UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

Memo No. 424

Interpolation formulas

By

Dr. Abbas I. Abdel Karim

(April 1964)

Contents

Interpolation formulas

Interpolation Formulas

1. Introduction

Interpolation has been said to be the art of reading between the lines of tabulated values of a function. We may now make a distinction between interpolation and extrapolation. The latter is the art of reading before the first line or after the last line of a tabulated function. More specifically, we may define interpolation as the process of finding the values of a function for any value of the independent variable within an interval for which some values are given and extrapolation as the process of finding the values outside of this interval.

The process of interpolation becomes very important in advanced mathematics when dealing with functions which either are not known at every value of the independent variable within an interval, or the expression of which is so complicated that the evaluation of the function is prohibitive. It is then that the function is replaced by a simple function which assumes the known values of the given function and from which the other values may be computed to the desired degree of accuracy. This is the broader sense of interpolation.

In precise mathematical language we are concerned with a function, $y = y(x)$, whose values $y_0, y_1, \dots , y_n$, are known for the values $x_0, x_1, \dots , x_n$ of the independent variable. Interpolation now seeks to replace $y(x)$ by a simpler function, $P(x)$, which has the same values as $y(x)$ for $x_0, x_1, \ldots, x_n$ and from which other values can easily be calculated. The function $P(x)$ is said to be an interpolating formula or interpolating function. In many engineering application this function is called a smoothing function.

A desired characteristic of interpolating functions is that they be simple. Consequently, the most frequently employed forms are the polynomial and the finite trignometric series. In these cases we refer to the process as polynomial interpolation or trignometric interpolation. The latter is used if the given values indicate that the function is periodic. The interpolating function can, of course, be arbitrarily chosen and can take any form; thus it could be exponential, logarithmic, etc. One such form which is frequently used is that of a rational fraction. However, it should always be as simple as possible.

The use of the polynomial and trignometric series is based on Weierstrass' Theorems

I. Every function, $F(x)$, which is continuous in an interval (a,b) can be represented there, to any degree of accuracy, by a polynomial $P(x)$, i.e.,

$$|F(x) - P(x)| < \varepsilon$$

for all $a < x < b$ and where ε is any preasigned positive quantity.

II. Every continuous function, $F(x)$, of period $2\,\Pi$ can be represented by a finite trignometric series:

$$T(x) = a_o + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx$$

such that

$$|F(x) - T(x)| < \varepsilon$$

for $a < x < b$ and $\varepsilon > 0$

2 - General Interpolation Formulas
Arbitrarily Spaced Data

2.1 Interpolation Polynomials

A polynomial of degree n

$$(1) \qquad y(x) = a_o + a_1 x + a_2 x^2 + \dots + a_n x^n = \sum_{k=o}^{n} a_k x^k$$

has n + 1 coefficients and it can be required to pass through the n + 1 points (x_i, y_i) $i = 0, 1, 2, \dots, n$ with $x_i \neq x_j$. To find the polynomial we see that we can always solve for the coefficients a_k by Cramers rule or by any other method.

Putting the values (x_i, y_i) $i = o, 1, 2, \dots, n$ into the polynomial (1), we have

$$y_i = \sum_{k=o}^{n} a_k \, x_i^k \quad , \; i = o, 1, 2, \ldots , n$$

this means

$$(2) \quad \begin{cases} y_o = a_o + a_1 x_o + a_2 x_o^2 + \ldots + a_n x_o^n \\ y_1 = a_o + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = a_o + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n \end{cases}$$

The determinant of the unknowns a_k of these $n + 1$ linear equations is the Vandermonde determinant:

$$(3) \quad D = \begin{vmatrix} 1 & x_o & x_o^2 & \ldots & x_o^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{vmatrix}$$

$$= (x_1 - x_o)(x_2 - x_o)(x_3 - x_o) \ldots (x_n - x_o)$$
$$(x_2 - x_1)({}_3 - x_1) \ldots (x_n - x_1)$$
$$(x_3 - x_2) \ldots (x_n - x_2)$$
$$\ldots\ldots\ldots$$
$$(x_n - x_{n-1})$$

$$= \prod_{i > k} (x_i - x_k)$$

Thus the Vandermonde determinant vanishes if and only if any two of the x_i coincide.

Equations (1) and (2) can be regarded as n + 2 homogeneous equations in the n + 2 quantities $-1, a_o, a_1, \dots, a_n$. Hence their determinant must vanish:

$$(4) \quad \begin{vmatrix} y(x) & 1 & x & x^2 & \dots & x^n \\ y_o & 1 & x_o & x_o^2 & \dots & x_o^n \\ y_1 & 1 & x_1 & x_1^2 & \dots & x_1^n \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ y_n & 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = 0$$

This can be regarded as an equation in y(x).

If we expand this determinant by elements of the first row, the first term will be D y(x), and every other term will be equal to some x^j multiplied by its cofactor, which is a constant. Hence when we solve for y(x), we shall have y(x) expressed as a linear combination of the functions x^j in just the form

$$y = a_o + a_1 + \dots + a_n x^n$$

as required. Also if we set $x = x_i$ in (4) and subtract the i + 2 from the first row we get

$$D\,(y(x_i) - y_i) = 0$$

which shows that $y(x_i) = y_i$, and the values of y at the points x_i agree with those of f(x).

It is possible that the coefficients of x^n is zero and that the polynomial is of degree less than n. To cover this detail, the statement that a polynomial is of degree n often means of degree n or less. In the trivial case when all the y_i are equal the polynomial is of degree zero, and y(x) = C (constnat).

2.2. The lagrange Method of Interpolation

The basic idea behind the method is first to find a polynomial which takes on the value 1 at a particular sample point and the value 0 at all the other sample points i.e.

$$L_i(x) = \begin{cases} 1 & \text{for } x = x_i \\ 0 & \text{for } x = x_j \quad j \neq i \end{cases}$$

It is easy to see that the function

$$(5) \quad L_i(x) = \frac{(x-x_o)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_o)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

$$= \prod_{j=o}^{n} \frac{(x-x_j)}{(x_i-x_j)} = \prod_{\substack{j=o \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

(where the prime on the product means " excluding the ith value") is such a polynomial of degree n_i it is 1 when $x = x_i$ and 0 when $x = x_j \quad j \neq i$.

The polynomial $L_i(x)\, y_i$ takes on the value y_i at the sample point x_i and is zero at all other sample point. It then follows the well, known lagrange formula:

$$y(x) = L_o\,(x)\, y_o + L_1(x)\, y_1 + \ldots + L_n(x)\, y_n = \sum_{i=o}^{n} L_i(x) y_i,$$

where

$$L_o(x) = \frac{(x-x_1)\,(x-x_2)\,(x-x_3)\;\ldots\;(x-x_n)}{(x_o-x_n)(x_o-x_2)(x_o-x_3)\;\ldots\;(x_o-x_n)}$$

$$L_1(x) = \frac{(x-x_o)\,(x-x_2)\,(x-x_3)\;\ldots\;(x-x_4)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)\;\ldots\;(x_1-x_4)}$$

$$L_2(x) = \frac{(x-x_o)\,(x-x_1)\,(x-x_3)\;\ldots\;(x-x_n)}{(x_2-x_o)(x_2-x_1)(x_2-x_3)\;\ldots\;(x_2-x_n)}$$

. . . .

$$L_n(x) = \frac{(x-x_o)\,(x-x_1)\,(x-x_2)\;\ldots\;(x-x_{n-1})}{(x_n-x_o)(x_n-x_1)(x_n-x_2)\;\ldots\;(x_n-x_{n-1})}$$

With other method we can write the solution of (4) in the form:

$$D\, y(x) = \begin{vmatrix} o & 1 & x & x^2 & \dots & x^n \\ y_o & 1 & x_o & x_o^2 & & x_o^n \\ y_1 & 1 & x_1 & x_1^2 & & x_1^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ y_n & 1 & x_n & x_n^2 & & x_n^n \end{vmatrix} = - D^{*}$$

Thus

$$y(x) = - D^{-1} D^{*}$$

which coincides with (1) if we expand along the first row, but has the form,

$$y(x) = \sum_{i=o}^{n} L_i(x)\, y_i \qquad (7)$$

when we expand along the first column. The L_i are themselves polynomials with coefficients which depend only upon the x_j. These polynomials are

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \qquad (8)$$

They can be obtained by direct expansion of the determinant, or we can verify that they satisfy the necessary conditions if we not that $L_i(x_j) = \delta_{ij}$

with δ_{ij} the Kronecker δ. From this it follows that with $L_i(x)$ defined by (8) and $y(x)$ by (7) we have $y(x_i) = y_i$.

