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Computer Package Programs for Mathematical programming techniques to solve linear programming models

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Contents

Preface

Planning, industry, and business firms and their operations continue to increase in size and complexity. Accordingly, planners and managers must turn to the new tools and techniques to cope with the many critical decisions which must be made.

Mathematical programming is one of the newer scientific techniques to planning and managerial decision making. The ability to make long term plans and the cost reduction are often the objective of the mathematical programming objectives.

For this purpose, we present here a computer package programs for many mathematical programming techniques used for solving Lp Models.

A brief discription of the different techniques have been presented. The computer programs have been written in BASIC language and tested on the HP-9830A calculator. Throughout the programs, many remark statements were written to describe in details the model parameters and the comming steps of the algorithms procedures.

The flow charts are introduced to assist the user to code the computer programs for his model in any computer language he wishes.

The first section introduces the Original simplex algorithm that will be used for solving a Lp models in extended form (OSA/E). In these extended forms, the simplex table includes an identity matrix, the columns of the initial basic variables.

Section 2 presents the original simplex algorithm but in compact form since we note that the columns of unit vectors included in the previous algorithm offer no significant information other than to designate the initial basic-variables and it will be convenient to omit these columns and place instead the subscripts of the basic variables in only one column to the left of the simplex table (OSA/C).

The third section introduces the revised simplex algorithm(RSA) although it may appear to be unjustified since the OSA is much more simple from the theoritical point of view and numerically both algorithms appear to be identical but this algorithm is very useful specially in the economic activities.

Section 4 presents the dual simplex algorithm in extended form (DSA/E) to help the user to solve his model without using artificial variables. It is very useful for example in games and strategies used by players to find their optimal strategies and game values.

The dual simplex algorithm in compact form (DSA/C) has been introduced in section five.

By a combination of the OSA and the DSA, artifical variables may be avoided completely which reduces the size of the simplex table and makes a marked reduction in the number of iterations necessary to optimize the objective function of the model. For this purpose, the primal-dual algorithm in compact form (PDA/C) has been included in the last section.

Section: 1

The Original Simplex Algorithm for Solving Lp Models in Extended form (OSA/Extended):

Introduction:

In developing the simplex algorithm, G.Dantzig made use of the classical Gauss-Jordan elimination method which is familier to anyone solved a system of linear equations. The key idea is to take a multiple of one equation and add it to or subtract it form another equation in order to eliminate one of the unknowns from the second equation hoping to change the original system to an equivalent one easier to solve for the remaining unknowns.

The simplex algorithm is an efficient mehtod which is routinely used to solve the Lp models on today's computers.

Outline of the Simplex Algorithm:

Our Lp model has the following general standard form:

$$\text{Max}^{*}. \qquad Z = \sum_{j=1}^{n} c_j x_j$$

S.to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \,,\; i=1,2,\ldots,m$$

$$\& \; b_i > o$$

$$\& \qquad x_j > 0$$

ii, After converting this model to equality form, the initial simplex table takes the form:

eq.no:	Basic Variables	Coeffs of $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}$	r.h.s.
0	Z	$-c_1 \; -c_2 \ldots -c_n \; 0 \; 0 \ldots 0$	0
1	X_{n+1}	$a_{11} \; a_{12} \ldots a_{1n} \; 1 \; 0 \ldots 0$	b_1
2	X_{n+2}	$a_{21} \; a_{22} \ldots a_{2n} \; 0 \; 1 \ldots 0$	b_2
.	.	. .	.
.	.	. .	.
.	.	. .	.
m	X_{n+m}	$a_{mi} \; a_{m2} \ldots a_{mn} \; 0 \; 0 \ldots 1$	b_m

iii, now the simplex algorithm consists of the following 3 steps:

I- <u>The Initialization Step(Iteration no.o):</u>

It starts at a corner-point feasible solution (origin). This is equivalent to selecting the original variables (x_1, x_2, x_n) to be the initial non-basic variables (equal to zero), and the slack variables $(x_{n+1}, x_{n+2}, \ldots x_{n+m})$ to be the initial Basic variables (equal to r.h.s).

II. <u>The Stopping Rule:</u>

The algorithm stops when the current corner-point feasible solution is better than all its adjacent corner point feasible solutions. In this case the solution is optimal. The current basic feasible solution is optimal if and only if every coefficient in eq o is non-negative. If it is, stop; otherwise, go to the following iterative step to obtain a better basic feasible solution which involves changing one of the non-basic variable with one of the basic variables and vice-versa and then solving for the new solution.

III- The Iterative Step (Successive Iterations):

Move to a better adjacent corner-point feasible solution. This involves replacing one Non-basic variable (called the entering basic variable (EBV) by one of the old basic variables (called the leaving basic variable (LBV). This can be done through the following 3 parts.

1. Determine the EBV: Select the variable (automatically a Non-basic variable) with the greatest-ve coeff., in eq.0. Let it be X_{J1}; J1 refers to the column below this coeff. and called the Pivot-column.

2. Determine the LBV: The leaving basic variable X_{I1} can be determined such that:

$$\frac{b'_{I1}}{a'_{I1,J1}} = \min_{i} \left(\frac{b'_i}{a'_{i,J1}} \right) \quad a'_{i,J1} > 0$$

I1 refers to the pivot-row no.,

$a'_{I1,J1}$ is called the pivot-element.

