UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

Memo. No. 779

FOREIGN LOANS AND ECONOMIC DEVELOPMENT

PART I

RESTATEMENT OF QAYUM'S MODEL

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June, 1967

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I- Introduction

Growth models usually single out the factor "capital" as the major determinant of economic growth. Since capital has to be produced, its scarcity makes itself severely felt at the early stages of development. The natural deduction is that it has to be imported, provided that it can "pay itself back" later. Capital importation can take one of many forms:

1) It can be imported against exports of non-capital goods. This possibility is quite limited, since it assumes the ability of the economy to produce a sufficient surplus over and above its current consumption needs.
2) Grants provided free of charge form a net addition. In principle, grants for consumption purposes can play the same role since they can free other resources for investment purposes. However, these are usually given under conditions of difficulty in meeting current consumption requirements. Hence, unless coupled with careful planning, they might eventually vanish through increased consumption.
3) During the last two centuries the accelerated rate of capital formation in the prospering industrial and commercial countries has set private capital wandering around looking for investment opportunities in the underdeveloped world. The success of such movement went hand in hand with expansionary imperialistic activities. The history of colonialism is partly that of foreign capital movements. This source went on the downswing towards the middle of the present century.
4) After the end of the Second World War, the trend shifted towards borrowed capital. This took place at the same time when many ex-colonies gained their independence, and started more systematic action geared towards economic and social development of their economies.

Thus the desire to establish high rates of growth has initiated a world-wise flow of borrowed capital tied up to development plans. It seems, however, that in less than two decades, some sort of a policy reconsideration is taking place both in the creditor and debtor countries. The fact that loans are usually negotiated on governmental or semi-governmental levels has eventually given the lead to political factors. Even if we dismiss this side of the problem, the flow of borrowed foreign capital need not conform exactly with the plans of the developing countries. Criteria for choice of investments which creditor countries use in judging their loans, normally differ from those obtaining first priority in developing countries.

It is not our intention to discuss the political issues involved, important as they may be. Further, we shall also assume that the choice of investment projects is not unduely affected by the whims of creditor countries. Granting that loans are concluded under the most reasonable conditions both from the political point of view, and with no economic disadvantages as far as the selection of projects is concerned, the question can be raised: What are the effects of the terms of the loan on the development of the borrowing country? Which is the party really gaining from the loan activity?

The answers to these questions are of vital importance for the long-run development of the developing world. It is not sufficient, however, to investigate criteria for judging foreing loans. What is really needed is to introduce the effects of both the loans and of their service explicitly in growth models. Elementary growth models on the Harrod-Domar lines, are usually presented to developing countries as a manageable device for building up the general frames of their plans.[1)] Such models emphasize the contribution of a deficit in the balance of payments, i.e., loans. But they usually leave it at that, without indicating the necessary surplus which has to follow in order to provide for the debt service, and the adverse effects of that surplus on the rate of growth.

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1) See for example Ichimura's model; Appendix to ch. II of, ECAFE: Programming Techniques for economic development.

The I.N.P.C. has already made a start towards the investigation of the role of foreign loans in long-term development. During his attachment to the Staff of the Institute, Mr. Qayum prepared two memos. dealing with some aspects of foreign loans:

- Memo.570: "Economic Criteria for Foreign Loans" (May 1965)
- Memo.579: "Size of Foreign Loan, Annual Repayment, and Exchange Rate in Programs of Economic Development" (June 1965).

The titles chosen indicate that the main purpose was the discussion of foreign loans in the first place. Although he produced another memo (563) at the same time on "Long term growth of a developing economy", he failed to integrate the various aspects of loans in a more elaborate model of economic growth.[2)]

The present memo. is a review of Qayum's approach, meant to rectify certain fundamental errors involved. It is the first in a series which aim at investigating the implications of foreign loans in order to pave the way for more elaborate models of economic growth. In a following part we shall deal with the problem of loans under the fixed instalment assumption. A third one will deal with the terms of trade factor which is quite important when dealing with a span of time such as that involved in loan repayments. This will be followed by a study of the relative advantage accruing to both the debtor and creditor countries. Sophistication of the behaviour of the parameters of the economy will be introduced at a later stage.

II- Notation:

In this and the following memos we shall adopt a system of notation whose main elements are summarized as follows:

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2) See section III below. The three memos were reproduced in a recent book by Qayum: Numerical Models of Economic Development. Rotterdam University Press 1966. Memo.570 was further published in the E.J., June, 1966 pp.358-369, with some minor variations, as will be shown later.

Y = G.D.P.

S = Gross domestic savings

I = Gross domestic capital formation

$J = S_0$ = Base year savings

L = The Size of loan in national currency

M = Increase in G.D.P. due to base year savings

N = Increase in G.D.P. caused directly by the loan

A = Annual instalment for loan repayment

Δ = Cumulative domestic investments created by the loan during the repayment period

V = The increase in G.D.P. at the end of repayment period due to the loan and to Δ.

r = Rate of interest per annum

e = Repayment period, in years

f = Gestation period of loan-financed investments

h = Repayment lag in years

k = Replacement period for loan-financed capital

γ = Gross capital coefficient (marginal)

σ = Overall marginal propensity to save

α = Marginal propensity to save out of incomes generated by domestic investments

β = Marginal propensity to save out of income created directly by the loan

s = Base year average propensity to save

a = Ratio of annual instalment to the base year G.D.P.

λ = Ratio of the loan to base year G.D.P.

δ = Ratio of Δ to the loan

v = Ratio of V to the output of the loan

$\varepsilon = \delta + 1$ = total increase in domestic capital due to the loan related to its size

η = Value of δ adapted for existence of gestations and repayment lags

μ = Value of ε similarly adapted $= \eta + 1$

ρ = Ratio of r to $\frac{\sigma}{\gamma}$

b = Ratio of annual instalment to the loan

In equation (32) we introduce the following abbreviations:

$$d = s\left(1 + \frac{\sigma}{\gamma}\right)$$

$$p = e - \frac{\gamma}{\sigma}$$

$$q = \frac{\sigma}{\gamma} - r$$

and for $\left(1 + \frac{\sigma}{\gamma}\right)$ raised to any power, the corresponding capital letter is used:

$$E = \left(1 + \frac{\sigma}{\gamma}\right)^{e}$$

$$F = \left(1 + \frac{\sigma}{\gamma}\right)^{f}$$

$$H = \left(1 + \frac{\sigma}{\gamma}\right)^{h}$$

$$K = \left(1 + \frac{\sigma}{\gamma}\right)^{k}$$

Further in (61) we have :

$$k^{*} = k + f - h$$

$$K^{*} = \frac{KF}{H} = \left(1 + \frac{\sigma}{\gamma}\right)^{k+f-h}$$

For investment, output and savings, the values in the no-loan case are denoted by the same symbols primed : I', Y', S'

III - Mr. Qayum's Approach

In a footnote to his article on "Economic Criteria for Foreign loans" (Memo.570), Mr. Qayum stated that he was "motivated to study this problem by the various statements made by the U.A.R. leaders about the terms of West German loans to this country". His findings were in line with those statements.

Further, he defined his objective as follows:[3)]

"The purpose of this note is to study the terms and conditions that are generally involved in these (development) loans, from whatever sources they may be, in the light of the long term economic advantage that accrues to the borrowing country".

His main findings can be summarized as follows:

The benefits of the loan are positively correlated with the magnitude of:

- the marginal propensity to save
- the repayment period
- the repayment lag

and they are negatively correlated with:

- the capital/output ratio
- the gestation period of investments financed by the loan
- the rate of interest charged on the loan.

Mr. Qayum believes that the most effective elements are the marginal propensity to save, (net) capital coefficients and the rate of interest[4)]. His own "guess" is that the marginal propensity to save seems to affect the situation more than others.

These findings conform with what can be expected a priori. The model used is quite simple based on a fixed propensity to save and a fixed capital coefficient. In memo.579, he used a more sophisticated model including a Cobb-Douglas production function, thus allowing for a varying capital/

3) See for example, his "Numerical Models" pp.63-64

4) E.J., 1966, p.368

output ratio, as well as a changing marginal propensity to save. Although he used that model to determine the size of foreign loans necessary to achieve a certain income target, thus taking one step towards the formulation of a development model which allows for loans explicitly, he did not account at the same time for the repayment phase. "Once the size of the foreign loan has been decided, the question about its repayment arises."[5) At this stage he shifted to another problem, which is the necessary change in the rate of exchange required to encourage exports in order to provide for repayment. It may be noticed here that this treatment is not systematic, especially as the original model assumed savings to be equal to investments, i.e., a balanced balance of payments. To account for the difference he appended the discussion with a consideration of the impact on final consumption. This means that his propensity to save is in fact a propensity to invest. The resultant saving propensity has still to be tested for feasibility, in spite of the fact that the change in the exchange rate has been already determined.