Note:

The Lagrange interpolation formula is not very practical for computations. Its form can be modified somewhat to make it more tractable.

Consider the special case when all the $y_i = 1$. Then $y(x) = 1$ for all x, that is

$$L_o(x) + L_1(x) + \ldots + L_n(x) = 1$$

is an identity.

We may now divide the right-hand side of the Lagrange formula (7)

$$y(x) = \sum_{i=o}^{n} L_i(x) \; y_i$$

by

$$\sum_{i=o}^{n} L_i(x) = 1 \tag{9}$$

and defining

$$u_i = \frac{1}{(x-x_i) \prod\limits_{\substack{i \\ (i \neq j)}} (x_i - x_j)} \tag{10}$$

we get,

when we divide numerator and denomiator by

$\prod_j (x-x_j)$:

(11) $$y(x) = \frac{\sum_{i=o}^{n} u_i \, y_i}{\sum_{i=o}^{n} u_i}$$

This is sometimes called the "barycentric formula" and is easier to use than the Lagrange formula. See the flow chart (1).

2.3 Newton's general interpolation formula

Newton's interpolation formula, which we now develop, is simply another way of writing the interpolating polynomial. It is useful because the number of points being used can easily be increased or decreased without repeating all the computation. Let the polynomial passing through the n + 1 points (x_i, y_i) , i=o,1,...,n be

(12) $$y(x) = \gamma_o + \gamma_1 (x-x_o) + \gamma_2 (x-x_o)(x-x_1) + \dots + \gamma_n (x-x_o)(x-x_1)\dots(x-x_{n-1})$$

It is now desired to determine the coefficients

γ_i, $i = 0, 1, \ldots, n$

$$(13)\quad\begin{cases} x = x_0 : & y_0 = \gamma_0 \\ x = x_1 : & y_1 = \gamma_0 + \gamma_1 (x_1 - x_0) \\ x = x_2 : & y_2 = \gamma_0 + \gamma_1 (x_2 - x_0) + \gamma_2 (x_2 - x_0)(x_2 - x_1) \\ \cdots & \cdots \\ x = x_n : & y_n = \gamma_0 + \gamma_1 (x_n - x_0) + \gamma_2 (x_n - x_0)(x_n - x_1) \\ & \quad + \ldots + \gamma_n (x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1}) \end{cases}$$

we can solve (13) for the coefficients γ_i as follows:

$$\gamma_0 = y_0,$$

$$\gamma_1 = \frac{y_1 - y_0}{x_1 - x_0} = [x_1 \; x_0]$$

$$\gamma_2 = \frac{y_2 - y_0 - \dfrac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\dfrac{y_2 - y_0}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}\,\dfrac{x_2 - x_0}{x_2 - x_1}}{x_2 - x_0}$$

$$= \left\{ \frac{y_2 - y_1}{x_2 - x_1} + \frac{y_1 - y_0}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\,\frac{x_2 - x_0}{x_2 - x_1} \right\} / (x_2 - x_0)$$

$$\gamma_2 = \left\{ \frac{y_2 - y_1}{x_2 - x_1} + \frac{y_1 - y_0}{x_1 - x_0} \right\} / (x_2 - x_0)$$

$$= \frac{[x_2 \; x_1] - [x_1 \; x_0]}{x_2 - x_0}$$

$$= [x_2 \; x_1 \; x_0]$$

Finaly we have :

$$\gamma_0 = y_0$$

$$\gamma_1 = [x_1 \; x_0] = \frac{y_1 - y_0}{x_1 - x_0}$$

(14) $$\gamma_2 = [x_2 \, x_1 \, x_0] = \frac{[x_2 \, x_1] - [x_1 \, x_0]}{x_2 - x_0}$$

$$\gamma_3 = [x_3 \, x_2 \, x_1 \, x_0] = \frac{[x_3 \, x_2 \, x_1] - [x_2 \, x_1 \, x_0]}{x_3 - x_0}$$

. . . .

$$\gamma_n = [x_n \, x_{n-1} \cdots x_0] = \frac{[x_n \, x_{n-1} \cdots x_1] - [x_{n-1} \, x_{n-2} \cdots x_0]}{x_n - x_0}$$

$\gamma_1, \gamma_2, \ldots \gamma_n$ are called the divided differences 1st, 2nd, ... , nth orda. It can be shown by induction that the divided differences are always symmetric functions of their argument.

The divided difference table which lies at the heart of Newtons interpolation formula may be sometimes more useful form.

This calculation of divided differences is best carried out in tabular form. For n = 3 the scheme is illustrated in the table below

x	y	$[x_i\ x_k]$	$[x_i\ x_k\ x_l]$	$[x_i\ x_k\ x_l\ x_m]$
x_o	y_o			
		$\frac{y_1-y_o}{x_1-x_o}=[x_1\ x_o]$		
x_1	y_1		$\frac{[x_2x_1]-[x_1x_o]}{x_2-x_o}=[x_2x_1x_o]$	
		$\frac{y_2-y_1}{x_2-x_1}=[x_2\ x_1]$		$\frac{[x_3x_2x_1]-[x_2x_1x_o]}{x_3-x_o}=[x_3x_2x_1x_o]$
x_2	y_2		$\frac{[x_3x_2]-[x_2x_1]}{x_3-x_1}=[x_3x_2x_1]$	
		$\frac{y_3-y_2}{x_3-x_2}=[x_3\ x_2]$		
x_3	y_3			

The divided difference in the top row of this table (the underlined values are the coefficients in the following function.

$$(15)\qquad y(x) = y_o + [x_1x_o](x-x_o) + [x_2x_1x_o](x-x_o)(x-x_1) + \ldots + [x_nx_{n-1}\ldots x_o](x-x_o)(x-x_1)\ldots(x-x_{n-1})$$

See the flow chart (2).

As a specific example consider the log table:

x	log x	[']	["]	[''']
1	0.0000			
2	0.3010	0.3010	-0.06245	+0.01230
3	0.4771	0.1761	-0.02555	
4	0.6021	0.1250		

$$y(x) = o + 0.3010\,(x-1) - 0.06245\,(x-1)\,(x-2) + 0.01230\,(x-1)\,(x-2)\,(x-3)$$

In particular,

$$y(2.5) = 0.3010.\ \frac{3}{2} - 0.06245.\ \frac{3}{2}\cdot\cdot\ \frac{1}{2} - 0.01230.\ \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= 0.40001$$

The correct value for log 2.5 is 0.3979

2.4. Error of the interpolation formula

Given a function $y(x)$, we have been taking n+1 points (x_i, y_i), $i = 0,1,\ldots,n$ and finding a polynomial $P_n(x)$ through these points. We then intend to use this polynomial in place of the original function, and it is therefore important to examing the question of how much the function and the polynomial can differ at points other than the sample points (where they agree within round off error).

