interruptions in the production system. In other words, T represents the time when the tool is being used exclusively for manufacturing the product.

To summarize the above assumptions, the decision problem can be illustrated as shown in Figure 1. The values S_u and S_L represent the specification limits of the process. If at any time the value of x falls outside these limits, the produced item is classified as defective and it is either reworked or scrapped, depending upon its condition. Figure 1 also shows that during the manufacturing process, a maintenance action is applied to the tool every T time units, where T, as defined above, is the decision variable to be determined by models developed below.

DEFINITION OF THE SYMBOLS

The following is a summary of the symbols and their definitions which will be used in this study. Let:

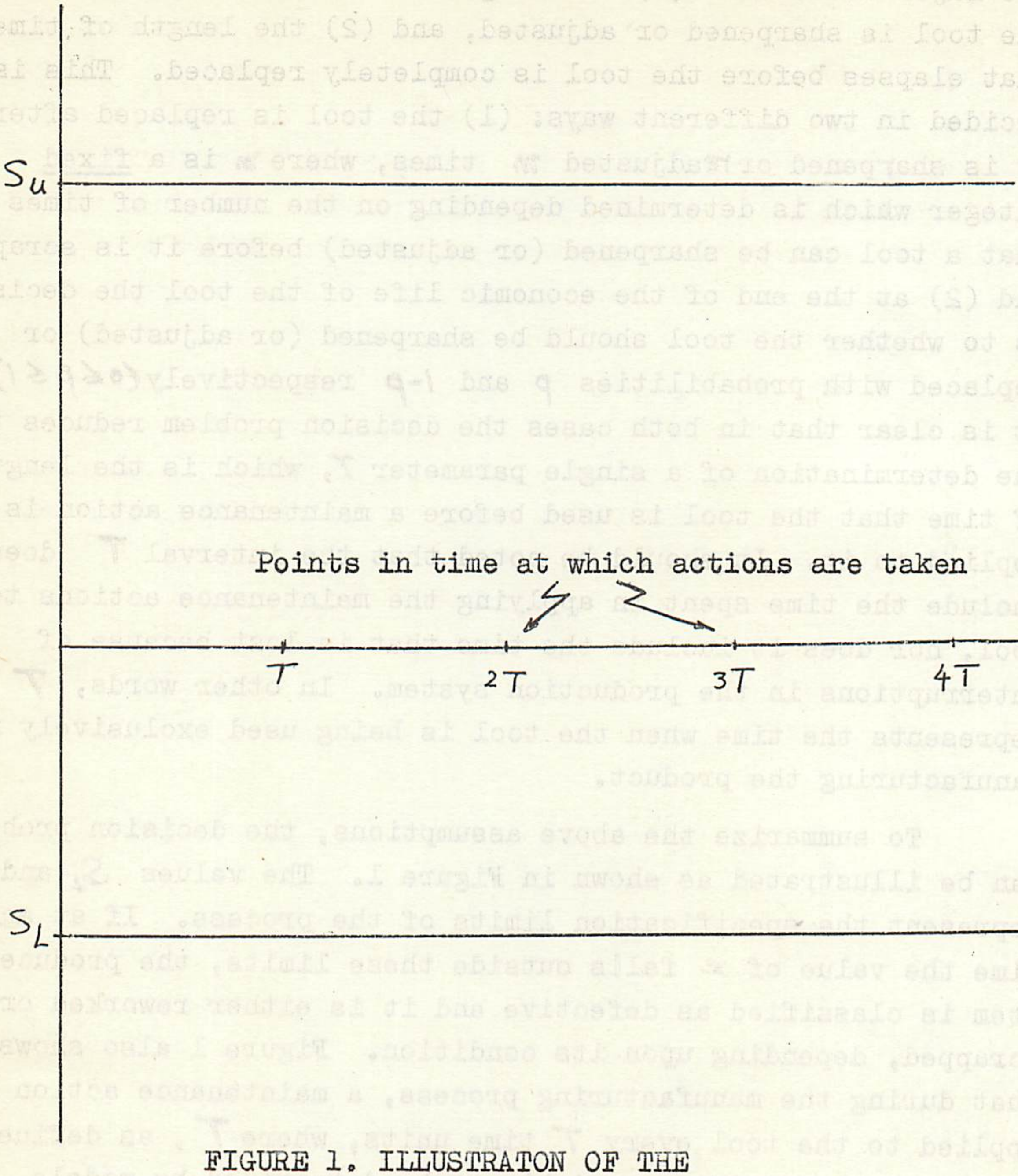

FIGURE 1. ILLUSTRATON OF THE DECISION PROBLEM

x value of the variable under control

$f(x; \mu(t), \sigma(t))$ probability density function (p.d.f.) of x given its mean μ and its standard deviation σ at time t.