3. Determine the New basic feasible solution by constructing a new simplex table as follows:

i - New pivot-row $= \dfrac{\text{Old Pivot row}}{\text{Pivot element}}$

ii- Any other row = Old row - "Pivot-Column coeff" X new pivot-raw

; "pivot-column coeff" is the element in this row that is in the pivot-column.

(note that the pivot column except the pivot element became zero after this pivoting operation)

iii- The subscript of the leaving basic variable is replaced by the subscript of the Entering basic variable

Illustrative Example:

(This ex is due to F.S.Hillier (1))

$$\text{Max.} \quad = 3x_1 + 5x_2$$

$$\begin{aligned} \text{S.to:} \quad x_1 \qquad & \leq 4 \\ 2x_2 & \leq 12 \\ 3x_1 + 2x_2 & \leq 18 \end{aligned}$$

&

$$x_1, x_2 \geq 0$$

The Equality form of this ex is:

max. Z

S.To:

$$
\begin{aligned}
Z \; -3x_1 \; -5x_2 \qquad\qquad\qquad & = 0 \\
x_1 \qquad +x_3 \qquad\qquad & = 4 \\
2x_2 \qquad +x_4 \qquad & = 12 \\
3x_1 + 2x_2 \qquad\qquad +x_5 & = 18
\end{aligned}
$$

&

$$x_j \geq 0 \;\forall\; j = 1,2,3,4,5$$

The Initial Simplex Tableau is:

eq.no:	Basic Variables	Coeffs of					r.h.s.
		x_1	x_2	x_3	x_4	x_5	
0	Z	-3	-5	0	0	0	0
1	X_3	1	0	1	0	0	4
2	X_4	0	2	0	1	0	12
3	X_5	3	2	0	0	1	18

II. The Initialization Step (Iteration No. 0)

From the above table, the Initial basic feasible solution is:

(0,0,4,12,18) & Z = 0

Now, go to the stopping rule to determine if this solution is optimal or not?

II. <u>The Stopping Rule:</u>

The ex has 2 -ve coeffs in eq o, -3 for x_1 & -5 for x_2, which means that this current solution is not optimal; so go to the iterative step.

III. <u>The Iterative Step:</u>

1. <u>To determine the EBV;</u> the largest -ve coeff in eq o is -5 for x_2, so x_2 is the EBV and J1, the pivot-column no., is equal 2.

2. <u>To determine the LBV;</u> it can be seen from the table that the LBV is x_4 which is associated with the minimum ratio indicated in the table, and there fore I1 is equal 2 also.

3. The new basic feasible solution can be determined by constructing a new simplex table as follows:

i,

$$\text{the new pivot-row} = \frac{\text{the old pivot-now}}{\text{the pivot-element}}$$

so, the table for this ex at this point has the appearance shown:

eq.no.	Basic Variable	Coeffs of x_1	x_2	x_3	x_4	x_5	r.h.s.	
0	Z	-3	-5	0	0	0	0	
1	X_3	1	0	1	0	0	4	
2	X_4	0	2	0	1	0	12	
3	X_5	3	2	0	0	1	18	
0	Z							
1	X_3							
2	X_2	0	1	0	½	0	6	←
3	X_5							

ii. To eliminate the new basic variable from the other equations, every row in the table, except the pivot-row, is changed for the new table by using the formula:-

new row = old row-"Pivot-Column coeff" X new pivot row,

for ex, row o becomes:

$$
\begin{array}{rrrrrrr}
 & (-3 & -5 & 0 & 0 & 0 & 0\ ) \\
-(-5) & (\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6\ ) \\
 = & (-3 & 0 & 0 & \frac{5}{2} & 0 & 30)
\end{array}
$$

and so on for the remaining rows, i.e, the new simplex table becomes:

Iteration no:	eq.no.	Basic Variable	Coeffs of x_1	x_2	x_3	x_4	x_5	r.h.s.	
0	0	Z	-3	-5	0	0	0	0	
	1	X_3	1	0	1	0	0	4	
	2	X_4	0	2	0	1	0	12	
	3	X_5	3	2	0	0	1	18	
1	0	Z	(-3)	0	0	$\frac{5}{2}$	0	30	
	1	X_3	1	0	1	0	0	4	$\frac{4}{1} = 4$
	2	X_2	0	1	0	$\frac{1}{2}$	0	6	
	3	X_5	3	0	0	1	1	6	$\frac{6}{3} = 2$ ←

The solution at this iteration is:

$$(0,6,4,0,6) \quad \& \quad Z = 30$$

Now, we have to go to the stopping rule to check whether this solution is optimal or not?

<u>The stopping rule:</u> Since the new eq.0 still has a -ve ceeff (-3 for x_1), this solution is not optimal and we return to the Iterative step to determine the next basic feasible solution.