A theoretical expression for the difference between the original function y(x) and the approximating polynomial $p_n(x)$ can be found as follows:

$$(16) \qquad R_{n+1}(x) = y(x) - P_n(x)$$

consider a point $x \neq x_i \quad i = 0, 1, \ldots, n$ then

$$[x\, x_o] = \frac{y(x) - y_o}{x - x_o}$$

$$[x\, x_o x_1] = \frac{[x\, x_o] - [x_o x_1]}{x - x_1}$$

$$[x\, x_o\, x_1 x_2] = \frac{[x\, x_o x_1] - [x_o x_1 x_2]}{x - x_1}$$

. . . .

$$[x\, x_o x_1 \cdots x_n] = \frac{[x\, x_o x_1 \ldots x_{n-1}] - [x_o x_1 \ldots x_n]}{x - x_n}$$

From these equations we get:

$$y(x) = y_o + [x\, x_o](x-x_o)$$

$$[x\, x_o] = [x_o x_1] + [x\, x_o x_1](x-x_1)$$

$$[x\, x_o x_1] = [x_o x_1 x_2] + [x\, x_o x_1 x_2](x-x_2)$$

. . . .

$$[x\, x_o x_1 \ldots x_{n-1}] = [x_o x_1 \ldots x_n] + [x\, x_o \ldots x_n](x-x_n)$$

Put the second equation in the first, the third in the second a.s.o. we have

$$y(x) = y_o + [x_o x_1](x-x_o) + [x_o x_1 x_2](x-x_o)(x-x_1) + \dots$$

$$+ [x_o x_1 \dots x_n](x-x_o) \dots (x-x_{n-1})$$

$$+ [x\ x_o x_1 \dots x_n](x-x_o) \dots (x-x_n)$$

(17) $$y(x) = p_n(x) + [x\ x_o x_1 \dots x_n](x-x_o)(x-x_1) \dots (x-x_n)$$

Thus we have:

(18) $$R_{n+1}(x) = (x-x_o)(x-x_1) \dots (x-x_n) . [x\ x_o x_1 \dots x_n]$$

$$= Q_{n+1}(x)\ S_{n+1}(x)$$

The error R_{n+1} appears as the product of two polynomials

Q_{n+1} and S_{n+1}

where

$$Q_{n+1}(x) = (x-x_o)(x-x_1) \dots (x-x_n)$$

$$S_{n+1}(x) = [x\ x_o x_1 \dots x_n]$$

But $R_{n+1}(x) = y(x) - p_n(x)$ vanishes at least n+1 times

by Roll theorem[1]: $y'(x) - p_n'(x)$ " " " n "

$y''(x) - p_n''(x)$ " " " n-1 "

. . . .

$y^{(n)}(x)-p_n^{(n)}(x)$ " " " one time

At a point ξ in the interval $x_o x_1 \ldots x_n$:

$$y^{(n)}(\xi) - p_n^n(\xi) = 0$$

$$y^{(n)}(\xi) - n! \left[x_o x_1 \ldots x_n\right] = 0$$

or

$$(19) \quad \left[x_o x_1 \ldots x_n\right] = \frac{1}{n!} y^{(n)}(\xi)$$

From (18) and (19) we get:

$$(20) \quad R_{n+n}(x) = (x-x_o)(x-x_1) \ldots (x-x_n) \frac{y^{(n+1)}(\xi)}{(n+1)!} \quad ; x_o \leq \xi \leq x_n$$

---

1) Rolle's Theorem: If f(x) is continuous in the closed interval $a \leq x \leq b$ and differentiable in the open interval $a < x < b$, and if f(a) = f(b) = o, it is possible to find at least one point ξ inside the interval, such that $f'(\xi) = 0$, $(a < \xi < b)$.

2.5. Linear Interpolation

The simplest of all interpolation is the case in which the interpolating polynomial is linear.

Thus

$$y = y_o + \frac{y_1 - y_o}{x_1 - x_o} (x - x_o) \tag{21}$$

One of the most convenient methods of writing the linear interpolating function for machine calculations is given by

$$y(x) = \frac{1}{x_1 - x_o} \begin{vmatrix} y_o & x_o - x \\ y_1 & x_1 - x \end{vmatrix} \tag{22}$$

$$= \{ y_o(x_1 - x) - y_1(x_o - x) \} / (x_1 - x_o)$$

which is linear in x.

The right hand side of formula (22) is easily evaluated since it is the difference of two products, $y_o(x_1-x) - y_1(x_o-x)$, followed by a division by x_1-x_o and is thus accomplished on a calculator without recording any intermediate

2.6 Aitken's Repeated Process

Formula (22) presents a method for linear interpolation which is convenient for machine calculations. As written the formula interpolates in the interval from x_o to x_1. It could also be written in a form to interpolate in the interval from x_o to x_2 , thus

$$(23) \qquad y(x) = \frac{1}{x_2 - x_o} \begin{vmatrix} y_o & x_o - x \\ y_2 & x_2 - x \end{vmatrix}$$

Some distinction in the notation y(x) must be made in the expressions (22) and (23). To accomplish this in a logical manner let us order the values for the independent variable x in the sequence $x_o, x_1 x_2, \ldots, x_n$ We shall always consider the beginning of the interval for interpolation to be at x_o. Then we may write

$y_{11}(x) = y(x)$ in the interval x_o to x_1

$y_{21}(x) = y(x)$ " " " x_o to x_2

or in general

$$(24) \qquad y_{i1}(x) = y(x) = \frac{1}{x_i - x_o} \begin{vmatrix} y_o & x_o - x \\ y_i & x_i - x \end{vmatrix}$$

denotes linear interpolation using the values (x_o, y_o) and (x_i, y_i). The function $y_{i1}(x)$ is a linear function of x. By changing the interval we can thus build up a set of values $y_{i1}(x)$, $(i = 1,2,\ldots,n)$. If linear interpolation is exact these values would all be alike. On the other hand if the function $y = f(x)$ is not linear, these values would all differ by some amount.

Let us list these values together with the difference $x_i - x$ in the following manner:

$$
\begin{array}{lll}
x_0 & & \\
x_1 & y_{11}(x) & x_1 - x \\
x_2 & y_{21}(x) & x_2 - x \\
x_3 & y_{31}(x) & x_3 - x \\
\vdots & \vdots & \\
x_i & y_{i1}(x) & x_i - x
\end{array}
$$

we could then apply the linear interpolation formula to these entries. Thus

(25) $$y(x) = \frac{1}{x_2 - x_1} \begin{vmatrix} y_{11}(x) & x_1 - x \\ y_{21}(x) & x_2 - x \end{vmatrix} = y_{22}(x)$$

It is clear that $y_{22}(x)$ satisfies all the criteria for a second degree interpolating polynomial. The process may be applied to the intaval from x_1 to x_i so that in general we may write

(26) $$y_{i2}(x) = \frac{1}{x_i - x_1} \begin{vmatrix} y_{11}(x) & x_1 - x \\ y_{i1}(x) & x_i - x \end{vmatrix}$$

We may now compute a set of values, $y_{i2}(x)$, $i = 2, \ldots, n$, and form the table

$$\begin{array}{lll} x_2 & y_{22}(x) & x_2-x \\ x_3 & y_{32}(x) & x_3-x \\ \vdots & \vdots & \vdots \\ x_i & y_{i2}(x) & x_i-x \end{array}$$

Again applying the linear interpolation formula we could obtain.

$$(27) \qquad y_{i3}(x) = \frac{1}{x_i-x_2}\begin{vmatrix} y_{22}(x) & x_2-x \\ y_{i2}(x) & x_i-x \end{vmatrix}$$

which yields a third interpolating polynomial. The Process can be repeated until all of the entries of the original table of values have been consumed. It is easily seen that the general formula is given. by

$$(28) \qquad y_{ik}(x) = \frac{1}{x_i - x_{k-1}}\begin{vmatrix} y_{k-1,k-1}(x) & x_{k-1}-x \\ y_{i,k-1}(x) & x_i \; -x \end{vmatrix}$$

in which k denotes the number of times linear interpolation has been applied and also the degree of the polynomial. The subscript i assumes the values k, k+1, ..., n. The process is know as Aitkens method or, more appropriately, as Aitkens' Repeated Process. It is a very useful process since the calculations are easily performed on a calculating machine, and furthermore, it provides its own criterion of when the process has been carried for enough. The work should be arranged as shown below.

x_0	y_0				x_0-x
x_1	y_1	$y_{11}(x)$			x_1-x
x_2	y_2	$y_{21}(x)$	$y_{22}(x)$		x_2-x
x_3	y_3	$y_{31}(x)$	$y_{32}(x)$	$y_{33}(x)$	x_3-x
x_4	y_4	$y_{41}(x)$	$y_{42}(x)$	$y_{43}(x)$	x_4-x

Computational Form for Aitken's Process.