$$P\{a \le x \le b \mid \mu(t), \sigma(t)\} = \int_a^b f(x; \mu(t), \sigma(t))\, dx$$

S_u, S_L upper and lower specification limits of the process

Q size of the lot in number of items which is to be manufactured on the machine

q production rate in number of items per unit time

T length of time that a tool is used before a maintenance or corrective action is applied to it, i.e., length of the economic life of the tool

C_w average cost of reworking or scrapping one defective item.

C_a cost of a single sharpening or adjustment of the tool

C_s machine set-up cost

C_r cost of replacing one tool

$C_{sa} = C_s + C_a$

$C_{sr} = C_s + C_r$

TC_1 total cost of sharpening, adjusting, and/or replacing the tool during manufacture of the whole lot

TC_2 total cost of reworking and/or scrapping the defective items in a lot of size Q items

$TC = TC_1 + TC_2$

MODEL 1

In this model it is assumed that the replacement of the tool takes place after $m-1$ sharpenings or adjustments, where m

is defined as a fixed integer which is determined a priori depending on the number of times that a tool can be sharpened or adjusted before it is scarapped. As mentioned above the optimal life of the tool is determined so as to minimize two types of costs: (1) costs resulting from applying maintenance actions to the tool, and (2) costs resulting from reworking and/or scrapping the defective items resulting from the malfunction of the tool. In the following paragraphs expressions for these types of costs are derived. The sum of these costs is then differentiated with respect the parameter T and the result is equated to zero in order to obtain the optimal life of the tool.

For a production rate of q items per unit time, the total maintenance costs incurred as a result of using the tool to manufacture a lot of size Q is given by,

$$TC_1 = \left[\frac{\left[\frac{Q}{qT}\right]}{m}\right](C_r+C_s) + \left\{\left[\frac{Q}{qT}\right] - \left[\frac{\left[\frac{Q}{qT}\right]}{m}\right]\right\}(C_s+C_a)$$

$$= \left[\frac{\left[\frac{Q}{qT}\right]}{m}\right] C_{sr} + \left\{\left[\frac{Q}{qT}\right] - \left[\frac{\left[\frac{Q}{qT}\right]}{m}\right]\right\} C_{sa}$$

where $K=[Z]$ is defined as the largest integer such that $K \leq Z$

For the same conditions as above, the cost of reworking and/or scrapping the defective items resulting from the poor condition of the tool is given by,

$$TC_2 = \left[\frac{Q}{qT}\right] C_w qT \left\{1 - \frac{\int_0^T P\{S_L \leq x \leq S_U / \mu(t), \sigma(t)\}\, dt}{T}\right\}$$

Where $$\frac{\int_0^T P\{S_L \leq x \leq S_U \mid \mu(t), \sigma(t)\}\, dt}{T} =$$

= average ratio of <u>good</u> items which are produced during the period T.

It then follows that the total cost function is given by ,

$$TC = TC_1 + TC_2$$
$$= \left[\frac{\left[\frac{Q}{qT}\right]}{m}\right] C_{sr} + \left[\frac{Q}{qT}\right] - \left[\frac{\left[\frac{Q}{qT}\right]}{m}\right] C_{sa}$$
$$+ \left[\frac{Q}{qT}\right] C_w qT \left\{1 - \frac{\int_0^T P\{S_L \leq x \leq S_U / u(t), \sigma(t)\} dt}{T}\right\} \quad (1)$$

The TC-function in its present form is not differentiable as it is not continuous over its domain. This function, however, can be made differentiable by approximating the step function $[Z]$ by the continuous function Z. Applying this to the TC-function above gives,

$$TC = \frac{Q}{qT}\left\{\frac{C_{sr} + (m-1)C_{sa}}{m}\right\} + QC_w\left\{1 - \frac{\int_0^T P\{S_L \leq x \leq S_U / \mu(t), \sigma(t)\} dt}{T}\right\} \quad (1)$$