Following the instructions of the Iterative step, we find that x_1 is the EBV and X_5 is the LBV. Then, the new simplex table for this iteration becomes:

Iteration	eq.no.	Basic Variables	Coeffs of x_1	x_2	x_3	x_4	x_5	r.h.s.
	0	Z	-3	-5	0	0	0	0
0	1	X_3	1	0	1	0	0	4
	2	X_4	0	2	0	1	0	12
	3	X_5	3	2	0	0	1	18
1	0	Z	-3	0	0	$\frac{5}{2}$	0	30
1	1	X_3	1	0	1	0	0	4
	2	X_2	0	1	0	$\frac{1}{2}$	0	6
	3	X_5	3	0	0	-1	1	6
	0	Z	0	0	0	$\frac{3}{2}$	1	36
2	1	X_3	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	2	X_2	0	1	0	$\frac{1}{2}$	0	6
	3	X_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

The new basic feasible solution at this iteration is:

(2, 6, 2, 0, 0) & Z = 36

Going to the stopping rule, we find that the solution is optimal because none of the coefficients in eq o is -ve, so the algorithm is finished and the optimal solution for the problem is:

$$\left.\begin{array}{l} X_1 = 2 \\ X_2 = 6 \end{array}\right\} \text{ the ordinary variables}$$

& $Z = 36$.

The Breaking in the Simplex Algorithm:

1. Case of Degeneracy:

One difficulty arises when two or more basic variables tie for being the LBV in the iterative step. Special procedures have been constructed for breaking all ties in a way to make cycling impossible. One procedure, for ex, consists of changing the r.h.s. column slightly so that ties do not occur. The program to be presented here resolves such problem by using the charnes-cooper method (4). Anyhow, the rest of programs written for other algorithms do not bother with the possibility of cycling because cycling usually donot occur in practical situations.

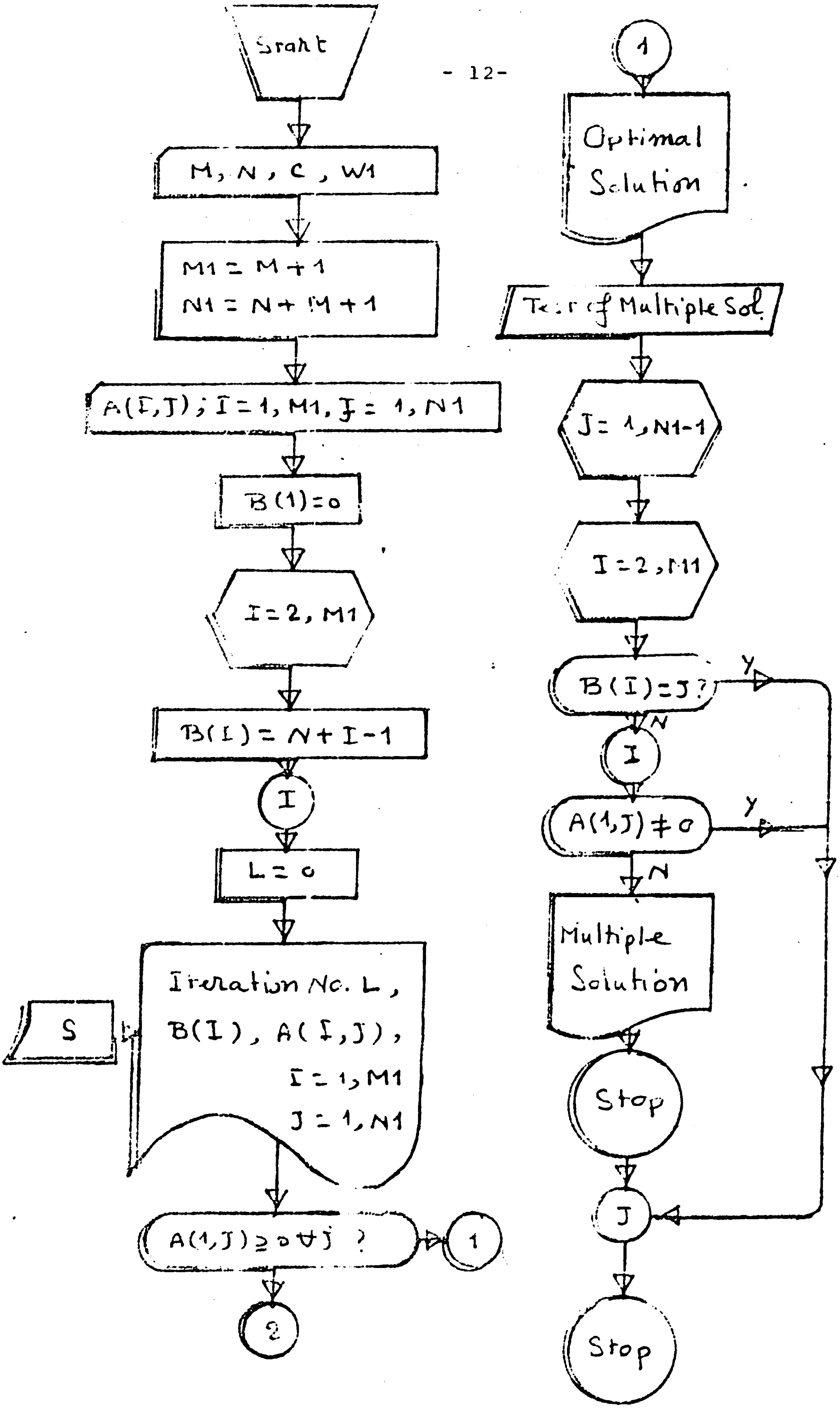
2. Case of No Leaving Basic Variable (Unbound Solution):