2.7 Inverse Interpolation

This subject matter deals with the process of finding the value of the argument corresponding to a given value of the function which is between two tabulated values. An easy method of doing inverse interpolation is Aitkens Repeated Process applied to data after interchanging the roles of the dependent and independent variables. Furthermore:

Lagranges formula adapts itself very nicely to inverse interpolation since formula (6)

$$y(x) = L_o(x)\, y_o + L_1(x)\, y_1 + \dots + L_n(x)\, y_n$$

is simply a relation between two variables, either of which may be considered the independent variable. Thus, we can write x as a function of y:

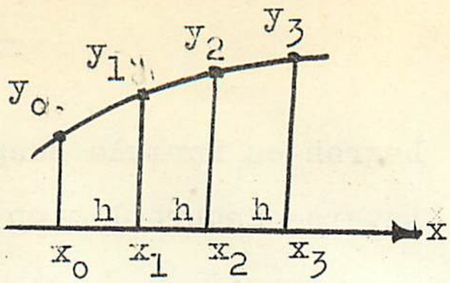
$$(29) \quad x(y) = L_o(y)\, x_o + L_1(y) x_1 + \dots + L_n(y)\, x_n$$

Normally, the values of y_i will be unequally spaced and the method just described must ingeneral be employed. Furthermore, it will be more difficult to find a linear transformation which will reduce the size of the numbers.

3. Special Interpolation formulas Interpolation formulas for equidistant function values.

3.1 Newton's formula for Interpolation

Very often our information about a function is given at a set of equally spaced values of x: $x_{i+1} - x_i = h$ $(i = o, 1, \dots, n-1)$. This simplifies much of the notation and computation as well as the ideas involved. For equally spaced data it is customary to use the operator Δ such that



(30) $$\Delta y_n = y_{n+1} - y_n$$

This is the familiar notation of the difference calculus except that we have fixed $\Delta x = h$ (the interval-length) in all times:

$$\Delta x_n = x_{n+1} - x_n = h$$

We also have

$$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n = y_{n+2} - 2y_{n+1} + y_n$$

(31) $$\Delta^3 y_n = \Delta^2 y_{n+1} - \Delta^2 y_n$$

$$\Delta^\nu y_n = \Delta^{\nu-1} y_{n+1} - \Delta^{\nu-1} y_n$$

These differences are related to the divided differences (14) as follows:

$$\gamma_0 = y_0$$

$$\gamma_1 = [x_1x_0] = \frac{y_1-y_0}{x_1-x_0} = \frac{\Delta y_0}{h}$$

$$\gamma_2 = [x_2x_1x_0] = \frac{[x_2x_1] - [x_1x_0]}{x_2 - x_0} = \frac{\Delta y_1 - \Delta y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}$$

(32) $$\gamma_3 = [x_3x_2x_1x_0] = \frac{[x_3x_2x_1] - [x_2x_1x_0]}{x_3 - x_0} = \frac{1}{2}\,\frac{\Delta^2 y_1 - \Delta^2 y_0}{3h^3} \equiv \frac{\Delta^3 y_0}{3!h^3}$$

$\vdots$

$$\gamma_\nu = [x\ x_{-1} \cdots x_1x_0] = \frac{\Delta^\nu y_0}{\nu!h}$$

From (32) and (19) we get:

$$\frac{\Delta^{\nu} y_o}{h^{\nu}} = y^{(\nu)}(\xi)$$

Newtons formula in this new notation for equidistant function :

$$y(x) = y_o + \frac{\Delta y_o}{h}(x-x_o) + \frac{\Delta^2 y_o}{2!h^2}(x-x_o)(x-x_1) + \dots + \frac{\Delta^n y_o}{n!h^n}(x-x_o)(x-x_1)\dots(x-x_{n-1}) \quad (33)$$

See the flow chart (3).

If we suppose that $x_o = o$, we get

$$y(x) = y_o + \frac{\Delta y_o}{h} x + \frac{\Delta^2 y_o}{2!h^2} x(x-h) + \frac{\Delta^3 y_o}{3!h^3} x(x-h)(x-2h) + \dots \quad (34)$$

and if we further assume h = 1 we have

$$y(x) = y_o + \Delta y_o x + \frac{\Delta^2 y_o}{2!} x(x-1) + \frac{\Delta^3 y_o}{3!} x(x-1)(x-2) + \dots \quad (35)$$

The work should be arranged as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_o	y_o				
		Δy_o			
x_1	y_1		$\Delta^2 y_o$		
		Δy_1		$\Delta^3 y_o$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_o$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$	$\vdots$	$\vdots$
		Δy_3	$\vdots$		
x_4	y_4	$\vdots$			
$\vdots$	$\vdots$				

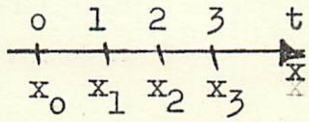
3.2. Gregory-Newtons Interpolation formula

Since the values x_i $(i = o, 1, \dots, n)$ are chosen at equidistant values we have $x_i - x_o = ih$, $i = 1, \dots, n)$, where h is the interval length, x. Let us now make a transformation on the variable x by letting

(36)
$$\boxed{t = \frac{x - x_o}{h}}$$

and note that

$$\frac{x-x_i}{h} = \frac{x-(x_o+ih)}{h} = \frac{x-x_o}{h} - i = t - i$$



for $i = 1, \dots, n-1$. It is also seen that each term of formula (3.3) contains an h in the denominator for each parenthetical expression $(x-x_i)$ so that by (36) we have from (33):

GNI

(37)
$$\boxed{y(t) = y_o + t\Delta y_o + \binom{t}{2}\Delta^2 y_o + \binom{t}{3}\Delta^3 y_o + \dots + \binom{t}{n}\Delta^n y_o}$$

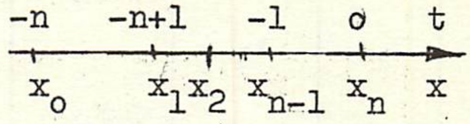
where $\binom{t}{i}$ is the binomial coefficient sysmbol. (see the flow chart (4).

This is Newtons Forward Interpolation Formula. It is called the forward interpolation formula since it utilizes y_o and higher-order differences of y_o. Consequently, it is used when it is desired to find values of y at the begining of a table. If we make a transformation on the variable x be letting

(38) $$\boxed{t = \frac{x - x_n}{h}}$$

so that

$$\frac{x - x_i}{h} = \frac{x - (x_n - ih)}{h} = t + i$$



then we have :

GNII

(39) $$y(t) = y_n + t\nabla y_n + \binom{t+1}{2}\nabla^2 y_n + \dots + \binom{t+n-1}{n}\nabla^n y_n$$

where the operator ∇ is defined such that:

$$\nabla y_1 = y_1 - y_o \;,\; \nabla y_2 = y_2 - y_1, \;\dots$$

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \; \nabla^2 y_3 = \nabla y_2 - \nabla y_1, \dots$$

$$\vdots$$

$$\nabla^{\nu} y_k = \nabla^{\nu-1} y_k - \nabla^{\nu-1} y_{k-1}$$

This is Newtons Backward Interpolation Formula. The formula is used when it is decired to find values of the function near the end of a table.

The work should be arranged as follows:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_o	y_o				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3			...
x_3	y_3		$\nabla^2 y_4$	$\nabla^3 y_4$	
		∇y_4	...	...	
		...			
x_4	y_4				
...	...				