The same article suffers from another error. Table III gives unacceptable values for annual values, which led him to the absurd result that a loan of 1000 at 4% can be repaid over 20 years at the rate of 44 annually, thus being paid back at 88% of its initial value. The reason is the introduction of an extra term (tE) which is redundant. The correct formula would be (in our notation):

$$b_t = b = \frac{(1+r)^e}{(1+r)^e - 1} \qquad (t = 1, \ldots, e)$$

Table III has to be corrected accordingly.[6) But the question has still to be asked whether the compound interest rule is the suitable basis for calculation of instalments.

Let us, therefore, go back to memo.570, and its E.J. version. Consider

5) Memo.579, p.7; or "Numerical Models" p.57

6) p.7. The correction given in the text was made in the reprint in "Numerical Models", and Table III was corrected accordingly.

first the criterion chosen for judging the benefits of the loan. In our terminology he first used the criterion Δ ,i.e., the cumulative investments created by the loan over the repayment period. This was replaced by another criterion $\Delta + L$ in the E.J. version with the gross concepts replaced by net concepts.[7] In both instances the rule was that the given criterion chosen should be non-negative if the loan is to be "worthwhile from the very long-term point of view". Thus in one case the loan is considered favourable if after repayment it leaves the economy in possession of a capital capacity exceeding that which would have been created in the absence of the loan by at least the amount of the loan. In the second the loan is worthwhile if it leaves to the economy the same capital stock that could be obtained without the loan; the capital due to the loan being eaten up by repayments. A more careful discussion of the choice of the criterion seems warranted, and we shall take it in the following section.

The most serious error, however, is in the formula used for expressing the advantage criterion. While it follows the cumulation rule in the first two rounds, it falls back simply to the additional effects in the following rounds. This can be seen as follows:

Let the loan L yield the output N

$$N = \frac{L}{\gamma} \qquad (1)$$

With a marginal propensity to save σ, an amount σN is saved. If A_t is the annual instalment to be repaid in year t, the residual for capital formation in any year t is, say

$$R_t = \sigma N - A_t$$

This raises a new income $\left(\frac{1}{\gamma} R_t\right)$ in year t + 1, out of which $\left(\frac{\sigma}{\gamma} R_t\right)$ is saved. Of course we have to add to this R_{t+1} defined in the same manner. This is correct so far. The slip starts from the consideration of year t+2. Instead of considering the direct and indirect effects of R_t, i.e., the $\left(\frac{\sigma}{\gamma} R_t\right)$ itself obtained in t+1, hence continuing subsequently

7) E.J., 1966, p.361. These were the main corrections made there, probably after consultation with Prof. R.C.O. Matthews who "pointed out very crucial slips"

(the same as with N itself), plus the saving out of the extra income generated by it, namely $(\frac{\sigma}{\gamma})$ out of it, or $(\frac{\sigma}{\gamma})^2 R_t$; Mr. Qayum takes into consideration this latter extra effect only. His definition of the total investments due to the effect R_t is [8)]

$$T_t = R_t \left[1 + \frac{\sigma}{\gamma} + (\frac{\sigma}{\gamma})^2 + \dots + (\frac{\sigma}{\gamma})^{e-t} \right]$$

$$= R_t \times \frac{\gamma}{\gamma - \sigma} \times \left[1 - (\frac{\sigma}{\gamma})^{e-t} \right]$$

The increase in the capital stock due to the investments generated by the loan, excluding the loan itself is denoted by S^*,

$$S^* = \sum_{t=1}^{e} T_t$$

No attempt was made to express this sum as a function of the data and parameters, since numerical solutions were in view.

Now, since R_t always included σN, we have to investigate the extra savings generated. In year t+1 they are $\frac{\sigma}{\gamma} R_t$. In year t+2, we still have this $\frac{\sigma}{\gamma} R_t$ plus $\frac{\sigma}{\gamma}$ out of it or $\frac{\sigma}{\gamma}(1 + \frac{\sigma}{\gamma})R_t = \frac{\sigma}{\gamma}R_t + (\frac{\sigma}{\gamma})^2 R_t$. In t+3 we again have this latter plus $\frac{\sigma}{\gamma}$ out of it, or $\frac{\sigma}{\gamma}(1 + \frac{\sigma}{\gamma})^2 R_t$ and so on. In year e, the effect will be $\frac{\sigma}{\gamma}(1+\frac{\sigma}{\gamma})^{e-t-1} R_t$

Thus the total for any year's investments is

$$T'_t = R_t \left[1 + \frac{\sigma}{\gamma} + \frac{\sigma}{\gamma}(1 + \frac{\sigma}{\gamma}) + \dots + \frac{\sigma}{\gamma}(1+\frac{\sigma}{\gamma})^{e-t-1} \right]$$

$$= R_t (1 + \frac{\sigma}{\gamma})^{e-t} \qquad (2)$$

The sum S^*, or (in our notation) Δ, can be obtained by summation. First we notice that the annual instalment is:

$$A_t = \frac{L}{e} \left[(1 + (e-t+1)r \right] \qquad (3)$$

Putting $j = e - t$, then $t = e - j$ Hence

$$R_t = L \frac{\sigma}{\gamma} - \frac{L}{e} \left[1 + re + r - r(e-j) \right]$$

$$= \frac{L}{e} (c - rj)$$

where, $c = \frac{\sigma}{\gamma} e - 1 - r$

8) There is a printing error in the power of the last term in memo.570, which was corrected in the "Numerical Analysis". It was given as (e-t-1) instead of (e-t).

Further, we notice that:

$$\sum_{j=0}^{e-1} j(1+\frac{\sigma}{\gamma})^j = \sum_{1}^{e-1} j(1+\frac{\sigma}{\gamma})^j = \frac{\gamma}{\sigma}\left[e(1+\frac{\sigma}{\gamma})^e - (1+\frac{\sigma}{\gamma})\right]$$

$$- (\frac{\gamma}{\sigma})^2 \left[(1+\frac{\sigma}{\gamma})^{e+1} - (1+\frac{\sigma}{\gamma})^2\right]$$

$$= \frac{\gamma}{\sigma}\left[(1+\frac{\sigma}{\gamma})^e - 1\right]\left[e - \frac{\gamma}{\sigma} - 1\right] + \frac{\gamma}{\sigma} e$$

$$\Delta = \sum_{t=1}^{e} T'_t = \sum_{1}^{e} R_t (1+\frac{\sigma}{\gamma})^{e-t}$$

$$= \frac{L}{e}\left[\sum_{j=0}^{e-1} \sigma (1+\frac{\sigma}{\gamma})^j - \sum_{1}^{e-1} r j (1+\frac{\sigma}{\gamma})^j\right.$$

$$= \frac{L}{e}\frac{\gamma}{\sigma}\left[(1+\frac{\sigma}{\gamma})^e - 1\right]\left[\frac{\sigma}{\gamma} e + \frac{\gamma}{\sigma} r - 1 - er\right] - \frac{L}{e}\frac{\gamma}{\sigma} er$$

$$\Delta = \frac{L}{e}\frac{\gamma}{\sigma}\left\{\left[(1+\frac{\sigma}{\gamma})^e - 1\right](e - \frac{\gamma}{\sigma})(\frac{\sigma}{\gamma} - r) - er\right\} \quad (4)$$

Qayum's other formulae for the cases where gestation periods and payment lags are involved, have to be corrected in the same manner. On the other hand if we choose the alternative criterion, we have to add L to Δ. This brings us back to the problem of choice of the rule suitable for judging the benefit of the loan.