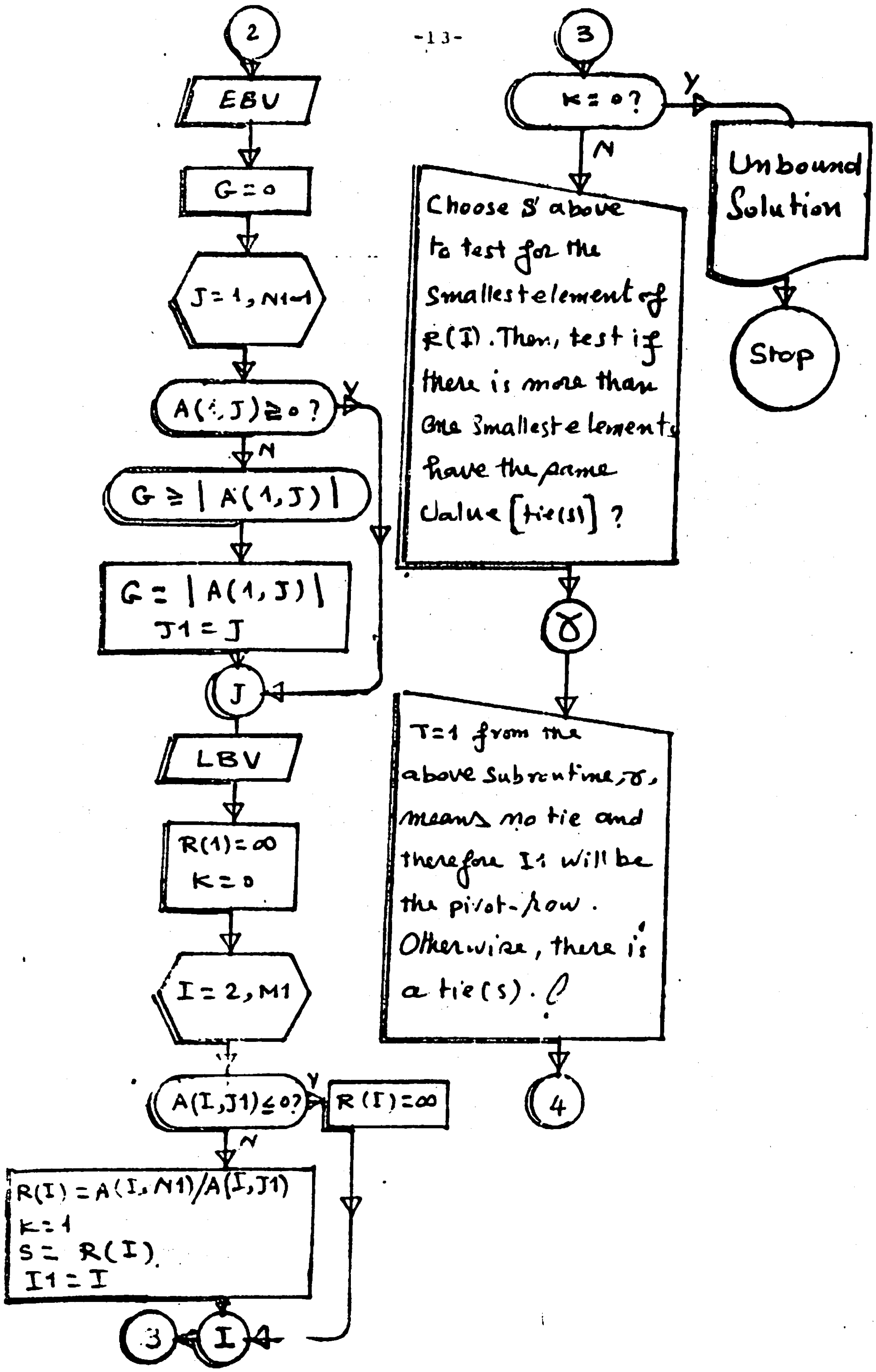
This would occur if the EBV can be increased indefinitely without giving zero to any of the current basic variables. This means that every element in the pivot-column (excluding eq 0) is either negative or zero. In such cases, the algorithm would stop with the message that the solution is unbounded.