It is clear that $\nabla^2 y_3 = \nabla^2 y_1$

and in general $\nabla^\nu y_k = \nabla^\nu y_{k-}$

3.3 Gauss Interpolation formulas

Central Difference Schema

We define the Operator S such that:

$$\delta y_{1/2} = y_1 - y_o \quad , \quad \delta y_{3/2} = y_2 - y_1, \; \ldots.$$

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2} \quad , \quad \delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2} \; , \; \ldots$$

$$\delta^\nu y_k = \delta^{\nu-1} y_{k+1/2} - \delta^{\nu-1} y_{k-1/2}$$

We can prove that:

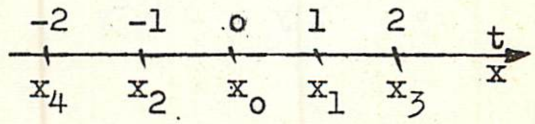
$$(40) \qquad \delta^{\nu} y_k = \nabla^{\nu} y_{k+\nu/2} = \Delta^{\nu} y_{k-\nu/2}.$$

Instead of x we can use the variable t such that:

$t = \ \dots \ -2 \ \ -1 \ \ 0 \ \ 1 \ \ 2 \ \dots \qquad \Delta t = 1, \ t = \frac{x - x_o}{h}$

$x = \ \dots \ x_4 \ \ x_2 \ \ x_o \ \ x_1 \ \ x_3 \ \dots \qquad \Delta x = h$

G I

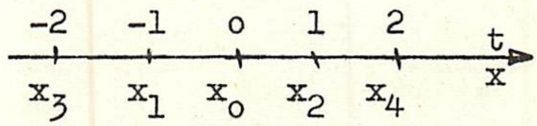


or

$t = \ \dots \ -2 \ \ -1 \ \ o \ \ 1 \ \ 2 \ \dots \qquad t = 1$

$x = \ \dots \ x_3 \ \ x_1 \ \ x_o \ \ x_2 \ \ x_4 \ \dots \qquad x = h$

G II



From (40) and the Newton formula (33)

we get the following Gauss forumaes:

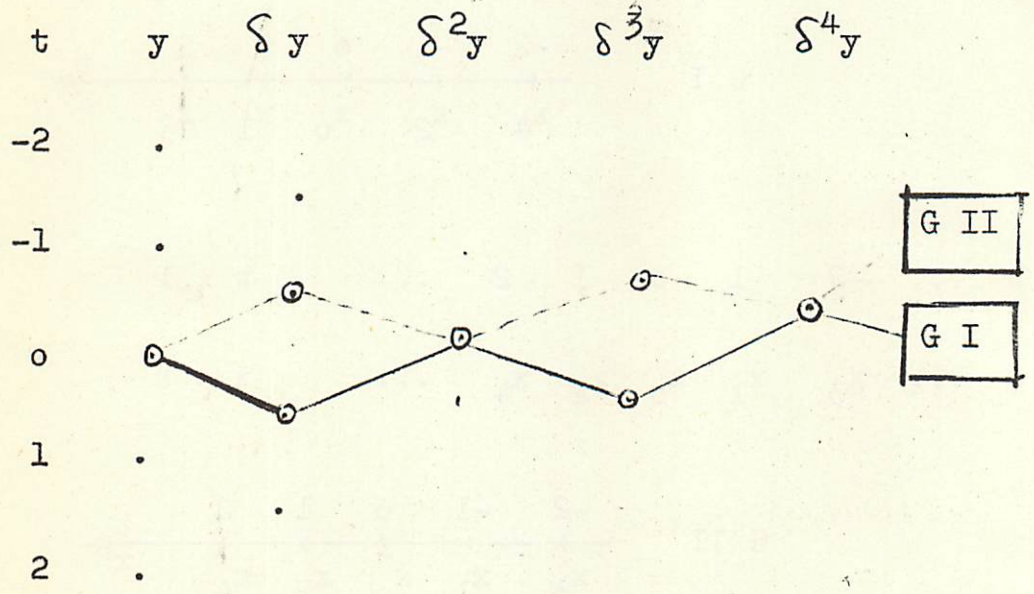
G I

$$(41) \qquad y(t) = y_o + \binom{t}{1} \delta y_{1/2} + \binom{t}{2} \delta^2 y_o + \binom{t+1}{3} \delta^3 y_{1/2} + \binom{t+1}{4} \delta^4 y_o + \binom{t+2}{5} \delta^5 y_{1/2} + \dots$$

G II

$$(42) \quad y(t) = y_0 + \binom{t}{1}\delta y_{-1/2} + \binom{t+1}{2}\delta^2 y_0 + \binom{t+1}{3}\delta^3 y_{-1/2} + \binom{t+2}{4}\cdot \delta^4 y_0 + \binom{t+2}{5}\delta^5 y_{-1/2} + \ldots$$

For the calculations the following scheme is more convenient:



It is advisable for the practical calculation to use the following Everett-Laplace formula.

3.4. Everett-Laplace formula

From (41) we have:

$$y(t) = y_o + \binom{t}{1}\delta y_{1/2} + \binom{t}{2}\delta^2 y_o + \binom{t+1}{3}\delta^3 y_{1/2} +$$

$$\binom{t+1}{4}\delta^4 y_o + \binom{t+2}{5}\delta^5 y_{1/2} + \dots$$

$$= y_o + \binom{t}{1}(y_1 - y_o) + \binom{t}{2}\delta^2 y_o + \binom{t+1}{3}(\delta^2 y_1 - \delta^2 y_o)$$

$$+ \binom{t+1}{4}\delta^4 y_o + \dots$$

Put $s = 1 - t$ and

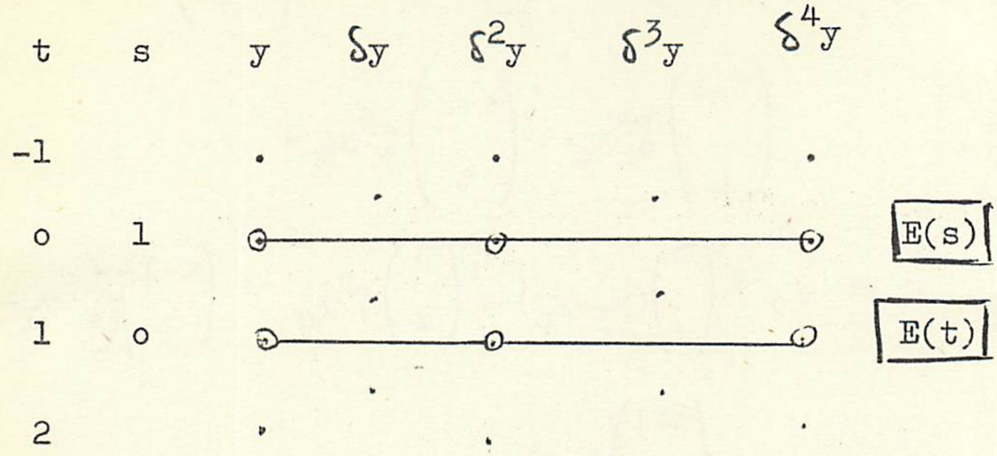
$$\binom{t}{2} - \binom{t+1}{3} = \binom{s+1}{3}$$

$$\binom{t+1}{4} - \binom{t+2}{5} = \binom{s+2}{5}$$

then we get Everett-Laplace formula:

$$y(t) = sy_o + \binom{s+1}{3}\delta^2 y_o + \binom{s+2}{5}\delta^4 y_o + \dots$$
$$+ ty_1 + \binom{t+1}{3}\delta^2 y_1 + \binom{t+2}{5}\delta^4 y_1 + \dots \tag{43}$$

For the calculation it is suitable to use the following scheme:



