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Decision Model For One Year Planning

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Introduction

The following problem had been raised to us:

For a given year, find the optimum allocation of given foreign resourses (imports), among the different sectors of the economy, provided the following conditions are satisfied:

1. The productions of the different sectors are allowed only to change between given lower & upper bounds.

2. For the sector of agriculture and ginning their productions are given constants.

3. There is no competitive imports in the sector of ginning.

4. There are given upper & lower bounds on the consumptions both private & government, the investment's sinking, exports from different sectors & on total labour.

5. The total value added arizing from productions of the services sectors does not exceed given ratio of the total value added due to the production of all the sectors of the economy.

6. The optimum allocation of foreign resoures is defined as the one which gives maximum labour plus national income, weighted differently

The problem so defined above, can be solved easily using the well known interflow analysis & the technique of linear programming. In the following section the outline of the solution is given.

The mathematical formulation:

Let n : Number of sectors of our economy

X_i : the production of sector i. (i= 1,2, ... , n)

X_{kh} : the technical input output coefficient
k the delivering sector, takes the values 1— n
h the receiving sector, takes the values 1— n

C_k , G_k : the private & government consumption from sector k

S_k : the investment sinking in sector k

A_k^+ : the exports from sector k

A_k^- : the competitive imports from sector k

B_h' : the noncompetitive import coefficient of sector h

B_c, B_G & B_S : the noncompetitive imports for private consumption, government consumption & investment sinking

V_h' : the value added coefficient of sector h

L_h' : the labour coefficient of sector h

From the equilibrium relation of the interflow table we have

$$\sum_{h=1}^{n} X'_{kh} X_h + C_k + G_k + S_k + A_k^+ - A_k^- = X_k \quad k=1,2,\ldots,n \quad (1)$$

The above equation takes the following form

$$A_k^- = -\sum_{h=1}^{n} (\delta_{kh} - X'_{kh}) X_h + C_k + G_k + S_k + A_k^+ \quad k=1,2,\ldots,n \quad (2)$$

where $\delta_{kh} = 1$ for $k = h$ & o for $k \neq h$

Let I defines the total imports or, foreign resource required. The expression for I is given by the following equation

$$I = \sum_{k=1}^{n} A_k^- + \sum_{h=1}^{n} B_h' X_h + B_c + B_G + B_S \qquad (3)$$

Let C, G, S & L define the total private consumption, total government consumption, total sinking, total labour, then by definition we have the following equations.

$$C = \sum_{k=1}^{n} C_k + B_c$$

$$G = \sum_{k=1}^{n} G_k + B_G \qquad \ldots \qquad (4)$$

$$S = \sum_{k=1}^{n} S_k + B_S$$

$$L = \sum_{h=1}^{n} L_h' \quad X_h$$

Economically, there is relation between the following quantities:

Total exports $\sum_{k=1}^{n} A_K^+$,

Total imports I ,

Gross borrowing R and

Repayments P

That economic relation states the following

Imports + Gross borrowing

= Exports + ,repayment.

Which symbolically is written in the following form

$$I + R = \sum_{k=1}^{n} A_k^{+} + P$$

$$\text{i.e.} \qquad \sum_{k=1}^{n} A_k^{+} - I = R - P = Z \tag{5}$$

The quantity $Z = R-P$ is assumed given in our problem.

Let $\alpha, \beta, \ldots, \gamma$ denote the services sectors. Then, as mentioned in the introduction (paragraph labelled 5) we have

$$\frac{\sum_{h=\alpha,\beta,\ldots,\gamma} V_h' \, X_h}{\sum_{h=1}^{n} V_h' \, X_h} \leq r$$

which can be written in the following form :

$$\sum_{h=1,2,\ldots)\alpha,\beta,\ldots,\gamma(\ldots n}^{n} r \, V_h' \, X_h \; - \sum_{h=\alpha,\beta,\ldots,\gamma} (1-r) \, V_h' \, X_h \geq 0 \tag{6}$$

where $)\alpha, \beta, \ldots \gamma($ means exculding the subscripts $\alpha, \beta, \ldots, \gamma$