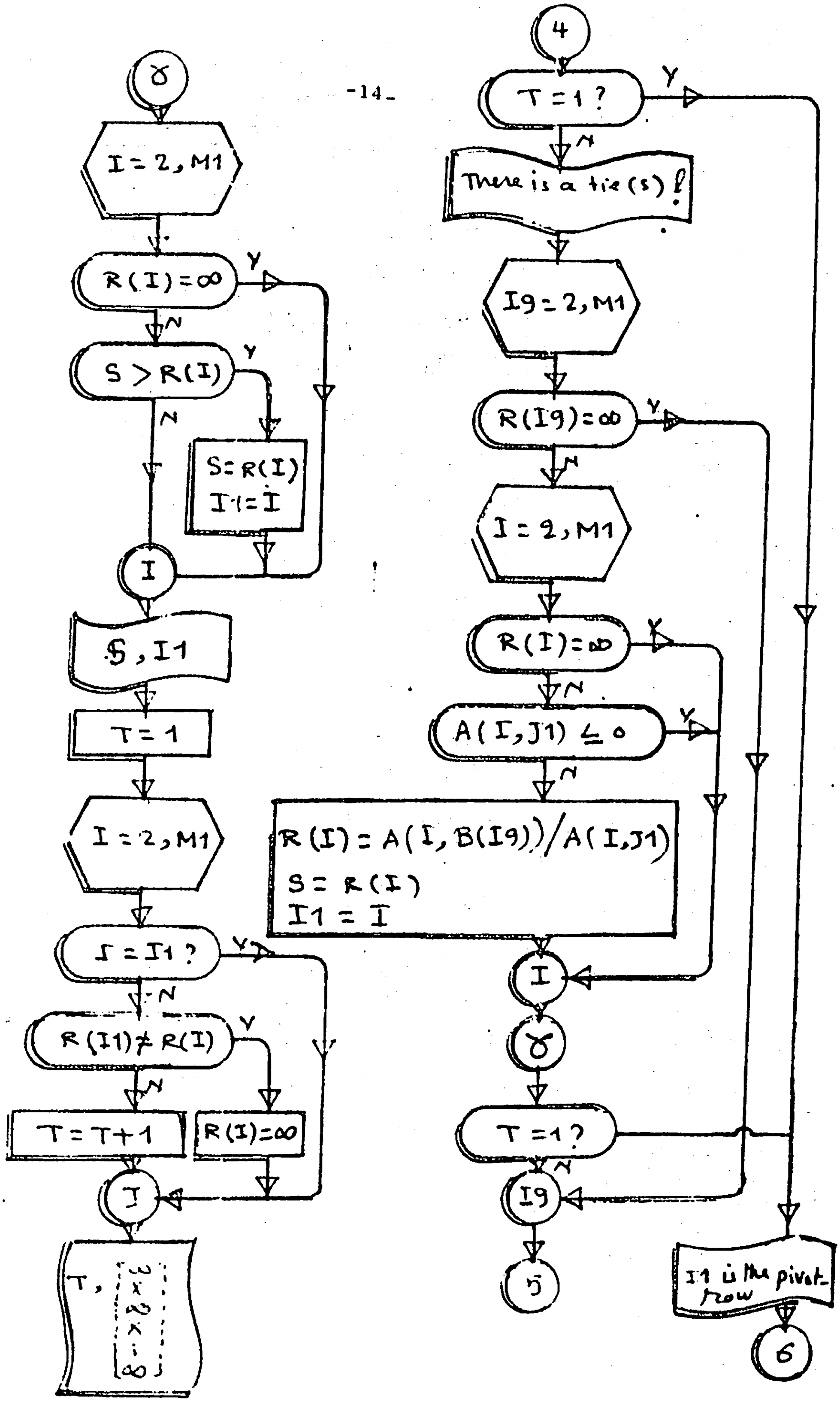
3. Case of Multiple Optimal Solutions:

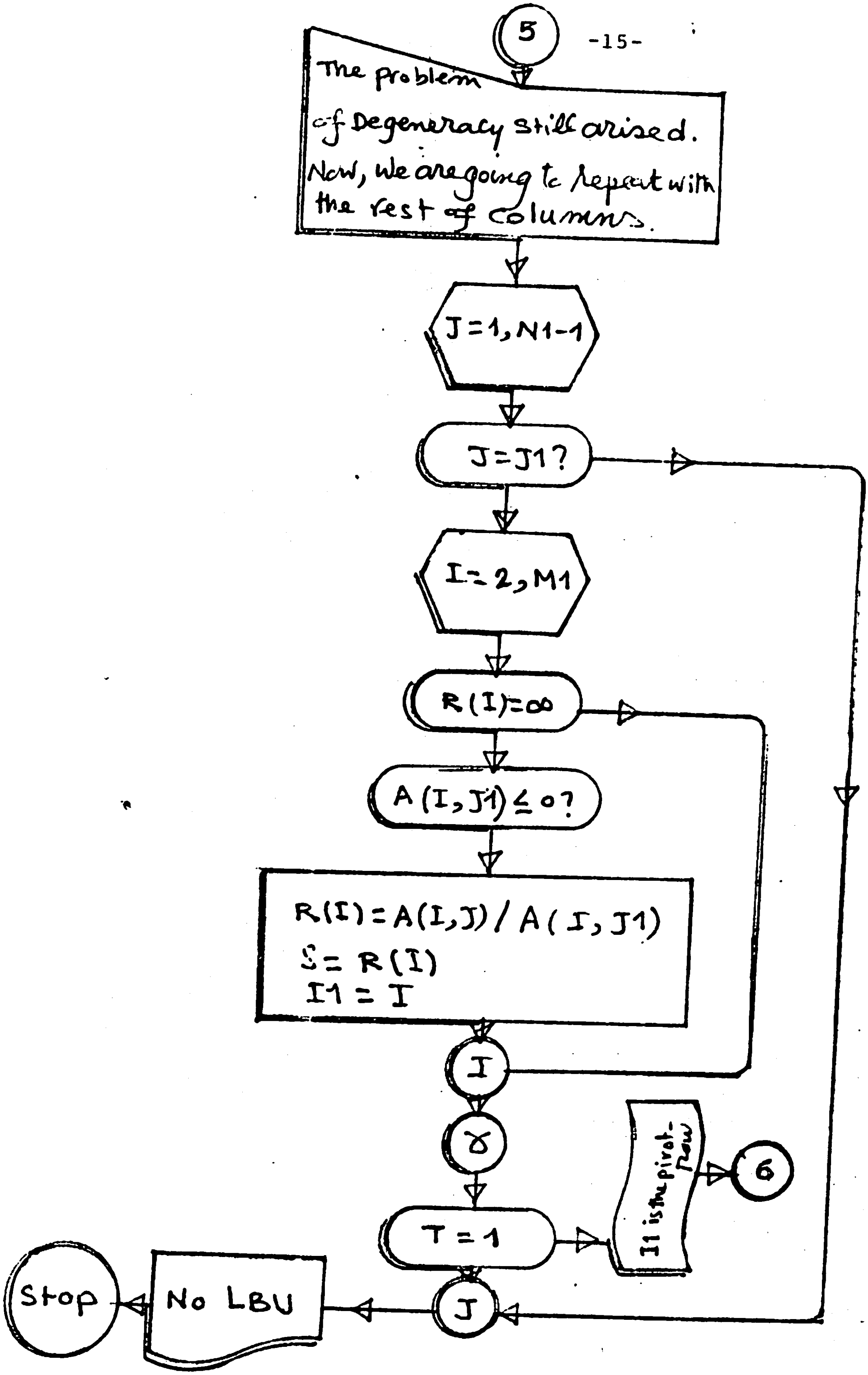
Somethimes, the problem has more than one optimal basic feasible solutions since at least one of the Non-basic variables has a coefficient of zero in the final (optimal) table, so increasing such variables not change the value of Z. Therefore, other optimal solutions usually can be identified by performing additional iterations of the simplex algorithm, each time choosing a non-basic variable with a zero coefficient as the EBV.