IV- The Advantage Criterion

The immediate effect of the loan is to raise G.D.P. by its full output. Later on, this extra income initiates a series of domestic investments which help to increase G.D.P. further. On the other hand, repayments work in the adverse direction. This latter ceases when the full repayment is over, i.e., starting year (e+1). From there onwards, the economy develops at its own inertia, determined by the ratio ($\frac{\sigma}{\gamma}$). Five cases can be distinguished; as indicated in Fig.(1)

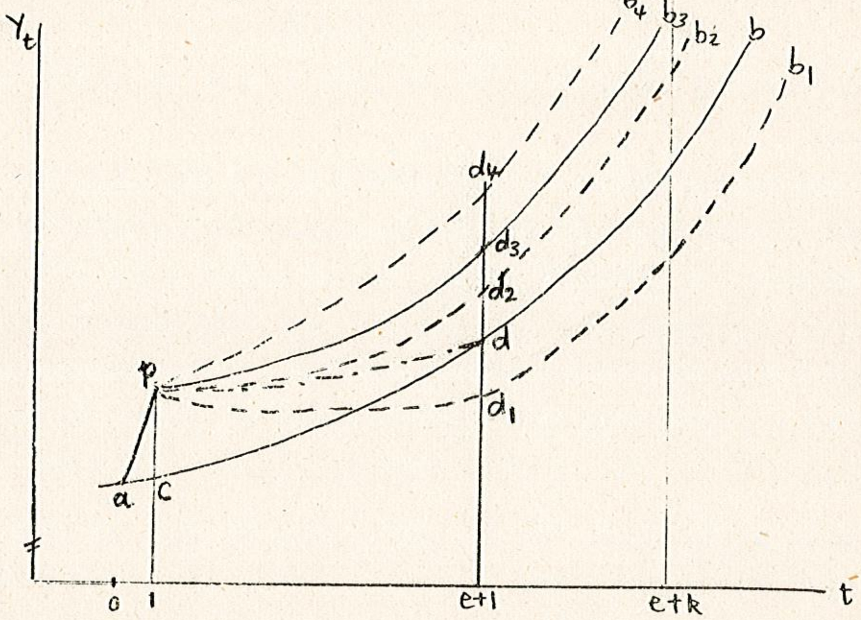

Figure (1)

Let a represent Y_o and c represent Y_1 if no loan was taken, while p is Y_1 including the output cp of the loan. The curve acdb represents the path of Y in the no-loan case, where d is the value of Y in year e+1 after repayment period. The five cases are indicated by the following paths:

1) path pd_1b_1. Here Y_{e+1} is d_1 which is less than d. For all $j > o$, the value of Y_{e+j} is systematically lower than the value obtained without the loan. The segment d_1b_1 is parallel to the segment db with equal horizontal distances which means increasing vertical

distances. Clearly the loan is harmful from the long-term point of view, since the loss will go on infinitely. If the loan terms are quite severe the loss dd_1 might be even greater than the output of the loan cp. In any case such a loss will occur at some future date, e+k say and afterwards.

2) path pdb. In this case the end of the repayment period leaves G.D.P. at the same levels which could have been obtained without the loan. According to the E.J. version this is the case where the loan is just worthwhile, with

$$\Delta + L = 0$$

In fact the loan is quite neutral as far as development is concerned.

3) path pd_2b_2 where dd_2 represents a net gain due to the loan, though not to the full amount of the loan. Since vertical differences increase in absolute terms, there exists some point e+k of time where the gain in Y is equal to the full output of the loan. According to the E.J. criterion, the loan is considered advantageous from the long-run development point of view. It keeps income always above the levels achieved without the loan.

4) path pd_3b_3 where dd_3 is just equal to cp. At the end of the repayment period, the economy is able to preserve the full output due to the loan. The conditions of the economy are such that, with the given terms of the loan, it is able to let the investments financed by the loan pay the loan back. This is the case where $\Delta = 0$, which was chosen by Qayum in his original version.

5) path pd_4b_4 which shows the ability of the economy to generate domestic investments equal to the loan, before its full repayment. This case is advantageous according to both criteria.

Whichever criterion we choose, the case where a clearcut decision can be taken is that of a neutral loan. But the whole situation is left to the conditions of the loan. The critical point chosen is determined at (e+1) which depends on the term of the loan. Thus if the loan is repaid in 10 years, we have to make sure that the critical point is satisfied in the 11th year. If it is repaid in 40 years, the critical point is delayed accordingly.

The change in the criterion used was accompanied by a switch from gross to net concepts. A little reflection shows that this switch is immaterial. In either case, the same formula leads to the same value of domestic product, allowing for the difference between the gross and net concepts. Therefore we can investigate the problem on the basis of the gross concept.

First, let us consider the case of an individual firm. We can distinguish two cases where the firm wants to raise capital from external resources. The first is that in which there is some temporary need which is expected to last for a limited period after which things "fall back to normal". The second is the case where an expansion is in view, and is expected to last indefinitely. In the former case the firm resorts to borrowing and arranges for the repayment of the debt in such a way that keeps its capital intact. In the latter it has to raise new capital from the market. But then it has to put aside reserves which would enable capital renewal after the period in which it is expected to be consumed.

In the case of the national economy a similar differentiation can be made. Loans held for purposes of meeting requirements of production without aiming at increasing the capacity of production have to satisfy the criterion that capital should be left intact after repayment (which usually occurs in a very short period). Thus we should have: $\Delta = 0$. A value smaller than that would mean that the loan has leaked into consumption purposes, and the rate of growth of production will be smaller than without the loan. Normally these loans are held with the hope of enabling the economy to export in the very near future an amount which is sufficient to repay them. The problem will disappear if exports are realized during the same year, and the loan - or credit facilities - will not figure out in the annual accounts of the country.

But if the loans are meant to increase the possibilities of growth of the economy, which is the main feature of development loans, something more has to be aimed at. We have to remember that such loans should essentially play the role of capital raised by the individual firm. Of course these loans should ipso facto satisfy the condition $\Delta \geqslant 0$. But there is a certain deceptive feature about investments financed by foreign loans: The projects thus constructed are tangible, and one is apt to forget those projects which will not be there because of the charges due to the repayment phase. According to the principle adopted by firms raising new capital, reserves accumulated at the end of the life of the capital goods thus procured should be equal to the same amount.

Therefore we can argue that a development loan is considered advantageous if conditions are such that at the end of the k years of the life of the project financed by it, the economy was capable of regaining in full the original capital, i.e., the loan. Here k depends on the type of investment rather than on the terms of the loan, and it is possible that $k \gtreqless e$. Since the behaviour of the rate of growth differs between the period of repayment and that following it, we have to investigate the implications of the two criteria $\Delta = 0$, and $\Delta + L = 0$, in order to locate the proper points. This will be indicated later.

V- Restatement of the Qayum Model

The corrections made in Section III can be considered as solution for the problem raised. However, it is advisable to work out another solution (which, incidentally, is much simpler) in order to pave the way for further investigation of the development of the economy. For the moment we shall stick to Qayum's basic assumptions. This means that:

1) Base year data are given. We further introduce the variation that the base year average propensity to save s need not necessarily

equal σ. This will not affect the argument as can be seen from (4).

2) All aggregates are expressed gross. Consequently, both the marginal propensity to save, σ, and the marginal capital/output ratio are on a gross-to-gross basis.[9]

3) Only one loan is concluded sometime before year 1, its ratio to base year G.D.P. being λ,

$$L = \lambda Y_o \qquad (5)$$

What is applied to a single loan can be extended to a series of loans taken consecutively.

4) The output of the loan is N defined by equation (1) or

$$N = \frac{\lambda}{\gamma} Y_o$$

In principle our analysis will not be greatly affected if the relevant capital/output ratio differs from that for the domestic economy.

5) At this stage we need not worry about the gestation period, since we shall assume that repayment starts in year 1, which is the first year in which full capacity is achieved. On the other hand, investments financed by internal resources are assumed to produce in the following year. Even if this is not so, we can approximate it by proper calculation of γ, under assumptions of systematic growth.

6) If the repayment period is e, and the rate of interest r, the annual instalment according to Qayum's assumptions are A_t given by (3). This can be related to the base year G.D.P. to obtain:

$$a_t = A_t / Y_o = \frac{\lambda}{e} \left[1 + (e+1)\, r - tr \right] \qquad (7)$$

According to these assumptions we can formally treat the loan as being additions to domestic investment in the base year, both starting to produce in year 1. Now according to the average propensity s the base year savings are :

$$J = S_o = s \cdot Y_o \qquad (8)$$

Then we can treat base year investments as follows:

$$I_o = J + L = (s + \lambda)\, Y_o \qquad (9)$$

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9) Strictly speaking, the assumption of a constant capital/output ratio would not fit exactly in this case. However, it can be accepted for a period long enough as probable values of e.

G.D.P. in year 1 will be therefore:

$$Y_1 = Y_o + \frac{1}{\gamma} I_o = (1 + \frac{s+\lambda}{\gamma}) Y_o \tag{10}$$

Savings will increase by a ratio σ of the increase in income :

$S_1 = S_o + \sigma(Y_1 - Y_o)$. Or, writing

$$c = s (1 + \frac{\sigma}{\gamma}) + \lambda \frac{\sigma}{\gamma} \tag{11}$$

we have:

$$S_1 = c\ Y_o \tag{12}$$

$$I_1 = (c - a_1) Y_o \tag{13}$$

where,

$$a_1 = \frac{\lambda}{e} (1 + er) \tag{14}$$

Thus given Y_o and the parameters $s, \sigma, \gamma, \lambda$ and a_1, we can determine the first year aggregates. In this treatment we assume that the balance of payments is balanced in the absence of the loan.