A variable with a lower dash means the lower bound of that variable & a variable with upper dash means the upper bound of that variable, for example $\bar{X}_h$ denotes the upper bound on X_h & $\underline{C}_k$ denote the lower bound on C_k & so on. With that definition of lower & upper bound we have the following inequalities to be satisfied by all the variables of our problem:

$$\left.\begin{array}{lllll}
\underline{X}_k & \leqslant & X_k & \leqslant & \bar{X}_k \\
\underline{C}_k & \leqslant & C_k & \leqslant & \bar{C}_k \\
\underline{G}_k & \leqslant & G_k & \leqslant & \bar{G}_k \\
\underline{S}_k & \leqslant & S_k & \leqslant & \bar{S}_k \\
\underline{A}_k^+ & \leqslant & A_k^+ & \leqslant & \bar{A}_k^+ \\
\underline{B}_c & \leqslant & B_c & \leqslant & \bar{B}_c \\
\underline{B}_G & \leqslant & B_G & \leqslant & \bar{B}_G \\
\underline{B}_S & \leqslant & B_S & \leqslant & \bar{B}_S \\
0 & \leqslant & A_k^- & & \\
\underline{L} & \leqslant & L & \leqslant & \bar{L}
\end{array}\right] \quad \text{-----} \quad (7)$$

where $k = 1, 2, \ldots, n$

Equations (2), (3), (4), (5) & inequalities (6) & (7) define for us the relations & constrains imposed on our problem.

We notice we have so far n+6 equations & 6n+5 inequalities.

The variables of our problem are $X_k, C_k, G_k, S_k, A_k^+, A_k^-, B_c, B_G, B_s, C,$ $G, S, L, I,$ i.e 6n+8

Our problem is to find the optimum allocation of our variables including the imports A_k^-, B_h, X_h, B_c, B_G, B_s which will maximize for us our preference function.

The preference function which will be optimized will be labour plus sum of value added, weighted differently.

Let α_1 = Weight given to labour

α_2 = Weight given to Total value added

Then the preference function will be

$$f = \alpha_1 \sum_{k=1}^{n} L'_k \; X_k + \alpha_2 \sum_{k=1}^{n} V'_k \; X_k$$

$$= \sum_{k=1}^{n} (\alpha_1 L'_k + \alpha_2 V'_k) X_k \qquad (8)$$

The weights α_1 & α_2 can be given different values, corresponding to different alternatives.

The problem so formulated can be solved with classical methods of linear programming using the electronic computer's facilities. Before procceding in outlining the steps towards its solution we shall introduce further assumptions towards more simplification of the problem.

<u>Further assumptions:</u> It will reduce the variables of our problem quite a bit if we assume certain patterns in consumption(private & government) and certain patterns in investment sinking.

Let these patterns be given respectively by C'_k , G'_k , S'_k (for $k = 1,2, \ldots , n$) and B'_c , B'_g , B'_s defined as following:

$$\left.\begin{aligned} C'_k &= \frac{C_k}{C} \\ G'_k &= \frac{G_k}{G} \\ S'_k &= \frac{S_k}{S} \\ B'_c &= \frac{B_c}{C} \\ B'_g &= \frac{B_g}{G} \\ B'_s &= \frac{B_s}{S} \end{aligned}\right\} \qquad (9)$$

From the above definitions we have the following relations

$$\left.\begin{aligned} \sum_{k=1}^{n} C'_k + B'_C &= 1 \\ \sum_{k=1}^{n} G'_k + B'_G &= 1 \\ \sum_{k=1}^{n} S'_k + B'_S &= 1 \end{aligned}\right\} \qquad (10)$$

The pattern assumption reduces our $3n+3$ variables C_k, G_k, S_k, B_c, B_g, B_s to only 3 variables C, G & S and reduces the corresponding $3n+3$ inequalities to only the following 3 inequalities.

$$\left.\begin{array}{ccccc} \underline{C} & \leqslant & C & \leqslant & \bar{C} \\ \underline{G} & \leqslant & G & \leqslant & \bar{G} \\ \underline{S} & \leqslant & S & \leqslant & \bar{S} \end{array}\right\} \qquad (11)$$

where $\underline{C}$, $\bar{C}$, $\underline{G}$, $\bar{G}$, $\underline{S}$ & $\bar{S}$ are assumed given.