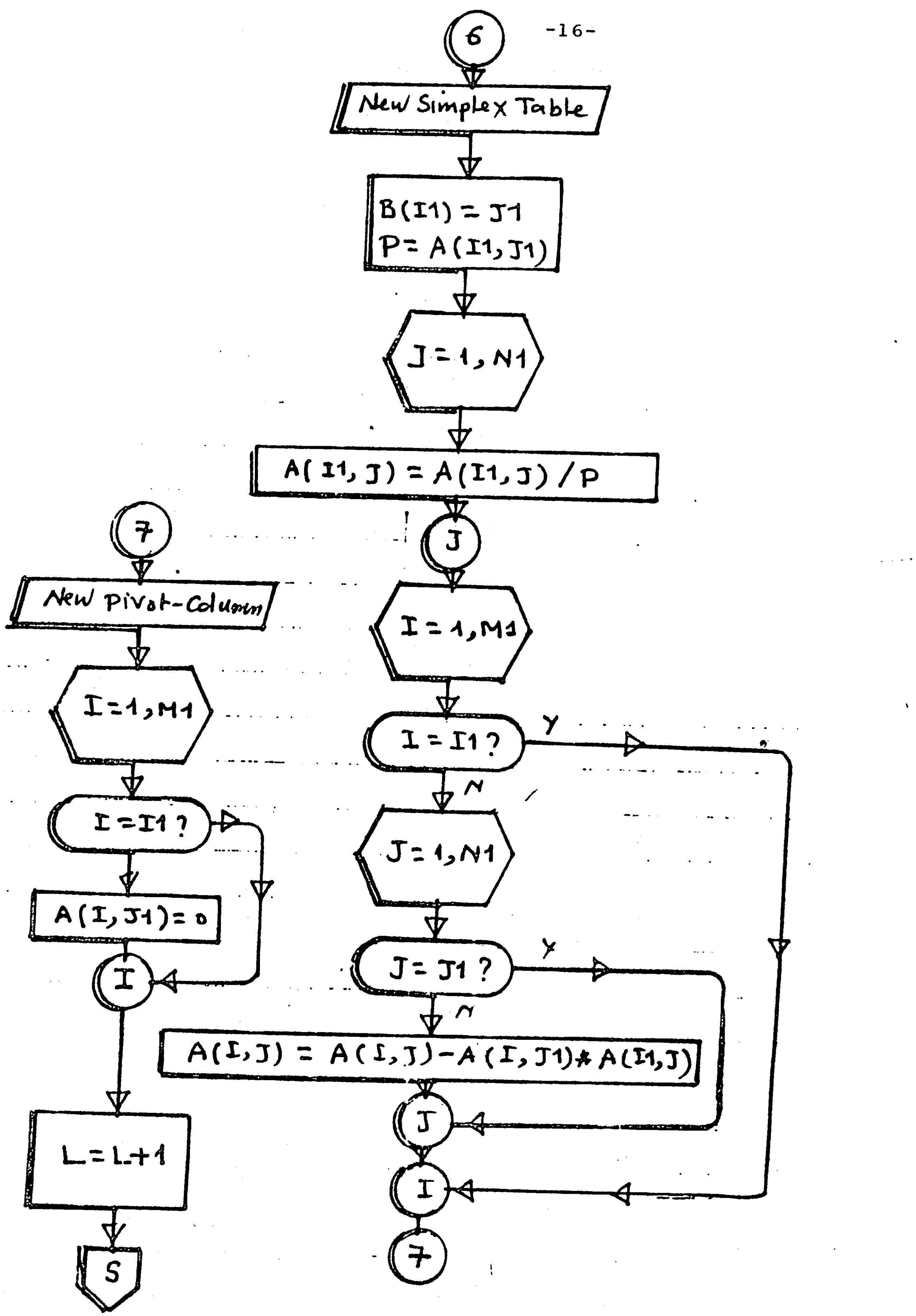
Anyhow, the algorithm breaks the tie among these optimal solutions by stopping with the first optimal (basic-feasible) solution it finds with a message of multiple solutions is written.