It is noted, that the term $\frac{C_{sr} + (m-1)C_{sa}}{m}$ is actually equal to the average cost of replacing and sharpening the tool over m periods. For simplicity the symbol C_{av} will be used to represent this average cost. Using this in Eq. 1 gives,

$$TC = \frac{C_{av} Q}{qT} - \frac{QC_w}{T}\int_0^T P\{S_L \leq x \leq S_U / \mu(t), \sigma(t)\} dt + QC_w \quad (2)$$

In order to obtain the value of T which minimizes the TC-function above, Eq. 2 is differentiated with respect to T, T being restricted by the inequality $0 < T \leq \frac{Q}{q}$ and the result is then equated to zero. This gives after simplification,

$$-C_{av} + qTC_w\left(\frac{\int_0^T P\{S_L \leq x \leq S_U / \mu(t), \sigma(t)\} dt}{T} - P\{S_L \leq x \leq S_U / \mu(T), \sigma(T)\}\right) = 0 \quad (3)$$

or,

$$\frac{C_{aV}}{C_W} = qT\left\{\left(1 - P\{S_L \le x \le S_U \mid \mu(T), \sigma(T)\}\right) - \left(1 - \frac{\int_0^T P\{S_L \le x \le S_U \mid \mu(t), \sigma(t)\}dt}{T}\right)\right\} \quad (4)$$

Simplifying this equation further, Eq. 4 can be put in the form,

$$\gamma = D_T - \bar{D}_T$$

where $\gamma = \frac{C_{aV}}{C_W}$ (5)

= cost ratio

$$D_T = qT\{1 - P\{S_L \le x \le S_U \mid \mu(T), \sigma(T)\}\}$$

= expected number of defectives at time T of the maintenance cycle

$$\bar{D}_T = qT\left\{1 - \frac{\int_0^T P\{S_L \le x \le S_U \mid \mu(t), \sigma(t)\}dt}{T}\right\}$$

= average number of defectives produced during the period T.

An interpretation of Eq. 5 can now be given. It is clear that if the excess in the expected number of defectives at time T of the maintenance cycle over the number of defectives during the period T, i.e., $D_T - \bar{D}_T$ increases due to deviations in μ and/or σ, the cost ratio, γ, should also increase to satisfy the optimal conditions of the system. This means that the average cost of a single sharpening or replacement of the tool, C_{aV}, should be higher relative to the average cost of reworking and/or scrapping a defective item, C_W, in such a way as to justify the expected increase in the number of defectives as detected by the trend of the expected number of defectives at time T (i.e., D_T). This actually implies that the expected number of defectives at time T is used in Eq. 4 to detect the future trend of the number of defectives that will be produced by the system. This, in turn, is used to set the cost ratio, γ, to the appropriate value which assures that optimal conditions are satisfied.

Before proceding to introduce the method for solving Eq. 4, it should be noted that an explicit solution for T in terms of the parameters of the system is not promising. We will thus avoid this difficulty by specifying appropriate numerical values for T and then solving Eq. 4 for the corresponding optimal values of the cost ratio, γ. Once the table giving the optimal values of T and γ is computed, one can use interpolation to determine the optimal value of T corresponding to any specific case where the cost ratio, γ, is known. It should be noted, however, that even with this procedure we still are confronted with the difficulty of determing the numerical value of the integral,

$$\int_0^T P\{S_L \leq x \leq S_u \mid \mu(t), \sigma(t)\} dt$$

In some classes of distributions, it may be possible to evaluate this integral directly. However, in other situations, where the output of the process is described by a distribution function which is so complex mathematically that the above integral cannot be evaluated directly, e.g., the important case of the normal distribution, it would be necessary to use an approximate numerical method such as the trapezoidal formula.[4]

This formula can be summarized as follows:

$$\int_a^b \phi(x)\,dx = \sum_{i=1}^{n-1} \frac{\phi(x_i) + \phi(x_{i+1})}{2} \Delta x_i \qquad (6)$$

Where

$$\Delta x_i = x_{i+1} - x_i \qquad i = 1, 2, \ldots, n-1$$

$$\sum_{i=1}^{n-1} \Delta x_i = b - a$$

It should be noted from the basic definition of the integration process, that the right hand side of Eq. 6 approaches the exact value of the integral as Δx_i approaches zero, for all values of i. This means that in using Eq. 6 above, it is desirable to select Δx_i as small as possible.