With these pattern assumptions, equations (2) & (3) take the following form:

$$A_k^- = -\sum_{h=1}^{n} (\delta_{kh} - X'_{kh})\, X_h + C'_k.\, C + G'_k.\, G + S'_k.\, S + A_k^+$$

$$k = 1, 2, \ldots, n \qquad (12)$$

$$I = \sum_{k=1}^{n} A_k^- + \sum_{h=1}^{n} B'_h X_h + B'_C.\, C + B'_G.\, G + B'_S.\, S \qquad (13)$$

Eliminating I from equation (13) & (5) we have

$$-\sum_{k=1}^{n} A_k^- = -\sum_{k=1}^{n} A_k^+ + \sum_{h=1}^{n} B'_h X_h + B'_C.\, C + B'_G.\, G + B'_S.\, S$$

$$+ Z \qquad (14)$$

Subtituting for A_k^- from (12) in (14) we have :

$$C = \sum_{h=1}^{n} V'_h X_h - G - S - Z \qquad (15)$$

Eliminating C between the set of equation (12) & equation (15) we have the following sets of equations:

$$A_k^- = - C_k' Z - \sum_{h=1}^{n} (\delta_{kh} - X_{kh}' - C_k' V_h') X_h + (G_k'-C_k')G+(S_k'-C_k')S+A_k^+$$

$$k = 1, 2, \ldots , n \qquad (16)$$

Our independent variables now, reduce to X_h $(h=1,\ldots,n)$, G, S, & A_k^+ $(k=1, \ldots ,n)$ while the dependent variables are A_k^- $(k=1, \ldots, n)$, C & L

Equation (6) implies another independent variables and another inequality defined as following

$$\xi = \sum \pi V_h' X_h - \sum_{h=\alpha,\beta,\ldots,\gamma} (1-\pi) V_h' X_h \qquad (17)$$

$$h=1,2,\ldots) \alpha,\beta,\ldots,\gamma(\ldots h$$

So far, we had not put the assumptions mentioned in paragraphs (2) & (3), in the introduction

Let, without losing any generality, the sector of agriculture and ginning be denoted by the subscripts, n-1, n. Constant production for these sectors & zero competitive imports for ginning imply the following

$$\left.\begin{array}{llll} & X_{n-1} & = X_{n-1}^{\circ} & = \text{const.} \\ & X_n & = X_n^{\circ} & = \text{const.} \\ \& & A_n^- & = 0 & \end{array}\right\} \qquad (18)$$

Substituting with equations (18) into equations (16) we have

$$\left.\begin{aligned} A_k^- = & - C_k' Z - \sum_{h=n-1}^{n} (\delta_{kh} - X_{kh}' - C_k' V_h') X_h \\ & - \sum_{h=1}^{n-2} (\delta_{kh} - X_{kh}' - C_k' V_h') X_h \\ & + (G_k' - C_k') G + (S_k' - C_k') S + A_k^+ \end{aligned}\right\} \quad (19)$$

$k=1,2, \ldots , n-1$

$$\left.\begin{aligned} \& - A_n^+ = & - C_n' Z - \sum_{h=n-1}^{n} (\delta_{nh} - X_{nh}' - C_n' V_h') X_h \\ & - \sum_{h=1}^{n-2} (\delta_{nh} - X_{nh}' - C_n' V_h') X_h \\ & + (G_n' - C_n') G + (S_n' - C_n') S \end{aligned}\right\} \quad (20)$$

define the following constants

$$\left.\begin{aligned} \beta_{ok} &= - C_k' Z - \sum_{h=n-1}^{n} (\delta_{kh} - X_{kh}' - C_k' V_h') X_h \\ \beta_{ik} &= -(\delta_{kh} - X_{kh}' - C_k' V_h') \\ \beta_{Gk} &= G_k' - C_k' \\ \beta_{Sk} &= S_k' - C_k' \end{aligned}\right\} \quad (21)$$