For the remaining years $t = 2, 3..., e+1$, we have the folloaing system of equations, in which $a_{e+1} = 0$.

$$Y_t = Y_{t-1} + \frac{1}{\gamma} . I_{t-1} \tag{15}$$

$$S_t = S_{t-1} + \frac{\sigma}{\gamma} I_{t-1} \tag{16}$$

$$I_t = S_t - a_t Y_o \tag{17}$$

Equation (15) means that $Y_t = Y_1 + \frac{1}{\gamma} \sum_1^{t-1} I_t$. In particular:

$$Y_{e+1} = Y_1 + \frac{1}{\gamma} \sum_1^{e} I_t \tag{18}$$

Solving (16) and (17) together we obtain :

$$I_t = (1 + \frac{\sigma}{\gamma}) I_{t-1} + (a_{t-1} - a_t)$$

From (7) it can be found that $(a_{t-1}-a_t) = \frac{\lambda}{e} r$.

Hence,

$$I_t - (1 + \frac{\sigma}{\gamma}) I_{t-1} = (\frac{\lambda}{e} r) Y_o \tag{19}$$

The solution of this first -order differnce equation can be easily found to be:

$$I_t = (1+\frac{\sigma}{\gamma})^{t-1}\ (I_1+\frac{\gamma}{\sigma}\frac{\lambda}{e} r\ Y_0) - \frac{\gamma}{\sigma}\frac{\lambda}{e} r\ Y_0 \quad (20)$$

The sum in (18) is:

$$\sum_1^e I_t = \frac{\gamma}{\sigma}\left[(1+\frac{\sigma}{\gamma})^e-1\right]\left[s\ (1+\frac{\sigma}{\gamma})+\frac{\lambda}{e}(\frac{\sigma}{\gamma}e+\frac{\gamma}{\sigma}r-er-1)\right]Y_0 - e\ (\frac{\gamma}{\sigma}\frac{\lambda}{e}r)\ Y_0 \quad (21)$$

Substituting in (18), Y_{e+1} is expressed as:

$$Y_{e+1} = \frac{1}{\sigma}\left[(1+\frac{\sigma}{\gamma})^e-1\right]\cdot\left[s\ (1+\frac{\sigma}{\gamma})+\frac{\lambda}{e}(\frac{\sigma}{\gamma}e+\frac{\gamma}{\sigma}r-er-1)\right]Y_0 + \left[(1+\frac{s}{\gamma})+\frac{\lambda}{\sigma}(\frac{\sigma}{\gamma}-r)\right]Y_0 \quad (22)$$

Similar expressions hold for Y_t, ($t = 1,\ldots,e$), with the power e changed to (t-1) in the first term.

If no loan was taken, a similar system can be constructed and solved in the same manner. Denoting the aggregates in this case by the same symbols with a prime, we can obtain the solution immediately by putting $\lambda = 0$ and $a_t = 0$ in the above formulae, while Y_0 and $J = S_0$ remain the same.

$$Y'_1 = (1+\frac{s}{\gamma})\ Y_0 \quad (23)$$

$$\&\quad Y'_{e+1} = Y'_1 + \frac{1}{\gamma}\sum_1^e I'_t$$

Further,

$$\sum_1^e I'_t = \frac{\gamma}{\sigma}\left[(1+\frac{\sigma}{\gamma})^e-1\right]\left[s(1+\frac{\sigma}{\gamma})\right]Y_0 \quad (24)$$

Hence,

$$Y'_{e+1} = \frac{s}{\gamma}(1+\frac{\gamma}{\sigma})\ (1+\frac{\sigma}{\gamma})^e\ Y_0 + (1-\frac{s}{\sigma})Y_0 \quad (25)$$

Subtracting (24) from (21) we obtain:

$$\Delta = \sum_1^e I_t - \sum_1^e I'_t = \delta\ L \quad (26)$$

which can be seen to be the same as (4), and where

$$\underline{\underline{\delta = \frac{1}{e}\ \frac{\gamma}{\sigma}\left\{\left[(1+\frac{\gamma}{\sigma})^e-1\right]\cdot(e-\frac{\gamma}{\sigma})\ (\frac{\sigma}{\gamma}-r)-er\right\}}} \quad (27)$$

Since Δ is proportionate to L, the ratio δ is independent of λ , i.e., of both L and Y_o. It is a function of the parameters of the economy, $\frac{\sigma}{\gamma}$, and the terms of the loan, e and r.

The total additional capital due to the loan is $L + \Delta$. Denoting its ratio to the size of the loan by ε, we can write:

$$\varepsilon = 1 + \delta \; , \qquad L + \Delta = (1 + \delta) L = \varepsilon L \qquad (28)$$

In fact the term "capital" is not strict unless we are using net concepts. In our case it denotes the cumulative of gross investments over the given period. To rationalize the use of such a gross cumulative, we introduce:

$$V = Y_{e+1} - Y'_{e+1} = N + \frac{1}{\gamma} \Delta = \frac{L}{\gamma} (1 + \delta) = \varepsilon \frac{L}{\gamma} \qquad (29)$$

This shows that if the original model is accepted, then the ratios δ and ε play the same role whether based on gross or net concepts. This latter criterion can be expressed as a ratio of Y_o :

$$v = \frac{V}{Y_o} = \frac{\lambda}{\gamma} (\delta + 1) = \varepsilon \frac{\lambda}{\gamma} \qquad (30)$$

The two advantage rules suggested by Qayum are:

1) The extra investments due to the loan are non-negative: $\Delta \geqslant 0$.
   This means that :

   $$\delta \geqslant 0, \qquad \varepsilon \geqslant +1 \qquad v \geqslant \frac{\lambda}{\gamma} \qquad (31/a)$$

2) The disinvestments due to the effects of repayment should not be greater than the size of the loan itself : $\Delta \geqslant -L$, or $\Delta + L \geqslant 0$

   $$\delta \geqslant -1, \qquad \varepsilon \geqslant 0 \qquad v \geqslant 0 \qquad (31/b)$$

This second condition is milder, thus setting absolute minima for the criteria chosen. The use of δ or ε involves zero minima which makes them more convenient than v.

VI- Gestations and Repayment Lags

The above treatment ignores two factors which might have important effects on the final results, though working in adverse directions.

The first is the possibility that there might be a lag in the realization of the gestation period. In fact we implicitly assumed the possibility of the existence of such a lag provided an equal lag is given for repayment. For a more general treatment, we have to introduce the effects of a gestation f and a repayment lag h. The period to be considered is h+e, which brings us to the end of the repayment period.

Starting again from a loan in year 0, we assume that the projects financed by it are completed in years 1 to f, so that their output begins in year f+1. We shall assume that they also reach their full capacity at that date. Thus the output $\frac{L}{\gamma} = \frac{\lambda}{\gamma} Y_o$ is to be added to Y_{f+1} instead of Y_1. Any I_t can be decomposed into three parts (some of which might be zero). First, there is the part which could be obtained without any loan:

$$I'_t = s \cdot (1+ \frac{\sigma}{\gamma})^t \; Y_o$$

Starting from year h+1, we have to subtract the annual instalment $a_t \; Y_o$ which has to be written now, $a_{t-h} Y_o$. The algebraic sum of these two parts is denoted by

$$J_t = I'_t - a_{t-h} \qquad (t > h)$$

At this stage we adopt the following notation:

$$\left.\begin{array}{ll} E = (1+ \frac{\sigma}{\gamma})^e, & F = (1+ \frac{\sigma}{\gamma})^f \\ H = (1+ \frac{\sigma}{\gamma})^h, & d = s \; (1+ \frac{\sigma}{\gamma}) \\ p = (e- \frac{\gamma}{\sigma}) , & q = (\frac{\sigma}{\gamma} - r) \end{array}\right\} \qquad (32)$$