3.5 Remarks on the formulas

1) How do the different formulas compare with each other and will the Lagrange formula found carbier? the value obtained in an interpolation depends on the polynomial used, and the polynomial depends on the sample points used. The error term has the form

$$\frac{(x-x_o)(x-x_1)(x-x_2)\ \ldots\ (x-x_n)\ y^{(n+1)}(\xi)}{(n+1)!}$$

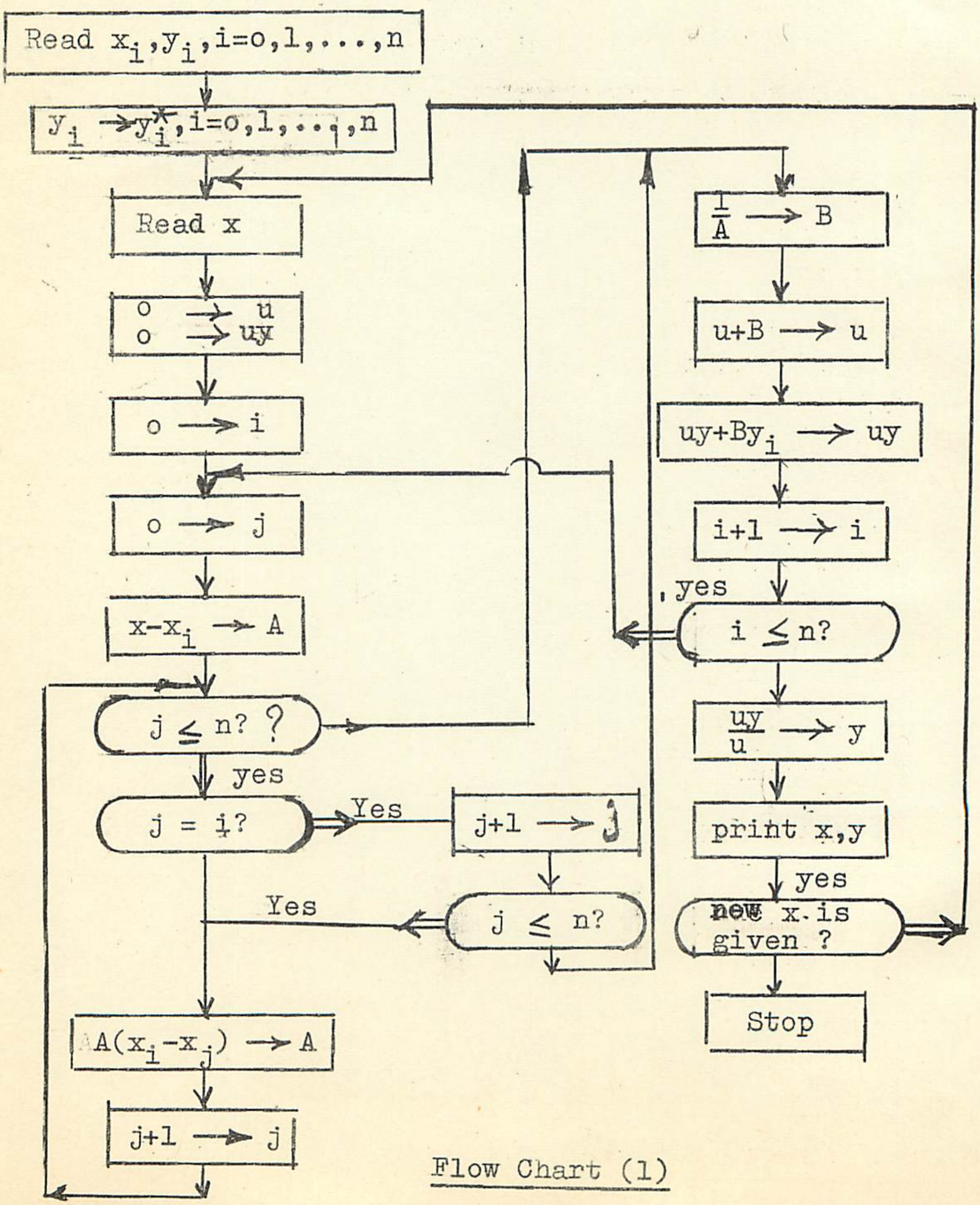
The coefficient of the derivative is minimized when x is in the middle of the range of samples. Thus there is a tendency to use an even number of samples when the interpolation point is in the middle of an interval and an odd number when it is near a sample point.

2) A large number of formulas which appear to be different are shown by Lozenge [5] that they are really all the same.

4. Flow Charts

4.1 Flow Chart for the lagrange-Method of Interpolation

$$y(x) = \frac{\sum_{i=o}^{n} u_i \, y_i}{\sum_{i=o}^{n} u_i} \quad , \qquad u_i = \frac{1}{(x-x_i)\prod_{\substack{j=o \\ j\neq o}}^{n} (x_i - x_j)}$$



Flow Chart (1)

Test for the Flow Chart (1)