Start
M, N, C, W1
M1 = M + 1
N1 = N + M + 1
A(I,J); I = 1, M1, J = 1, N1
B(1)=0
I=2, M1
B(I) = N + I - 1
I
L = 0
Iteration No. L, B(I), A(I,J), I = 1, M1 J = 1, N1
S
A(1,J) ≥ 0 ∀ J ?
1
2
1
Optimal Solution
Test of Multiple Sol.
J= 1, N1-1
I = 2, M1
B(I) = J ?
Y
N
I
A(1,J) ≠ 0
Y
N
Multiple Solution
Stop
J
Stop


2
EBU
G = 0
J = 1, N1-1
A(1,J) ≥ 0 ?
Y
N
G ≥ | A(1,J) |
G = | A(1,J) |
J1 = J
J
LBV
R(1) = ∞
K = 0
I = 2, M1
A(I,J1) ≤ 0 ?
Y
R(I) = ∞
N
R(I) = A(I, N1)/A(I,J1)
K = 1
S = R(I)
I1 = I
3
I
3
K = 0 ?
Y
N
Unbound Solution
Stop
Choose S above to test for the smallest element of R(I). Then, test if there is more than one smallest elements have the same value [tie(s)] ?
γ
T=1 from the above subroutine, γ, means no tie and therefore I1 will be the pivot-row. Otherwise, there is a tie(s).
4


γ
I = 2, M1
R(I) = ∞
Y
N
S > R(I)
Y
N
S = R(I)
I1 = I
I
S, I1
T = 1
I = 2, M1
I = I1 ?
Y
N
R(I1) ≠ R(I)
Y
N
T = T+1
R(I) = ∞
I
T, ∞ x ∞ x : ∞
4
T = 1 ?
Y
N
There is a tie(s) !
I9 = 2, M1
R(I9) = ∞
Y
N
I = 2, M1
R(I) = ∞
Y
N
A(I, J1) ≤ 0
Y
N
R(I) = A(I, B(I9))/A(I, J1)
S = R(I)
I1 = I
I
γ
T = 1 ?
N
I9
5
I1 is the pivot row
6


5
The problem of Degeneracy still arised. Now, we are going to repeat with the rest of columns.
J=1, N1-1
J=J1?
I=2, M1
R(I)=∞
A(I,J1)≤0?
R(I)=A(I,J)/A(I,J1)
S=R(I)
I1=I
I
γ
T=1
I1 is the pivot-row
6
J
No LBU
Stop