The situation is similar to that expressed by system (15)-(17), hence the solution is similar to (20):

$$J_{h+1} = I'_{h+1} - a_1 \; Y_o = \left[s \; (1+ \frac{\sigma}{\gamma})^{h+1} - \frac{\lambda}{e} . (1+ er) \right] Y_o$$

$$\therefore \; J_{h+1} + \frac{\gamma}{\sigma}\frac{\lambda}{e} \; r . Y_o = \left[s \; (1+ \frac{\sigma}{\gamma})^{h+1} - \frac{\lambda}{e} \left\{ 1 + (e - \frac{\gamma}{\sigma}) \; r \right\} \right] Y_o$$

$$= \left[d \; H - \frac{\lambda}{e} (1 + pr) \right] Y_o$$

The solution for J_t ($t = h + 1, \ldots, h + e$) is :

$$J_t = (1 + \frac{\sigma}{\gamma})^{t-h-1} \left[dH - \frac{\lambda}{e} (1 + pr) \right] Y_o - \frac{\gamma}{\sigma}\frac{\lambda}{e} r \; Y_o \qquad (33)$$

The cumulative value over the period of repayment is:

$$\sum_{h+1}^{h+e} J_t = \frac{\gamma}{\sigma}(E-1)\left[dH - \frac{\lambda}{e}(1+pr)\right] Y_0 - \frac{\gamma}{\sigma}\frac{\lambda}{e} er\, Y_0 \qquad (34)$$

This is equivalent to (21) with H = 1, and the output of the loan = 0. To this sum we add the sum of investments in the first h years $\sum_1^h I'_t$, which is the same as without loan.

$$\sum_1^h I'_t = \frac{\gamma}{\sigma}(H-1)\, d\, Y_0 \qquad (35)$$

Finally, we have to add the third part which is the series of investments due to incomes generated by the loan. The initial investment in year f + 1 is $\lambda\frac{\sigma}{\gamma} Y_0$, and it increases at the annual rate $(1+\frac{\sigma}{\gamma})$, which gives the cumulative total of :

$$\sum_{f+1}^{h+e} \lambda\frac{\sigma}{\gamma}\left(1+\frac{\sigma}{\gamma}\right)^{t-f-1} Y_0 = \lambda\left(\frac{EH}{F}-1\right) Y_0$$

$$= \lambda\left[(H-F)\frac{E}{F} + (E-1)\right] Y_0 \qquad (36)$$

Adding up the three components we obtain the cumulative of investments up to the end of repayment period :

$$\sum_1^{e+h} I_t = \frac{\gamma}{\sigma}(EH-1)\, d\, Y_0 + \frac{\gamma}{\sigma}\frac{\lambda}{e}\left[(E-1)\, pq + e\,\frac{\sigma}{\gamma}(H-F)\frac{E}{F} - er\right] Y_0 \qquad (37)$$

From this we subtract the cumulative in the non-loan case:

$$\sum_1^{e+h} I'_t = \frac{\gamma}{\sigma}(EH-1)\, d\, Y_0$$

The relative advantage corresponding to δ is :

$$\eta = \frac{1}{e}\frac{\gamma}{\sigma}\left[(E-1)\, pq - er\right] + \frac{E}{F}\cdot(H-F)$$

$$\eta = \delta + \frac{E}{F}(H-F) \qquad (38)$$

If h = f, the two criteria are equal. When they differ, the two criteria will differ by an amount which is approximately equal to

$$\eta - \delta \doteq (h-f)\left(1+\frac{\sigma}{\gamma}\right)^e \frac{\sigma}{\gamma} \qquad (39)$$

This approximation is on the lower side, since it ignores positive terms involving higher powers of $\frac{\sigma}{\gamma}$. If either h or f is zero the same formula still holds. Hence it can be considered as a general one.

VII - Advantage of the Loan:

The total cumulative savings created as a result of the loan was defined as a ratio ε of the loan in the case where h and f are both zero. This remains true if h = f. If they differ a new term emerges, which can be denoted by :

$$m = \frac{E}{F} \cdot (H - F) \qquad (40)$$

The ratio is altered from ε to :

$$\mu = \varepsilon + m = \delta + 1 + m = \eta + 1 \qquad (41)$$

The conditions (31) have to be expressed in terms of η and μ; while v has to be interpreted in terms of incomes in year (e + h + 1) rather than (e+1). Thus if m is positive, ε can fall below its zero minimum.

Let us investigate the sign of ε.

$$\varepsilon = \frac{1}{e} \frac{\gamma}{\sigma} \left[(E-1) pq - er \right] + 1$$

$$\therefore \quad \varepsilon = \frac{1}{e} \frac{\gamma}{\sigma} \left[(E-1) p + e \right] \cdot q \qquad (42)$$

Now,

$$\begin{aligned} (E-1)\, p+e &= Ep + \frac{\gamma}{\sigma} = e\,E - \frac{\gamma}{\sigma} (E-1) \\ &= e \left(1 + \frac{\sigma}{\gamma}\right)^e - \frac{\gamma}{\sigma}\left[\left(1+ \frac{\sigma}{\gamma}\right)^e - 1 \right] \\ &= \sum_{t=0}^{e-1} \left[\left(1 + \frac{\sigma}{\gamma}\right)^e - \left(1 + \frac{\sigma}{\gamma}\right)^t \right] \end{aligned}$$

This is the sum of e elements, all of which is positive for any positive $\frac{\sigma}{\gamma}$. Hence :

$$(E - 1) p + e > 0 \qquad (43)$$

This shows that the sign of ε depends on that of q. Hence :

$$\varepsilon \gtreqless 0 \qquad \text{if} \qquad \frac{\sigma}{\gamma} \gtreqless r \qquad (44)$$

The exact value of r can be obtained by solving (42):

$$r = \frac{\sigma}{\gamma}\left[1 - \varepsilon \frac{e}{e + (E-1)p} \right] \qquad (45)$$

where $\varepsilon = \mu - m$. If m is positive, ε can become negative in which case r can exceed $\frac{\sigma}{\gamma}$. At a zero m, the minimum value of $\mathcal{E}$ is zero and $r = \frac{\sigma}{\gamma}$ when $\mathcal{E} = 1$, the interest rate has to fall below $\frac{\sigma}{\gamma}$, only approaching it at $e = \infty$.

To investigate the behaviour of e, we notice that the quantity between brackets in (45) should be non-negative :

$$1 - \frac{\mathcal{E}e}{e + (E-1)P} \geqslant 0$$

This gives:

$$e \geqslant \frac{\gamma}{\sigma}\left[1 - \frac{1-\mathcal{E}}{E-\mathcal{E}}\right] \tag{46}$$

Since $E = (1 + \frac{\sigma}{\gamma})^e > (1 + e\frac{\sigma}{\gamma})$, inequality (46) can be reduced to :

$$e \geqslant \frac{\gamma}{\sigma}\mathcal{E} \tag{47}$$

Thus $\mathcal{E} = 0$ can be realized for any positive value of e. As $\mathcal{E}$ increases, the minimum period of repayment has to increase, reaching $\frac{\gamma}{\sigma}$ at $\mathcal{E} = 1$; thereafter increasing beyond that limit. On the other hand, inequality (47) provides a bound on the relative advantage criterion $\mathcal{E}$.

The characteristics of the case $\underline{\mathcal{E} = 1}$ can be investigated on the basis of the formula for r :

$$r = \frac{\sigma}{\gamma}\left[\frac{(E-1)p}{e + (E-1)P}\right] \tag{48}$$

Since the denominator is positive, the numerator should be non-negative, which means that

$p \geqslant 0$, or,

$$\frac{\gamma}{\sigma} \leqslant e < \infty \qquad \text{for which} \quad 0 \leqslant r < \frac{\sigma}{\gamma} \tag{49}$$

For example, if $\sigma = 0.10$ and $\gamma = 4$, the repayment period has to exceed 40 years. At e = 50 years, r = 0.8%; at e = 100 years, r = 2.15%; then it increases gradually to 2.5% at an infinite e. If $\frac{\sigma}{\gamma}$ is as high as the unrealistic value 8%, the loan should be repaid in more than 12.5 years. With e = 15 years, r = 2.13%; and with e = 40 years, r = 7.47%.

Higher rates of interest are compatible only with larger values of m, which enable $\mathcal{E}$ to fall below 1. Noticing that for many developing economies the ratio $\frac{\sigma}{\gamma}$ hardly exceeds 5%, it can be seen that with the familiar values of r, e, f and h involved in foreign investment loans, no serious contribution to development should be expected. The above analysis shows that unless a generous repayment lag is accorded, a rate of interest exceeding $\frac{\sigma}{\gamma}$ will sterilize a good part of the loan's contribution, especially at relatively small repayment periods.

This means that naive models based on assessing loan requirements as the difference between total resources necessary to achieve a predetermined target and the resources that could be raised out of domestic means beg the question. If we take loan conditions as granted we have to rely on ($\mu\lambda$) rather than λ itself. But this raises many difficulties, and it would be more advisable to put clear-cut conditions on μ, other than simply requiring it to be positive. For this purpose we suggest the replacement criterion as a working hypothesis.