Substituting the above definitions in equations 19 & 20 we have

$$A_k^- = \beta_{ok} + \sum_{h=1}^{n-2} \beta_{hk} X_h + \beta_{Gh\cdot} G + \beta_{Sk\cdot} S + A_k^+ \qquad k=1,2,\ldots,n-1 \qquad (22)$$

$$\& + A_n^+ = -\beta_{on} - \sum_{h=1}^{n-2} \beta_{hn} X_h - \beta_{Gn\cdot} G - \beta_{Sn\cdot} S \qquad (23)$$

Again substituting the assumption of constant X_{n-1}, X_n in the equations (17), (15), 4 & 8 defining ξ, C, L & f we have

$$\xi = \sum_{h=n-1}^{n} r V_h' X_h - \sum_{h=1,2,\ldots)\alpha,\beta,\ldots,\gamma(\ldots n-2} V_h' X_h - \sum_{h=\alpha,\beta\ldots,\gamma} (1-r) V_h' X_h \qquad (24)$$

$$C = \sum_{h=n-1}^{n} V_h' X_h - Z + \sum_{h=1}^{n-2} V_h' X_h - G - S \qquad (25)$$

$$L = \sum_{h=n-1}^{n} L_h' X_h + \sum_{h=1}^{n-2} L_h' X_h \qquad (26)$$

$$f = \sum_{h=1}^{n-2} (\alpha_1 L_h' + \alpha_2 V_h') X_h \qquad (27)$$

The constrains to our variables, dependent and independent are written in, the following form

$$\left.\begin{array}{ccccc} \underline{X}_h & \leqslant & X_h & \leqslant & \overline{X}_h \\ \underline{C} & \leqslant & C & \leqslant & \overline{C} \\ \underline{G} & \leqslant & G & \leqslant & \overline{G} \\ \underline{S} & \leqslant & S & \leqslant & \overline{S} \\ \underline{L} & \leqslant & L & \leqslant & \overline{L} \end{array}\right\} \quad \begin{array}{l} h=1,\ 2,\ \ldots,\ n-2 \\ k=1,\ 2,\ \ldots,\ n-1 \end{array} \qquad (28)$$

$$\left.\begin{array}{ccccc} 0 & \leqslant & A_h^- & & \\ \underline{A}_i^+ & \leqslant & A_i^+ & \leqslant & \overline{A}_i^+ \\ 0 & \leqslant & \xi & & \end{array}\right\} \quad i=1,\ 2,\ \ldots,\ n$$

Equations (22) $\rightarrow$ (26) & inequalities (28) & the preference function (27) define for us the mathematical formulation of our problem which will be solved by linear programming & electronic computer facilities.

<u>The programming equation</u>: The numerical preparation of the above problem is simple & straightforward. It can systematically be prepared using a desk calculator or more generally using a computer. In this section we shall outline the scheme for desk calculator's computation. The computer program right with the numerical result will be given in a further memo by Dr. Roshdi Amer who is in direct change with that phase of the problem.

Four tables 1,2,3 & 4 are needed for the preparation of the programming equation.

Table (1) gives the basic technical data & the upper & lower bounds of our variables whether dependent or independent

Table (2) is remanipulation of table (1) & in fact it expresses in a tabular form equation (12), (15), (4), & (8)

Tables (3) is the result of elimination of C using equation 15 & putting X_{n-1}, X_h as constants.

The programming equations as expressed in tabular form in table (3) will be sufficient in case we use the multiplex technique for solving linear programming. The multiplex method had not yet been coded on the I.B.M. 1620.