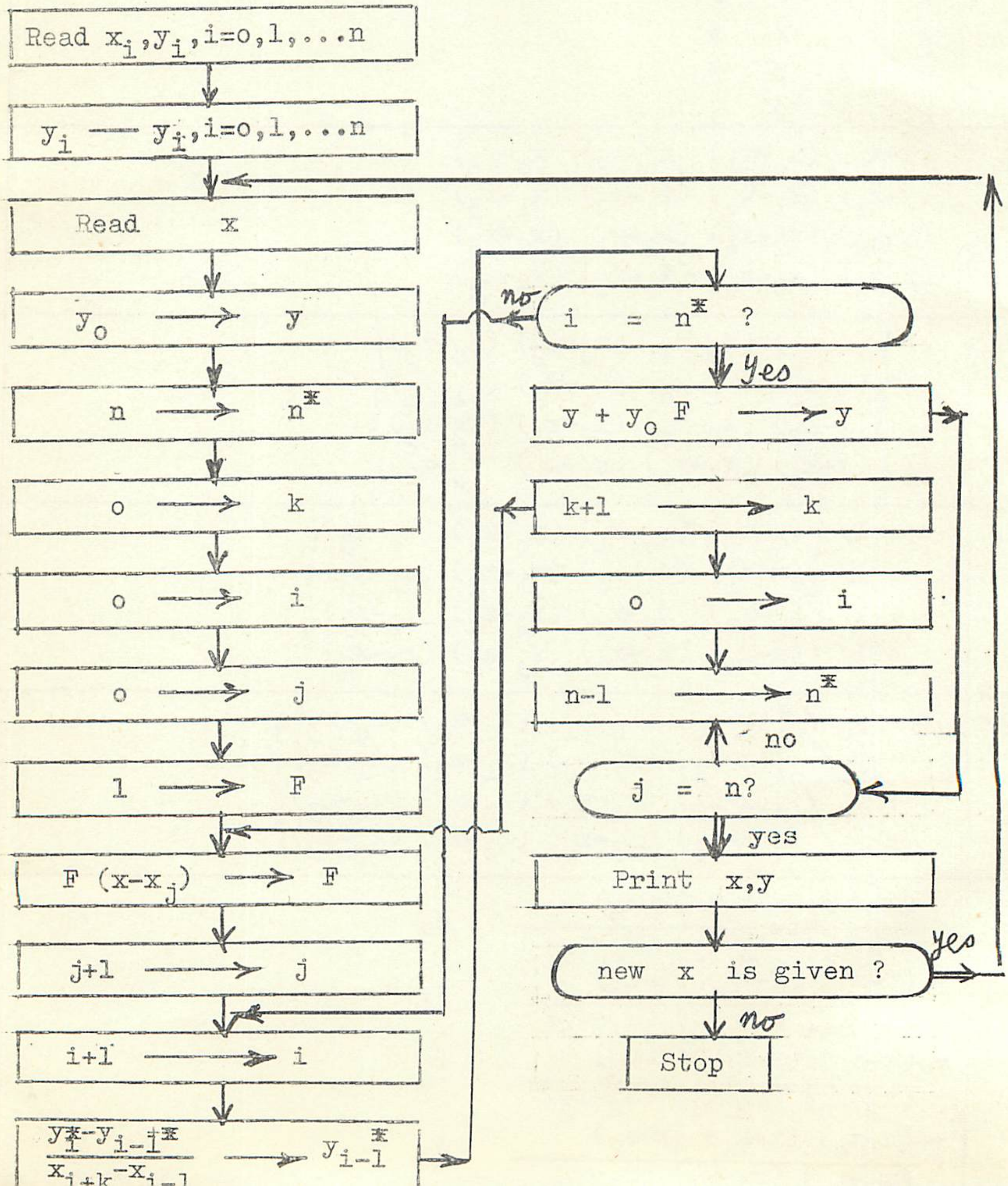
<u>n = 3</u>

i	~~0~~ ~~1~~ ~~2~~ 3
j	~~0~~ ~~1~~ ~~2~~ ~~3~~ ~~0~~ ~~1~~ ~~2~~ ~~3~~ ~~0~~ ~~1~~ ~~2~~ ~~3~~ ~~0~~ ~~1~~ ~~2~~ 3
A	$(x-x_0)\ (x_0-x_1)\ (x_0-x_2)\ (x_0-x_3)$ $(x-x_1)\ (x_1-x_0)\ (x_1-x_2)\ (x_1-x_3)$ $(x-x_2)\ (x_2-x_0)\ (x_2-x_1)\ (x_2-x_3)$ $(x-x_3)\ (x_3-x_0)\ (x_3-x_1)\ (x_3-x_2)$
B	$1/\{(x-x_0)\ (x_0-x_1)\ (x_0-x_2)\ (x_0-x_3)\}$ $1/\{(x-x_1)\ (x_1-x_0)\ (x_1-x_2)\ (x_1-x_3)\}$ $1/\{(x-x_2)\ (x_2-x_0)\ (x_2-x_1)\ (x_2-x_3)\}$ $1/\{(x-x_3)\ (x_3-x_0)\ (x_3-x_1)\ (x_3-x_2)\}$
u	$o + 1/\{(x-x_0)\ (x_0-x_1)\ (x_0-x_2)\ (x_0-x_3)\}$ $+ 1/\{(x-x_1)\ (x_1-x_0)\ (x_1-x_2)\ (x_1-x_3)\}$ $+ 1/\{(x-x_2)\ (x_2-x_0)\ (x_2-x_1)\ (x_2-x_3)\}$ $+ 1/\{(x-x_3)\ (x_3-x_0)\ (x_3-x_1)\ (x_3-x_2)\}$
uy	$o + y_0 / \{(x-x_0)\ (x_0-x_1)\ (x_0-x_2)\ (x_0-x_3)\}$ $+ y_1/ \{(x-x_1)\ (x_1-x_0)\ (x_1-x_2)\ (x_1-x_3)\}$ $+ y_2 / \{(x-x_2)\ (x_2-x_0)\ (x_2-x_1)\ (x_2-x_3)\}$ $+ y_3/ \{(x-x_3)\ (x_3-x_0)\ (x_3-x_1)\ (x_3-x_2)\}$
y	$\frac{(x-x_1)\ (x-x_2)\ (x-x_3)}{(x_0-x_1)\ (x_0-x_2)\ (x_0-x_3)}\ y_0$ $+ \frac{(x-x_0)\ (x-x_2)\ (x-x_3)}{(x_1-x_0)\ (x_1-x_2)\ (x_1-x_3)}\ y_1$ $+ \frac{(x-x_0)\ (x-x_1)\ (x-x_3)}{(x_2-x_0)\ (x_2-x_1)\ (x_2-x_3)}\ y_2$ $+ \frac{(x-x_0)\ (x-x_1)\ (x-x_2)}{(x_3-x_0)\ (x_3-x_1)\ (x_3-x_2)}\ y_3$

4.2 Flow Chart for Newtons' general Interpolation formula

$$y(x) = \gamma_0 + \gamma_1 (x-x_0) + \gamma_2 (x-x_0)(x-x_1) + \dots + \gamma_n (x-x_0)(x-x_1) \dots (x-x_{n-1})$$

where $\gamma_0 = y_0$; $\gamma_1 = [x_1 x_0]$, $\gamma_2 = [x_2 x_1 x_0]$, $\dots \gamma_n = [x_n x_{n-1} x_{n-2} \dots x_0]$



Test for the Flow Chart (2)

$n = 3$

n^*	~~3~~	~~2~~	1	
k	~~0~~	~~1~~	2	
i	~~0~~ ~~1~~ ~~2~~ ~~3~~	~~0~~ ~~1~~ ~~2~~	0 1	
j	~~0~~ ~~1~~	~~2~~	3	
F	$1(x-x_0)$	$(x-x_1)$	$(x-x_2)$	
x_0	y_0^*			
	$\frac{y_1-y_0}{x_1-x_0} = [x_1x_0]$			
		$\frac{[x_2x_1]-[x_1x_0]}{x_2-x_0} = [x_2x_1x_0]$		
			$[x_3x_2x_1x_0]$	
x_1	y_1^*			
	$\frac{y_2-y_1}{x_2-x_1} = [x_2x_1]$			
		$\frac{[x_3x_2]-[x_2x_1]}{x_3-x_1} = [x_3x_2x_1]$		
x_2	y_2^*			
	$\frac{y_3-y_2}{x_3-x_2} = [x_3x_2]$			
y_3	y_3^*			
y	$y_0+[x_1x_0](x-x_0)+[x_2x_1x_0](x-x_0)(x-x_1)+[x_3x_2x_1x_0](x-x_0)(x-x_1)(x-x_2)$			