VIII- The Replacement Rule:

The requirement that ε (or μ) should be unity to ensure that through distribution of repayment over time, domestic resources can take over from foreign resources sounds acceptable on its face value. However, the large values of e revealed by the previous analysis show that no control is availed to the debtor country on the total size of development resources at a reasonable future date.

It is only natural that such a date should not exceed the life of the projects financed by the loans. If we are working with gross concepts, as we suggested here, this means that the country should be able to replace those projects out of domestic means accumulated over and above any repayments made. Assuming out any changes in prices, this means that at the end of the replacement period, k , the cumulative investments should be equal to the value of the loan itself.

Let us consider the case where both f and h are equal to zero. Three possibilities are open which can be distinguished on the basis of the relationship between e and k. The first is that of $k = e$, which requires ε to be equal to unity. This has been already investigated. Let us turn to the other 2 cases.

a) $k < e$:

We construct an expression similar to Δ, replacing e by k :

$$\sum_{1}^{k} (I_t - I'_t) = \frac{\lambda}{e}\frac{\gamma}{\sigma}\left[(K-1) p q - k r \right] \cdot Y_o \qquad (50)$$

where,

$$K = (1 + \frac{\sigma}{\gamma})^k \qquad (51)$$

Equating this to zero, we can solve for r :

$$r = \frac{\sigma}{\gamma}\left[\frac{(K-1)p}{k + (K-1)p} \right] = \frac{\sigma}{\gamma}\left[1 - \frac{k}{k + (K-1)P} \right] \qquad (52)$$

Compared with (45), this shows that:

$$\varepsilon = \frac{k}{e} \left[\frac{e + (E-1)p}{k + (K-1)p} \right] \qquad (53)$$

Since,

$$E - K = (e-k)\frac{\sigma}{\gamma} + \text{positive terms}$$

$$(E-K) > (e-k)\frac{\sigma}{\gamma}, \text{ or, } \frac{E-1}{K-1} > \frac{e}{k}$$

This means that for non-negative values of p,

$$\varepsilon \geqslant 1$$

Given k, and for values of $e \geqslant \frac{\gamma}{\sigma}$, both ε and r can be calculated. For e at its lower bound, r = 0, and as e increases indefinitely, r approaches its limit $\frac{\sigma}{\gamma}$.

Let us denote the ratio of r to $\frac{\sigma}{\gamma}$ by ρ:

$$\rho = r/\frac{\sigma}{\gamma} = \frac{(K-1)p}{k+(K-1)p} \qquad (54)$$

This can be solved for e to give :

$$e = \frac{\gamma}{\sigma} + \left(\frac{\rho}{1-\rho}\right) \left(\frac{k}{K-1}\right) \qquad (55)$$

or, its approximation

$$e = \frac{\gamma}{\sigma}\left(\frac{1}{1-\rho} \right) \quad \text{app.} \qquad (56)$$

Given any rate of interest within the upper bound $\frac{\sigma}{\gamma}$, we calculate its ratio ρ to this latter and substitute in (56) to obtain a rough estimate of e, which can be precised by means of (55). It might be seen that (56) is slightly on the larger side; its discrepancy increasing as k increases. Notice that (56) is independent of k.

For example if $\frac{\sigma}{\gamma} = 0.04$, and $r = 2\%$, which means that $\rho = 0.5$, the approximate value of e is 50; i.e., it has to be somewhere below 50. For k = 15, it should be 43.7 years and as k increases to 20, the value of e drops further to 41.8. A rate of interest equal to 3%, requires a repayment period of 81.2 years for k = 15, and 75.4 for k = 20; while the approximate period is 100.

b) $k > e$:

In this case we have to account for the extra investment during the period e + 1 to k. Given that :

$$S_e = I_e + a_e Y_o = I_e + \frac{\lambda}{e} (1+r) Y_o$$

$$\therefore \quad I_e = (1+ \frac{\sigma}{\gamma})^{e-1} (I_1 + \frac{\gamma}{\sigma}\frac{\lambda}{e} r Y_o) - \frac{\lambda}{e}\frac{\gamma}{\sigma} r Y_o$$

Hence, $I_{e+1} = S_{e+1} = S_e + \frac{\sigma}{\gamma} I_e$, or

$$I_{e+1} = (1 + \frac{\sigma}{\gamma})^e \quad (d + \frac{\lambda}{e} p q) Y_o - \frac{\lambda}{e}\frac{\gamma}{\sigma} q Y_o$$

Starting year e + 1, no repayments are involved. Investments in each of these years can be expressed as follows:

$$I_t = (1 + \frac{\sigma}{\gamma})^{t-e-1} \quad I_{e+1}$$

$$= (1 + \frac{\sigma}{\gamma})^{t-1} (d + \frac{\lambda}{e} pq) Y_o + (1 + \frac{\sigma}{\gamma})^{t-e-1} \frac{\lambda}{e}\frac{\gamma}{\sigma} q Y_o$$

For I'_t (i.e., the no-loan case) the same formula applies, with $\lambda = 0$. Hence the cumulative over the years e+1 to k, of the extra investmensts is :

$$\sum_{e+1}^{k} (I_t - I'_t) = (K-E) \frac{\gamma}{\sigma}\frac{\lambda}{e} (pq + \frac{\gamma}{\sigma} q \frac{1}{E}) Y_o$$

To this we add Δ as defined by (4), in order to obtain the total of extra cumulative investments due to the loan in the k years.

Equating this to zero, we obtain :

$$0 = \frac{\gamma}{\sigma}\frac{\lambda}{e}\left[K\left(pq + \frac{\gamma}{\sigma}q\frac{1}{E}\right) - \left(pq + er + \frac{\gamma}{\sigma}q\right)\right]$$

$$= \frac{\gamma}{\sigma}\frac{\lambda}{e}\left[\frac{K}{E}\left(E\,pq + \frac{\gamma}{\sigma}q\right) - e\frac{\sigma}{\gamma}\right] \qquad (57)$$

Dividing both sides by $\frac{K}{K}\frac{\gamma}{\sigma}\frac{\lambda}{e}$, and noticing that

$$Epq + \frac{\gamma}{\sigma}q = \left[(E-1)p + e\right]\left(\frac{\sigma}{\gamma} - r\right)$$

it follows that :

$$\left[(E-1)p + e - \frac{E}{K}\cdot e\right]\frac{\sigma}{\gamma} = \left[(E-1)p + e\right]r$$

Solved for r, this gives :

$$r = \frac{\sigma}{\gamma}\left[1 - \frac{E}{K}\cdot\frac{e}{e + (E-1)p}\right] \qquad (58)$$

from which we deduce :

$$\varepsilon = \frac{E}{K} < 1 \qquad (59)$$

Using (47), we can assess the minimum value of e.

$$e \geqslant \frac{\gamma}{\sigma}\frac{E}{K} > \frac{e + \frac{\gamma}{\sigma}}{K}$$

Hence,

$$e > \frac{\gamma}{\sigma}\left(\frac{1}{K-1}\right) \qquad (60)$$

If $K > 2$, the minimum e can fall below $\frac{\gamma}{\sigma}$. Therefore, given k, we have to calculate (60) in order to make sure that e is smaller than k. Of course this assumes that $k > \frac{\gamma}{\sigma}$

In the previous example where $\frac{\sigma}{\gamma} = 0.04$, if k = 40 years, (60) gives $e > \frac{25}{3.801} = 7$ app. In fact the minimum value of e is about 12 years. The maximum value under this case is k = 40, at which r is found to be 2.02%. At $e = \frac{\gamma}{\sigma} = 25$, the value of r = 1.78%; and if e=15 only, r = 0.8%. If we are to allow some larger value for r, we have to resort to the formulae of the previous case. Thus (56) shows that for a rate of 3%, e = 71 years.

We can conclude that if $k > \frac{\gamma}{\sigma}$, we can check the values of r at $e = \frac{\gamma}{\sigma}$ and at e = k. If the value of r is not acceptable to the creditor country, a higher rate has to go side by side with a larger period calculated according to (56). In fact (56) helps to locate roughly the position of e at a given rate of interest with respect to k. For example if $r = 0.8$, hence $\rho = 0.2$, formula (56) shows that $e = 28$ years, which is smaller than k, and we have to shift to the relevant set of formulae.

In any case the few examples which we have considered have shown that unless the interests charged are very low, the period of repayment is apt to exceed that of replacement, probably several times.