In our library program we have a program for solving linear programming problems using the simplex technique. In that program the variables of our problem are assumed to be non-negative.

Problem with upper & lower bounded variables can be transferred to problems with a non negative variables with some simple transformation. To illustrate that consider the following simple example: Given the variables x_1, x_2, x_3, y_1, y_2 such that

$$\left.\begin{aligned}
y_1 &= a_1\, x_1 + a_2\, x_2 + a_3\, x_3 \\
y_2 &= b_1\, x_1 + b_2\, x_2 + b_3\, x_3 \\
\underline{x}_1 \leqslant\ & x_1 \leqslant \bar{x}_1 \\
\underline{x}_2 \leqslant\ & x_2 \leqslant \bar{x}_2 \\
\underline{x}_3 \leqslant\ & x_3 \leqslant \bar{x}_3 \\
\underline{y}_1 \leqslant\ & y_1 \leqslant \bar{y}_1 \\
\underline{y}_2 \leqslant\ & y_2 \leqslant \bar{y}_2
\end{aligned}\right\} \tag{29}$$

Put

$$\left.\begin{aligned}
\xi_1 &= x_1 - \underline{x}_1 \\
\xi_2 &= x_2 - \underline{x}_2 \\
\xi_3 &= x_3 - \underline{x}_3 \\
\eta_1 &= \bar{x}_1 - x_1 \\
\eta_2 &= \bar{x}_2 - x_2 \\
\eta_3 &= \bar{x}_3 - x_3 \\
\psi_1 &= y_1 - \underline{y}_1 \\
\psi_2 &= y_2 - \underline{y}_2 \\
\epsilon_1 &= \bar{y}_1 - y_1 \\
\epsilon_2 &= \bar{y}_2 - y_2
\end{aligned}\right\} \tag{30}$$

It is obvious that all these newly introduced variables $\xi_1, \xi_2, \ldots, \epsilon_2$ are non-negative.

The problem defined in the sets of equations & inequalities (29) can be transformed to the following problem with non-negative variables.

$$\Psi_1 = (a_1 \underline{\underline{x}}_1 + a_2 \underline{\underline{x}}_2 + a_3 \underline{\underline{x}}_3 - \underline{y}_1) + a_1 \xi_1 + a_2 \xi_2 + a_3 \xi_3$$

$$\Psi_2 = (b_1 \underline{\underline{x}}_1 + b_2 \underline{\underline{x}}_2 + b_3 \underline{\underline{x}}_3 - \underline{y}_2) + b_1 \xi_1 + b_2 \xi_2 + b_3 \xi_3$$

$$\epsilon_1 = (\bar{y}_1 - a_1 \underline{\underline{x}}_1 - a_2 \underline{\underline{x}}_2 - a_3 \underline{\underline{x}}_3) - a_1 \xi_1 - a_2 \xi_2 - a_3 \xi_3$$

$$\epsilon_2 = (\bar{y}_2 - b_1 \underline{\underline{x}}_1 - b_2 \underline{\underline{x}}_2 - b_3 \underline{\underline{x}}_3) - b_1 \xi_1 - b_2 \xi_2 - b_3 \xi_3 \qquad (31)$$

$$\eta_1 = \bar{x}_1 - \underline{\underline{x}}_1 \qquad - \xi_1$$

$$\eta_2 = x_2 - \underline{\underline{x}}_2 \qquad - \xi_2$$

$$\eta_3 = x_3 - \underline{\underline{x}}_3 \qquad - \xi_3$$

The simple transformation from bounded variables problem to non-negative variables one can be applied to our problem described by equations (22) $\rightarrow$ (26), inequalities (28) & preference function (27). In that case we have the following transformation of variables.