It is interesting to note that in a practical situation, the above intogral while represents the average fraction of defective items produced during the period T, can actually be determined by noticing the actual number of defectives that are produced by the system as a function of time. This must be determined by using an appropriate sampling method so as to obtain a good estimate of the value of the integral. It is moted, however, that in order to solve Eq. 4, it is still necessary to know the distribution function of x.

MODEL II

In this model it is assumed that at the end of the period T, the tool is either sharpened (adjusted), or replaced with probabilities p and 1-p respectively. This assumption is compared with that of Model I above where it is assumed that the tool is replaced after m-1 sharpenings or adjustments. Investigation shows that such a difference will only cause a change in the expression for TC_1, the total cost of sharpening and/or replacing the tool during the manufacturing of the whole lot of size Q. The expression for TC_2, the total cost of reworking and/or scrapping the defective items in the whole lot, on the other hand, will remain the same as in Model I. Hence in the present model,

$$TC_1 = \left[\frac{Q}{qT}\right]\left\{p c_{sa} + (1-p)\, c_{sr}\right\}$$

$$\simeq \frac{Q}{qT}\left\{p\, c_{sa} + (1-p)\, c_{sr}\right\}$$

Clearly, the value $\left\{p c_{sa} + (1-p)\, c_{sr}\right\}$ is equal to the average cost of a single replacement and/or adjustment of one tool. Thus, by using the symbol c_{av} to represent this average cost, Eq. 4 above can still be used to represent this model.

NUMERICAL EXAMPLE

The purpose of this example is to illustrate the method of computing he table which gives the optimal values of the maintenance period, T, and the cost ratio, γ. As mentioned above, the procedure in this model is to specify the values of T and then to compute the corresponding optimal values of γ.

In this example, it is assumed that the process is described by a normal distribution, with a time dependent mean, $\mu(t) = \mu_0 + t$, where μ_0 is the mean of the process at time t=0. The standard deviation of the process is assumed to be constant and independent of time; i.e. $\sigma(t) = \sigma$. For simplicity we will take $\sigma = 1$. Assume further that the specification limits of the process are symmetrical around its mean at time $t = 0$ i.e., around μ_0. This means,

$$S_L = \mu_0 - 3\sigma = \mu_0 - 3$$

$$S_U = \mu_0 + 3\sigma = \mu_0 + 3$$

To complete the list of parameters necessary for solving Eq. 4 above, it is assumed that the production rate of the process, q is equal to 10 items per unit time.

Table I gives the computations necessary for the determination of the optimal cost ratio γ! The value of T shown in Col. 1 are specified in advance. Using the formula $\mu(T) = \mu + T$ to determine the mean of the process at time T, the corresponding values of he probabilities, $P\{|x - \mu(T)| \leq 3\}$, can then be [illegible]

[illegible]

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[5]The idea of this example is taken from B. L. Grant, op. cit., pp. 121-123.

determined from the normal tables as shown in Col. 2.[6] Once these values are determined, Eq. 6 above can be used to determine the corresponding approximate value of the integral

$$\int_0^T P\{|x-\mu(t)| \leq 3 / \sigma = 1\}\, dt$$

The values of this integral are given in Col. 3. Equation 4 can now be used to determine the optimal values corresponding to the various values of T. These final results are given in Col. 4.

It is clear from the results in Table I that for the special case investigated here, the cost ratio, γ, is a monotone increasing function in T. This follows from he fact that as T increases, the percentage of defectives resulting from the deviation in the mean of the process also increases. It then follows that in order to justify the increase in T, the average cost of a single adgustment or replacement of a tool, c_{av} must be higher relative to the average cost of reworking and/or scrapping a defective item, c_w. It should be noted that, in general, the cost ratio, γ, is expected to be a monotone increasing function in T. This is suggested from the fact that as the tool is used longer,

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[6] For small increments of T, where intrapolation in the normal tables may not give satisfactory approximation or in the case where digital computers are used for computing the results, one can use the following approximation. Given

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2}\, dt \qquad 0 \leq x \leq \infty$$

where the value of (x) can be approximated by,

$$\phi^*(x) = 1 - \frac{1}{(a_1 + a_2 x + a_3 x^2 + a_4 x^3)^4}$$

where $a_1 = 0.278393$
$a_2 = 0.230389$
$a_3 = 0.000972$
$a_4 = 0.078108$

The maximum absolute error resulting from this approximation; i.e., $|\phi(x) - \phi^*(x)|$, does not exceed 0.0005. This formula along with many others for approximating the values of different functions can be found in Cecil Hastings, Jr., Approximations for Digital Computers, (Princeton University, N. J.: Princeton Univerisity Press. 1955).