IX - Gestations, Lags & Replacement:

We now introduce the same assumptions made before as regards the gestation period, f, and the repayment lag, h, together with the replacement period, k. The case where $k = h + e$, has been already investigated. We turn to the other two cases:

a) $k < h + e\ f$

Using equations (33), (35) and (36), we can find the extra investments due to the loan up to the end of the first k years following the gestation period. Let us introduce the following notation:

$$k^* = k - (h - f), \qquad K^* = \frac{KF}{H} \tag{61}$$

Hence,

$$\sum_{1}^{k+f} (I_t - I'_t) = \sum_{h+1}^{k+f} J_t + \sum_{f+1}^{k+f} \left(1 + \frac{\sigma}{\gamma}\right)^{t-f-1} \lambda \frac{\sigma}{\gamma} Y_o$$

$$= (K^* - 1)\frac{\lambda}{e}\frac{\gamma}{\sigma}\left(pq - e\frac{\sigma}{\gamma}\right) Y_o - \frac{\lambda}{e}\frac{\gamma}{\sigma} k^* r Y_o + (K-1)\lambda Y_o$$

This sum has to be equated to zero if the replacement condition is to be satisfied. Rearranging the terms and dividing by λY_o we obtain

$$0 = \frac{1}{e}\frac{\gamma}{\sigma}\left[(K^* - 1) pq - k^* r\right] + (K - K^*)$$

or,

$$\frac{1}{e}\frac{\gamma}{\sigma}\left[\left(\frac{KF}{H} - 1\right) pq - (k - h + f) r\right] + \frac{K}{H} . (H - F) = 0 \tag{62}$$

Compared with (38), it can be seen that the left hand side is similar to η, with E replaced by $K^* = KF/H$. On the other hand, if we ignore the last term, it can also be seen that the rest is similar to the expression (50) with k replaced by k^*. Since k,h, and f are given, the above expression can be solved for r in terms of e and vice versa.

b) $k > e+h-f$:

Following the technique used in the last section we can obtain an expression for the advantage criterion. First of all, we have the value of the part over the first e+h years by means of (38), namely

$$\lambda\eta = \frac{\lambda}{e}\frac{\gamma}{\sigma}\left[(E-1)pq - er\right] + \lambda\left(\frac{EH}{F} - E\right)$$

For the remainder of the period, we have :

$$(I-I')_{e+h} = \frac{\lambda}{e}\left(1+\frac{\sigma}{\gamma}\right)^{e-1}\cdot\left(pq - e\frac{\sigma}{\gamma}\right) - \frac{\lambda}{e}\frac{\gamma}{\sigma}r + \lambda\frac{\sigma}{\gamma}\left(1+\frac{\sigma}{\gamma}\right)^{e+h-f-1}$$

$$\therefore (I-I')_{e+h+1} = \frac{\lambda}{e}\left(1+\frac{\sigma}{\gamma}\right)^{e}\cdot\left(pq - e\frac{\sigma}{\gamma}\right) + \left(1+\frac{\sigma}{\gamma}\right)^{e-h-f}\lambda\frac{\sigma}{\gamma} + \frac{\lambda}{e}\frac{\gamma}{\sigma}q$$

It follows that between e+h+1 and k+f,

$$I_t - I'_t = \frac{\lambda}{e}\left(1+\frac{\sigma}{\gamma}\right)^{t-h-1}\left(pq - e\frac{\sigma}{\gamma}\right) + \left(1+\frac{\sigma}{\gamma}\right)^{t-f-1}\lambda\frac{\sigma}{\gamma} + \frac{\lambda}{e}\frac{\gamma}{\sigma}\left(1+\frac{\sigma}{\gamma}\right)^{t-h-e-1}q$$

Hence

$$\sum_{e+h+1}^{f+k}(I_t - I'_t) = \frac{\lambda}{e}\frac{\sigma}{\gamma}(K^* - E)\left(pq - e\frac{\sigma}{\gamma}\right) + \left(K - \frac{EH}{F}\right)\lambda + \frac{\lambda}{e}\frac{\gamma}{\sigma}\left(\frac{K^*}{E} - 1\right)\frac{\gamma}{\sigma}q$$

To this we add $\lambda\eta$,

$$\therefore \sum_{1}^{f+k}(I_t - I'_t) = \frac{\lambda}{e}\frac{\sigma}{\gamma}(K^* - 1)pq + (K - K^*)\lambda + \frac{\lambda}{e}\frac{\gamma}{\sigma}\left(\frac{K^*}{E} - 1\right)\frac{\gamma}{\sigma}q - \frac{\lambda}{e}\frac{\gamma}{\sigma}er$$

$$= \frac{\lambda}{e}\frac{\gamma}{\sigma}\left[\frac{K^*}{E}\left(E\,pq + \frac{\gamma}{\sigma}q\right) - \left(pq + \frac{\gamma}{\sigma}q + er\right)\right] + (K - K^*)\lambda$$

Equating to zero we get an expression similar to (57) with K replaced by K^*, and we also have the extra term in ($K-K^*$) as in the previous case. Dividing by λ we obtain :

$$\frac{1}{e}\,\frac{\gamma}{\sigma}\left[\frac{K^*}{E}\left(Epq+\frac{\gamma}{\sigma}q\right)-e\,\frac{\sigma}{\gamma}\right]+\left(K-K^*\right)=0 \qquad (63)$$

Since both K and K^* are given, the solution of this equation proceeds as before. It might be noticed that if $h = f$, then $K = K^*$ and we are back to the case considered in the previous section.

Conclusions:

In this memo., we have tried to study the full implications of the Qayum model. The basic idea was to bring the model much closer to the study of long-term development, with foreign loans explicitly introduced. The model was found to yield itself to a more straightforward expression of the growth pattern, when the fundamental process described by Qayum was corrected.

Leaving aside the complications arising from the differences that could exist between gestation periods and repayment lags, the main results have shown that:

(1) The direction of the effects of a given loan depends on the sign of the difference between the ratio $\frac{\sigma}{\gamma}$ and the rate of interest r. The case where the loan has a neutral effect on the long-run development of income is the one characterized by equality between the two rates. Like the Harrod-Domar model, the present one is characterized by a knife-edge equilibrium path, where a single point out of an infinity of alternatives satisfies the condition of coinciding paths.

(2) The effect of the repayment period length is the magnification of the effects, once their direction is determined according to the above rule. Since the magnification of a zero effect is itself

zero, the above results show that the length of the repayment period is irrelevant in the zero-effect case.

It was also found that an excess of the repayment lag over the gestation period produces a positive effect.

A prolongation of the repayment period is, generally, less useful than according a repayment lag by the same length. Let us ignore gestations which have the same negative effects in either case. A prolongation by a period h means that the repayment period is e+h, and the criterion ε becomes:

$$\varepsilon' = \frac{1}{e+h} \cdot \frac{\gamma}{\sigma} \left[(EH-1)(p+h) + (e+h) \right] \cdot q \tag{64}$$

On the other hand $\mu = \varepsilon + (EH-E) = \varepsilon + (EH-1) - (E-1)$. Hence the excess of μ over ε' is

$$\mu - \varepsilon' = \frac{1}{e+h} \frac{\gamma}{\sigma} (EH-1) \left[1+r(p+h) \right] - \frac{\gamma}{\sigma} q - \frac{1}{e} \frac{\gamma}{\sigma} (E-1) \left[1+rp \right] + \frac{\gamma}{\sigma} q$$

Approximating (E-1) by $e \frac{\sigma}{\gamma}$ and (EH-1) by $(e+h) \frac{\sigma}{\gamma}$, we find that

$$\mu - \varepsilon' \doteq rh > 0 \tag{65}$$

The excess increases as h increases; it is also more pronounced for higher values of r.

In this paper we have attempted to introduce some principle by which the point of time at which the advantage criterion is to be measured. Although this principle has the disadvantage of introducing a characteristic of the projects financed by the loans, such a disadvantage can be diluted by considering some sort of an average replacement period, or even a completely different basis for the determination of k. Once k has been determined the rest of the analysis remains valid. The outcome is, if we stick to a single set of parameters σ and γ, we have to face situations in which extremely long repayment periods have to be accepted.

The elaboration of Qayum's model helps to point out certain facts. One main result is that if loans have to receive any consideration from a developmental point of view, the behaviour of the parameters has to be reconsidered. This is treated in a forthcoming paper, together with the possibility of adopting the fixed instalment principle.

Appendix I.