Bounded Variables	Corresponding Non-negative variables
x_k $(k=1,2,\ldots,n-2)$	$x_k - \underline{x}_k$ $k=1,2,\ldots,n-2$
G	$G - \underline{G}$
S	$S - \underline{S}$
A_k^+ $k=1,2,\ldots,n$	$A_k^+ - \underline{A}_k^+$ $k=1,2,\ldots,n$
C	$C - \underline{C}$
L	$L - \underline{L}$

The above transformation takes care of the lower bounds . In order to allow for the upper bounds we introduce the following new dependent variables

$$\bar{x}_k - x_k \qquad k=1,2,\ldots,n-2$$

$$\bar{C} - C$$

$$\bar{G} - G$$

$$\bar{S} - S$$

$$\bar{L} - L$$

$$\bar{A}_k^+ - A_k^+ \qquad k=1,2,\ldots,n$$

With these transformed & newly introduced variables we have the following reformulation of our problem where all the variables, dependent or non-dependent are now non-negative

$$A_k^- = \gamma_{ko} + \sum_{h=1}^{n-2} \beta_{kh}\,(x_h-\underline{x}_h) + \beta_{kG}\cdot(G-\underline{G}) + \beta_{kS}\cdot(S-\underline{S}) + (A_k^+-\underline{A}_k^+)$$

$$k=1,2,\ \dots\ ,\ n-1 \qquad (32)$$

$$A_n^+-\underline{A}_n^+ = \gamma_{no} - \sum_{h=1}^{n-2} \beta_{nh}\,(x_h-\underline{x}_h) - \beta_{nG}\cdot(G-\underline{G}) - \beta_{nS}\cdot(S-\underline{S}) \qquad (33)$$

$$C-\underline{C} = \gamma_{C-\underline{C}} + \sum_{h=1}^{n-2} V_h'\,(x_h-\underline{x}_h) - (G-\underline{G}) - (S-\underline{S}) \qquad (34)$$

$$L-\underline{L} = \gamma_{L-\underline{L}} + \sum_{h=1}^{n-2} L_h'\,(x_h-\underline{x}_h) \qquad (35)$$

$$\bar{x}_k-x_k = \gamma(\bar{x}_k - x_k) - (x_k-\underline{x}_k) \qquad (36)$$

$$\bar{C}-C = \gamma_{\bar{C}-C} - \sum_{h=1}^{n-2} V_h'\,(x_h-\underline{x}_h) + (G-\underline{G}) + (S-\underline{S}) \qquad (37)$$

$$\bar{L}-L = \gamma_{\bar{L}-L} - \sum_{h=1}^{n-2} L_h'\,(x_h-\underline{x}_h) \qquad (38)$$

where

$$\gamma_{ko} = \beta_{ko} + \sum_{h=1}^{n-2} \beta_{kh}\,\underline{x}_h + \beta_{kG}\cdot\underline{G} + \beta_{kS}\cdot\underline{S} + \underline{A}_k^+$$

$$\gamma_{no} = -\beta_{no} - \underline{A}_n^+ - \sum_{h=1}^{n-2} \beta_{nh}\,\underline{x}_h - \beta_{nG}\cdot\underline{G} - \beta_{nS}\cdot\underline{S} \qquad (39)$$

$$\gamma_{C-\underline{C}} = V_{n-1}'\,X_{n-1} + V_n'\,X_n - Z + \sum_{h=1}^{n-2} V_h'\,\underline{x}_h - \underline{G} - \underline{S} - \underline{C}$$

$$\gamma_{L-\underline{L}} = L_{n-1}'\,X_{n-1} + L_n'\,X_n + \sum_{h=1}^{n-2} L_h'\,\underline{x}_h - \underline{L}$$

$$\gamma_{\bar{x}_k - x_k} = \bar{x}_k - \underline{x}_k$$

$$\gamma_{\bar{c} - c} = \bar{c} - \underline{c} - \gamma_{c - \underline{c}}$$

$$\gamma_{\bar{L} - L} = \bar{L} - \underline{L} - \gamma_{L - \underline{L}}$$

The non-negative variables problem now defined in equations (32) - (39) can be represented in a tabular form. This is done in table 4.

E.Z.