TABLE I

COMPUTATIONS OF THE OPTIMAL COST RATIO, γ, FOR THE VARIOUS VALUES OF THE MAINTENANCE PERIOD T, FOR A NORMAL PROCESS WITH MEAN $\mu(t) + \mu_0 + t$ AND STANDARD DEVIATION $\sigma(t) = \sigma' = 1$ AND FOR THE SPECIFICATION LIMITS $\mu_0 \pm 3$

(Production rate, q=10 items per unit time)

T (1)	$P\{\lvert X-\mu(T)\rvert \leq 3\}$ (2)	$\int_0^T P\{\lvert X-\mu(t)\rvert \leq 3\}dt$ (3)	γ (4)
0.00	0.997200	-	-
0.20	0.996757	0.199406	0.0005
0.40	0.995001	0.398582	0.0058
0.60	0.991643	0.597246	0.0226
0.80	0.986027	0.795013	0.0619
1.00	0.977217	0.991338	0.1412
1.20	0.964056	1.185465	0.2859
1.40	0.945194	1.376391	0.5311
1.60	0.919230	1.562834	0.9205
1.80	0.884900	1.743248	1.5042
2.00	0.841300	1.915868	2.3326
2.20	0.788100	2.078808	3.4499
2.40	0.725700	2.230188	4.8850
2.60	0.655400	2.368298	6.6452
2.80	0.579300	2.491768	8.6972
3.00	0.500000	2.599698	10.9970

its condition will become worse.

In order to show that the values in Table I satisfy the optimal conditions of the system, the two values, $\gamma = 0.9205$ and $\gamma = 4.8850$, are selected from Table I and the values of $\frac{TC}{c_w}$ (see Eq. 2) versus T are computed for a lot of size $Q = 100$ items (notice that in Eq. 4 the determination of the optimal value of T is independent of the value of Q) and for the same conditions as are given in the above example. These results are given in Table II. Figure 2 illustrates these results graphically. The minimum value of $\frac{TC}{c_w}$ or equivalently the minimum value of TC is shown in Fig. 2 to occur at $T = 1.6$ for $\gamma = 0.9205$ and at $T = 2.4$ for $\gamma = 4.8850$. This shows that the values in Table I correspond to optimal conditions.

CONCLUSIONS

In this paper a new procedure for determining the optimal policy for a machine tool replacement has been presented. It is noted that this method is more advantegeius to the available methods because it takes into consideration the costs incurred as a result of producing defective items, due to the poor condition of the tool, as well as the other costs that are associated with the methods now being used.

The models presented here, however, assume that the increase in the number of defectives is due to the malfunction of the tool only. This excludes the presence of any other assignable causes, such as the operator, the machine, and/or the raw material used for manufacturing the product. It may been be necessary to introduce some correction factors in the time-trends of the mean and the standard deviation of the distribution to allow for the effect of these external causes.

TABLE II

VALUES OF $\frac{TC}{c_w}$ VERSUS THE MAINTENANCE PERIOD T FOR TWO VALUES OF γ UNDER THE SAME CONDITIONS GIVEN IN TABLE I AND FOR Q = 100 ITEMS

	$\frac{TC}{c_W}$	
T	$\gamma = 0.09205$	$\gamma = 4.8850$
0.20	46.32	244.55
0.40	23.36	122.48
0.60	15.80	81.87
0.80	12.12	61.68
1.00	10.07	49.71
1.20	8.88	41.92
1.40	8.26	36.58
1.60	8.07	32.85
1.80	8.26	30.29
2.00	8.80	28.63
2.20	9.69	27.71
2.40	10.91	27.43
2.60	12.45	27.70
2.80	14.29	28.45
3.00	16.41	29.62

In order to make this procedure more useful, it will be necessary to conider the case where the tool is used to process more than one measurable dimension of the manufactured product rather than limiting it to one dimension only. This, however, would need a special consideration of the specific product to be manufactured.