Values of Criterion ε

The criterion $\varepsilon = \delta + 1$ can be calculated according to (42):

$$\varepsilon = \frac{1}{e}\,\frac{\gamma}{\sigma}\Big[\,(E-1)\,p + e\,\Big]\cdot q$$

$$= \frac{1}{e}\,\frac{\gamma}{\sigma}\Big[\Big\{\Big(1 + \frac{\sigma}{\gamma}\Big)^{e} - 1\Big\}\Big(e - \frac{\gamma}{\sigma}\Big) + e\Big]\Big(\frac{\sigma}{\gamma} - r\Big)$$

According to (43), $[(E-1)\,p + e]$ is positive always, for all relevant values of the parameters. It follows that the sign of ε is that of $q \cdot (\frac{\sigma}{\gamma} - r)$. The condition that $\varepsilon = 0$, simply means that $\frac{\sigma}{\gamma} = r$, whatever e is.

On the other hand, it is sufficient to calculate the expression to be multiplied into q. Given r we calculate q and hence the corresponding ε. Therefore instead of tabulating ε, we tabulate the multiplier of q, and the reader can obtain ε for any given r according to the above formula. Thus, if $\sigma = .10$, and $\gamma = 4$, then $\frac{\sigma}{\gamma} = 0.025$, and we can use the table given in this appendix to multiply it by (0.025 - r). For any $r > 0.025$ the value of ε is negative. Thus if e = 30, the tabulated value is 25.366 which when multiplied into (0.025-r) gives the following values, which are compared with Qayum's estimate, to indicate the amount of errors involved.

r	corrected values			Qayum's estimates		
	e = 20	30	40	e = 20	30	40
.02	+ .072	+ .127	+ .200	.779	1.199	1.620
.04	- .217	- .380	- .600	.558	.873	1.188
.06	- .506	- .888	-1.400	.337	.547	.757
.08	- .795	-1.395	-2.200	.116	.220	.325

It is clear that while r defines the sign of ε, the role of e is to magnify its value.

Table I - Values of $\left\{ \frac{\varepsilon}{\frac{\sigma}{\gamma} - r} \right\}$

$\frac{\sigma}{\gamma}$	e							
	5	10	15	20	25	30	35	40
.02	3.164	6.201	9.649	13.554	17.970	22.955	28.574	34.900
.025	3.206	6.390	10.113	14.455	19.505	25.366	32.153	40.000
.03	3.248	6.584	10.601	15.420	21.180	28.047	36.212	45.900
.035	3.291	6.785	11.113	16.452	23.007	31.029	40.818	52.729
.04	3.335	6.991	11.651	17.555	25.000	34.347	46.044	60.634
.045	3.379	7.203	12.215	18.737	27.174	38.039	51.975	69.791
.05	3.423	7.422	12.807	20.000	29.545	42.146	58.709	80.000
.055	3.468	7.647	13.428	21.352	32.133	46.717	66.355	92.694
.06	3.513	7.879	14.080	22.798	34.955	51.804	75.037	106.944
.065	3.559	8.118	14.765	24.344	38.034	57.465	84.899	123.466
.07	3.606	8.364	15.482	25.999	41.392	63.765	96.099	142.623
.075	3.653	8.618	16.235	27.768	45.056	70.777	108.823	164.838
.08	3.700	8.878	17.025	29.661	49.053	78.582	123.275	190.601

For intervening values of e or $\frac{\sigma}{\gamma}$, the above table can be easily interpolated.

Appendix II - Values of m & μ

The introduction of gestation period and repayment lags changes the criterion from ε to μ, according to (41):

$$\mu = \varepsilon + m$$

where,

$$m = E^{*} - E$$

$$E^{*} = \frac{EH}{F} = (1+\frac{\sigma}{\gamma})^{e} \times (1+\frac{\sigma}{\gamma})^{h} / (1+\frac{\sigma}{\gamma})^{f} = (1+\frac{\sigma}{\gamma})^{e+h-f}$$

To overcome the difficulty of tabulating m, which has four dimensions, we tabulate

$$N = (1+\frac{\sigma}{\gamma})^{n}$$

For the given values of e, f and h we read the table by substituting n for any of them then substituting to obtain E^{*}, hence m. This can be added to ε as obtained from table I.

Table II - Values of $(1+\frac{\sigma}{\gamma})^{n}$

$\frac{\sigma}{\gamma}$	n							
	2	3	4	5	6	7	8	9
.02	1.0404	1.0612	1.0824	1.1041	1.1262	1.1487	1.1717	1.1951
.025	1.0506	1.0769	1.1038	1.1314	1.1597	1.1887	1.2184	1.2489
.03	1.0609	1.0927	1.1255	1.1593	1.1941	1.2299	1.2668	1.3048
.035	1.0712	1.1087	1.1475	1.1877	1.2293	1.2723	1.3168	1.3629
.04	1.0816	1.1249	1.1699	1.2167	1.2653	1.3159	1.3686	1.4233
.045	1.0920	1.1412	1.1925	1.2462	1.3023	1.3609	1.4221	1.4861
.05	1.1025	1.1576	1.2155	1.2763	1.3401	1.4071	1.4775	1.5513
.055	1.1130	1.1742	1.2388	1.3070	1.3788	1.4547	1.5347	1.6191
.06	1.1236	1.1910	1.2625	1.3382	1.4185	1.5036	1.5938	1.6895
.065	1.1342	1.2080	1.2865	1.3701	1.4591	1.5540	1.6550	1.7626
.07	1.1449	1.2250	1.3108	1.4026	1.5007	1.6058	1.7182	1.8385
.075	1.1556	1.2423	1.3355	1.4356	1.5433	1.6590	1.7835	1.9172
.08	1.1664	1.2597	1.3605	1.4693	1.5869	1.7138	1.8509	1.9990

Table II, cont.

$\frac{\sigma}{\gamma}$	n						
	10	15	20	25	30	35	40
.02	1.2190	1.3459	1.4859	1.6406	1.8114	1.9999	2.2080
.025	1.2801	1.4483	1.6386	1.8539	2.0976	2.3732	2.6851
.03	1.3439	1.5580	1.8061	2.0938	2.4273	2.8139	3.2620
.035	1.4106	1.6753	1.9898	2.3632	2.8064	3.3336	3.9593
.04	1.4802	1.8009	2.1911	2.6658	3.2434	3.9461	4.8010
.045	1.5530	1.9353	2.4117	3.0054	3.7453	4.6673	5.8164
.05	1.6289	2.0789	2.6533	3.3864	4.3219	5.5160	7.0400
.055	1.7081	2.2325	2.9178	3.8134	4.9839	6.5138	8.5133
.06	1.7908	2.3966	3.2071	4.2919	5.7435	7.6861	10.2857
.065	1.8771	2.5718	3.5236	4.8277	6.6144	9.0623	12.4161
.07	1.9672	2.7590	3.8697	5.4274	7.6123	10.6766	14.9745
.075	2.0610	2.9589	4.2479	6.0983	8.7550	12.5689	18.0442
.08	2.1589	3.1722	4.6610	6.8485	10.0627	14.7853	21.7245

For example, suppose e = 20, f = 0, h = 5, and $\frac{\sigma}{\gamma}$ = .025, then : E' = EH/1 = 1.6386 x 1.1314 = 1.8539, which is the value at n = 25. Hence m = .215

If r = .02, then referring to the table given in App.I :

ε = .072 & μ = .282

The advantage is increased four times due to the allowance of a free five years before repayment. However, we have to remember that this advantage takes place in year 25 and not 20. What should be compared with μ is εH , and not ε. This gives (from Table II), .072 x 1.1314 = .081, instead of .072.

If h = 10, m = 1.6386X(1.2801 - 1.0000) = .459 hence for r = .06, μ = - .047, compared with εH = -.506 x 1.2801 = - .648. According to (65) the difference $\mu - \varepsilon'$ = r h = .06x10 = + .600; which is in fact equal to $\mu - \varepsilon H = -.047 - (-.648) = .601$. Finally suppose that h = 10, but f = 5. Then $E' = 1.6386 \times \frac{1.2801}{1.1314} = 1.8539$; which could have been calculated directly as N for n = 20 + 10 - 5 = 25. Hence m = .215. For r = .06, the value of μ is - .291. This has to be compared with the value of ε' calculated as before, but corrected for the existence of f = 5. $\varepsilon' = \varepsilon H + \left(\frac{EH}{F} - EH \right)$; the correction being calculated according to the formula for m with the period of repayment = (e + h), the repayment lag = 0, and the gestation period = f. Hence $\varepsilon' = -.648 + (1.8539 - 2.0976) = -.892$. The difference $\mu - \varepsilon'$ is again to be approximately equal to rh.

It might be noticed that while h postpones the date at which repayment is concluded, it causes a drop in the loss; while an equivalent prolongation of the repayment period only magnifies the loss.