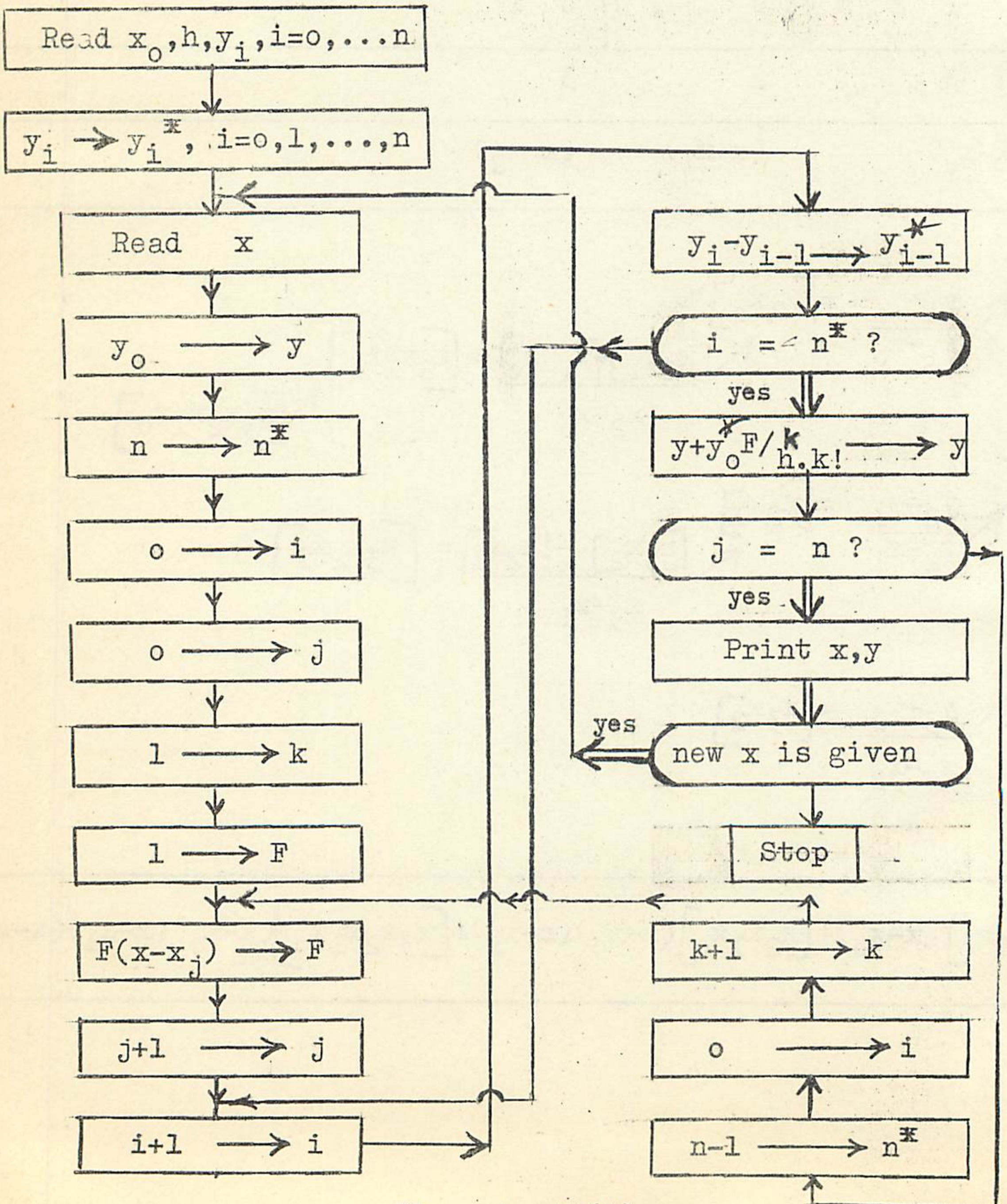
Flow Chart for

Newtons' Interpolution formula for

equidistant function

$$y(x) = y_o + \frac{\Delta y_o}{h} (x-x_o) + \frac{\Delta^2 y_o}{2!h^2} (x-x_o)(x-x_1) + \dots +$$

$$+ \frac{\Delta^n y_o}{n!h^n} (x-x_o)(x-x_1) \dots (x-x_{n-1})$$



Flow chart (3)

Test for the Flow Chart (3)

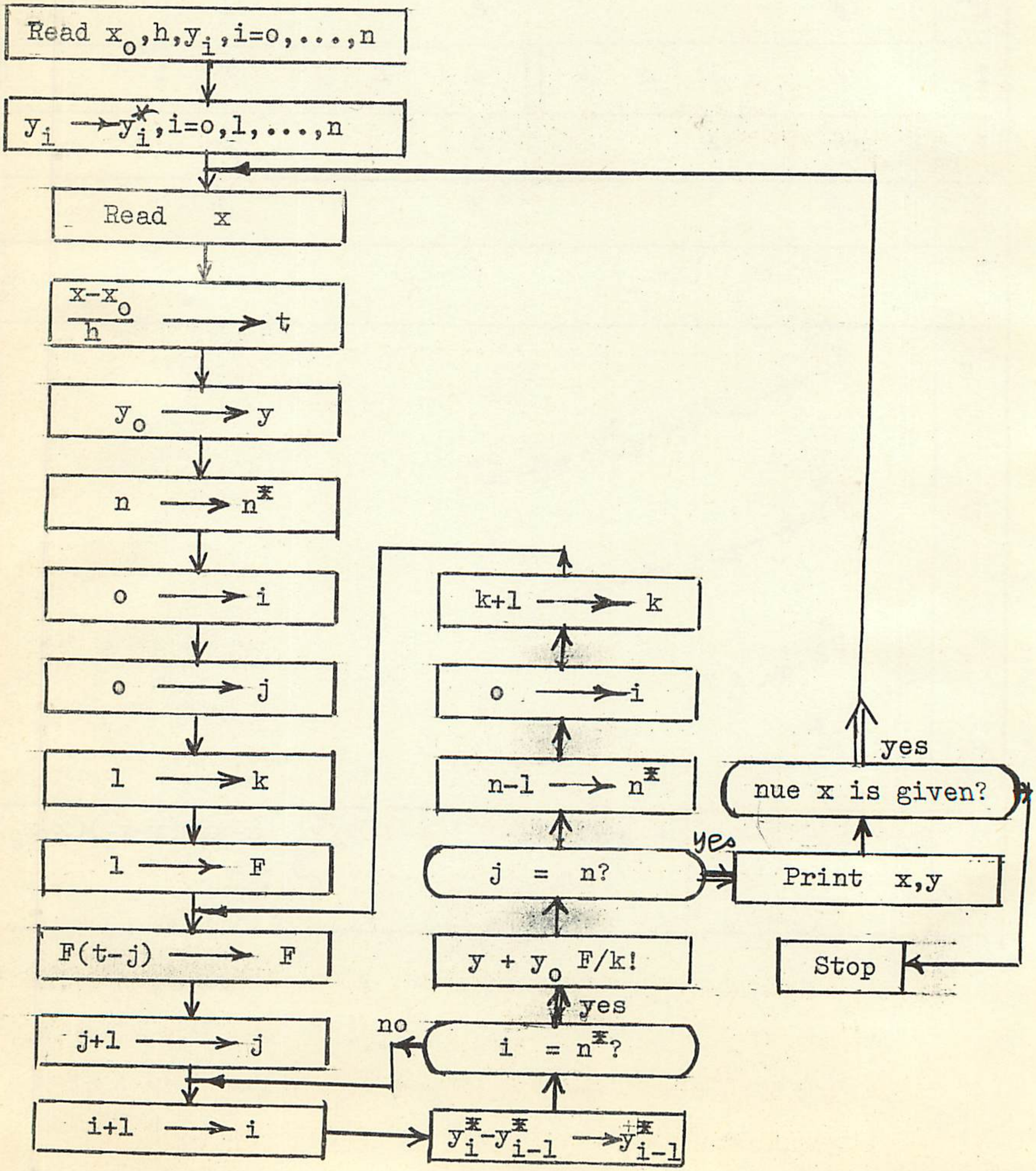
$n = 3$

n^*	~~3~~ ~~2~~ 1
i	~~0~~ ~~1~~ ~~2~~ ~~3~~ ‖ ~~0~~ ~~1~~ ~~2~~ ~~0~~ 1
j	~~0~~ ~~1~~ ~~2~~ 3
k	~~1~~ ~~2~~ 3
F	1 $(x-x_0)$ $(x-x_1)$ $(x-x_2)$
x_0	y_0^* ← Δy_0 ← $\Delta^2 y_0$ ← $\Delta^3 y_0$
x_1	y_1^* ← Δy_1 ← $\Delta^2 y_1$
x_2	y_2^* ← Δy_2
x_3	y_3^*
y	$y_0 + \Delta y_0 (x-x_0)/h + \frac{\Delta^2 y_0}{h^2\, 2!}(x-x_0)(x-x_1) + \frac{\Delta^3 y_0}{h^3\, 3!}(x-x_0)(x-x_1)(x-x_2)$

4.4. Flow - Chart for Gregory-Newtons' Interpolation formula for equidistant function

$$y(t) = y_0 + \binom{t}{1}\Delta y_0 + \binom{t}{2}\Delta^2 y_0 + \binom{t}{3}\Delta^3 y_0 + \dots + \binom{t}{n}\Delta^n y_0$$

where $t = \dfrac{x-x_0}{h}$



Test for the Flow Chart (4)

$n = 3$

n^*	~~3~~ ~~2~~ 1
i	~~0~~ ~~1~~ ~~2~~ ~~3~~ ‖ ~~0~~ ~~1~~ ~~2~~ ‖ ~~0~~ 1
j	~~0~~ ~~1~~ ~~2~~ 3
k	~~1~~ ~~2~~ 3
F	1.t (t-1) (t-2)
x_0	y_0^* ← Δy_0 ← $\Delta^2 y_0$ ← $\Delta^3 y_0$
x_1	y_1^* ← Δy_1 ← $\Delta^2 y_1$
x_2	y_2^* ← Δy_2
x_3	y_3^*
y	$y_0 + \binom{t}{1}\Delta y_0 + \binom{t}{2}\Delta^2 y_0 + \binom{t}{3}\Delta^3 y_0$

References

[1] R. Zurmühl, Praktische Mathematik für Ingenieur und Physiker, 3. Auflage, 1961, Springer-Verlag, Berlin

[2] J. Steffensen, Interpolation, 2. Edition 1950 New York.

[3] K. Nielsen, Methods in Numerical Analysis, 4. Edi. 1961, United States of America

[4] F. A. Willers, Practical analysis, 1947, United states of America

[5] R. W. Hamming, Numerical Methods for Scientists and Engineers, 1962 New York.