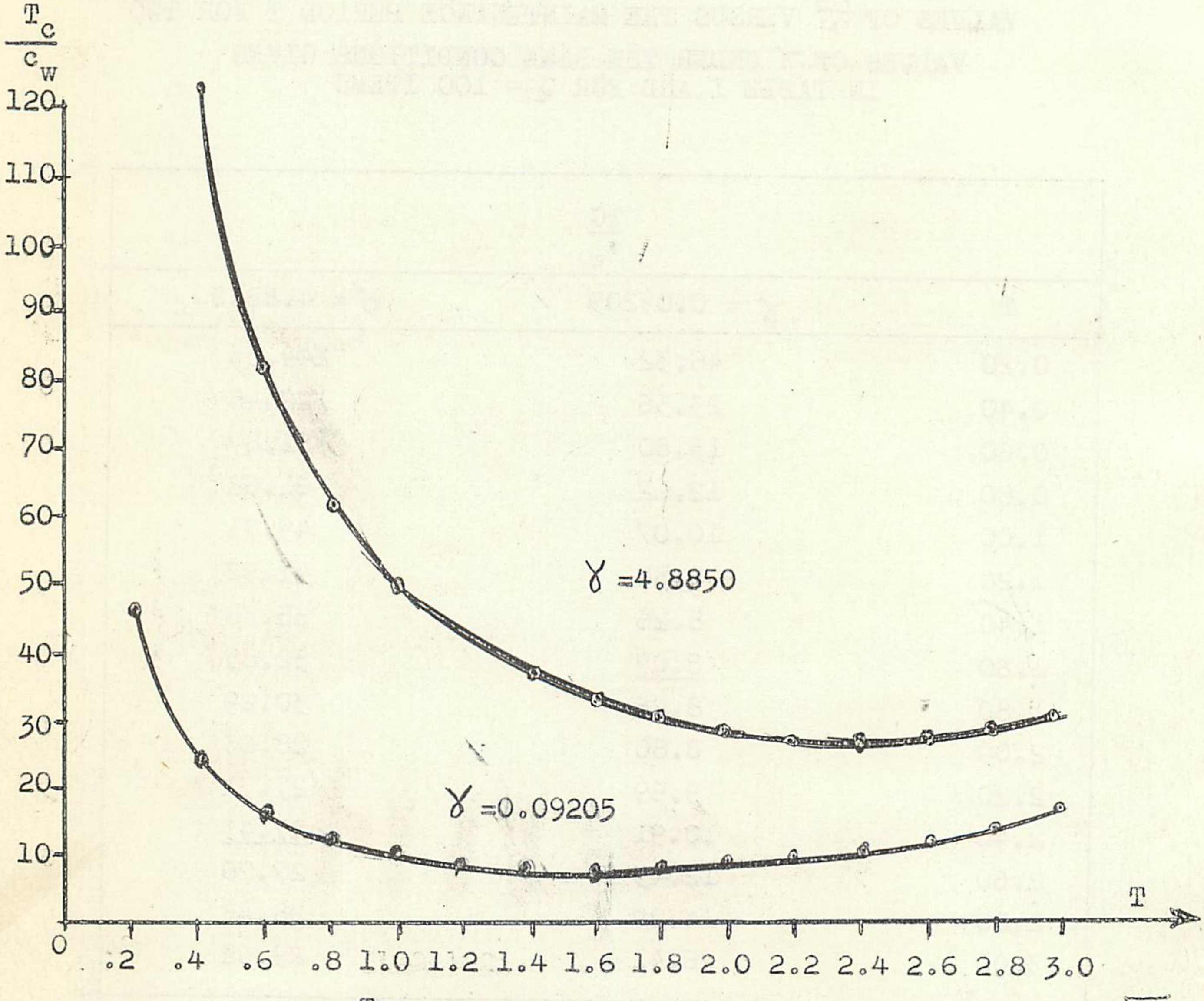

FIGURE 2: PLOT OF $\frac{T_c}{c_W}$ VERSUS T FOR γ = 0.09205 & γ = 4.8850

Source of data: Table II

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