6
New Simplex Table
B(I1) = J1
P = A(I1,J1)
J = 1, N1
A(I1, J) = A(I1,J) / P
J
I = 1, M1
I = I1 ?
Y
N
J = 1, N1
J = J1 ?
Y
N
A(I,J) = A(I,J)-A(I,J1)*A(I1,J)
J
I
7
7
New Pivot-Column
I = 1, M1
I = I1 ?
A(I, J1) = 0
I
L = L+1
S

```
10 REM A COMPLETE PROGRAM TO SOLVE A LP OF THE TYPE <=.
20 REM. THIS PROG. RESOLVES DEGENERACY WHEN THERE IS A TIE(S) FOR THE L.
30 REM.. L.B.V..
40 REM. PROGRAMMED BY THE STUDENTS STUDYING THE COORSE 1416 .
50 REM. SUPERVISED BY DR. ABDALLA EL-DAOUSHY..
60 REM. APRIL. 1981 ..
70 REM.......................................
80 DIM B(21),A(21,31),R(21)
81 OPEN "PR:",1,1
90 PRINT "ENTER M,N,CASE NO."
100 INPUT M,N,C
110 M1=M+1
120 N1=N+M+1
130 PRINT "ENTER THE INITIAL SIMPLEX TABLEAU"
140 FOR I=1 TO M1
150 FOR J=1 TO N1
160 INPUT A(I,J)
170 NEXT J
180 NEXT I
190 PRINT ON(1) "CASE NO.",C:PRINT ON(1)"================="
210 PRINT ON(1):PRINT ON(1)
230 REM. CREATING THE BASIS ......
240 B(1)=0
250 FOR I=2 TO M1
260 B(I)=N+I-1
270 NEXT I
280 L=0
290 REM. L DENOTE THE ITERATION NO. L=0 FOR THE I.S.TABLAU..
300 REM.
310 REM. THE OUTPUT OF THE ITERATION NO. AND THE SIMPLEX TAELEAU..
320 PRINT ON(1) "ITERATION NO.",L:PRINT ON(1) "-----------------"
340 PRINT ON(1)
341 PRINT ON(1) TAB(14)," ";
350 FOR J=1 TO N1-1
360 IF J>10 THEN 400
370 PRINT ON(1) J,
380 GO TO 410
400 PRINT ON(1)
410 NEXT J
420 PRINT ON(1)
430 FOR I=1 TO M1
450 PRINT ON(1) B(I),
460 FOR J=1 TO N1
470 IF J>10 THEN 510
480 PRINT ON(1) A(I,J),
490 GO TO 520
510 PRINT ON(1)
520 NEXT J
530 PRINT ON(1)
540 NEXT I
```

```
541 PRINT ON(1):PRINT ON(1)
550 REM.
560 REM TEST FOR OPTIMALITY .
570 REM.
580 FOR J=1 TO N1-1
590 IF A(1,J)<0 THEN 910
600 NEXT J
610 REM.
620 REM. OPTIMAL SOLUTION ....
630 REM.
640 PRINT ON(1)
650 PRINT ON(1)
660 PRINT ON(1) "        Z=   ",A(1,N1)
680 FOR I=2 TO M1
690 PRINT ON(1) "       X(",B(I),")= ",A(I,N1)
710 NEXT I
720 PRINT ON(1)
730 PRINT ON(1) "                    WHICH IS THE OPTIMAL SOLUTION"
750 REM.
760 REM. TEST FOR MULTIPLE SOLUTION .....
770 REM.
780 FOR J=1 TO N1-1
790 FOR I=2 TO M1
800 IF B(I)=J THEN 860
810 NEXT I
820 IF A(1,J)<>0 THEN 860
830 PRINT ON(1) TAB(46),"MULTIPLE SOLUTION"
850 STOP
860 NEXT J
870 STOP
880 REM.
890 REM. TO FIND THE E.B.V.....
900 REM.
910 G=0
920 FOR J=1 TO N1-1
930 IF A(1,J)>=0 THEN 970
940 IF G>=ABS(A(1,J)) THEN 970
950 G=ABS(A(1,J))
960 J1=J
970 NEXT J
980 REM.
990 REM. J1 IS THE PIVOT-COLUMN NO. AND X(J1) IS THE EBV...
1000 REM.
1010 GOSUB 1320
1020 REM.
1030 REM. THE NEW SIMPLEX TABLEAU....
1040 REM.
1050 REM. THE NEW BASIC VARIABLE & THE NEW ROWS OF THE TABLEAU.
1060 REM.
1070 P=A(I1,J1)
```

```
1080 B(I1)=J1
1090 FOR J=1 TO N1
1100 A(I1,J)=A(I1,J)/P
1110 NEXT J
1120 FOR I=1 TO M1
1130 IF I=I1 THEN  1180
1140 FOR J=1 TO N1
1150 IF J=J1 THEN 1170
1160 A(I,J)=A(I,J)-A(I,J1)*A(I1,J)
1170 NEXT J
1180 NEXT I
1190 REM... NEW PIVOT-COLUM. .
1200 FOR I=1 TO M1
1210 IF I=I1 THEN 1230
1220 A(I,J1)=0
1230 NEXT I
1240 REM.....
1250 REM
1260 L=L+1
1270 GO TO 320
1280 REM
1290 REM
1300 REM.
1310 REM. SUBROUTINE TO DETERMINE THE LEAVING BASIC VARIABLE ...
1320 R(1)=1E+38
1330 K=0
1340 FOR I=2 TO M1
1350 IF A(I,J1)<=0 THEN 1410
1360 R(I)=A(I,N1)/A(I,J1)
1370 K=1
1380 S=R(I)
1390 I1=1
1400 GO TO 1420
1410 R(I)=1E+38
1420 NEXT I
1430 IF K<>0 THEN 1480
1440 PRINT ON(1) "UNBOUND SOLUTION ......
1460 STOP
1470 REM. CHOOSE THE ABOVE S TO TEST FOR THE SMALLEST +VE RATIO. I.E.,
1480 FOR I=2 TO M1
1490 IF R(I)=1E+38 THEN 1530
1500 IF S<R(I) THEN 1530
1510 S=R(I)
1520 I1=I
1530 NEXT I
1540 REM. TEST IF THERE IS ONLY ONE SMALLEST VALUE.
1550 REM. SUPPOSE R(I1) IS THE ONLY SAMLEST+VE VALUE IN THIS COL,S RATIOS
1560 I=1
1570 FOR I=2 TO M1
1580 IF I=I1 THEN 1640
```

```
1590 IF R(I1)<>R(I) THEN 1630
1600 REM. THERE IS A TIE
1610 T=T+1
1620 GO TO 1640
1630 R(I)=1E+38
1640 NEXT I
1650 RETURN
1660 REM. . .
1670 IF T=1 THEN 2050
1680 REM. THERE IS A TIE(S). .
1690 FOR I9=2 TO M1
1700 IF R(I9)=1E+38 THEN 1830
1710 FOR I=2 TO M1
1720 IF R(I)=1E+38 THEN 1780
1730 I1=I
1740 IF A(I,J1) <=0 THEN 1780
1750 R(I)=A(I,B(I9))/ A(1,J1)
1760 S=R(I)
1770 I1=I
1780 NEXT I
1790 GOSUB 1480
1800 REM. . .
1810 IF T=1 THEN 2050
1820 REM. . .
1830 NEXT I9
1840 REM. THE PROBLEM OF DEGENERACY STILL ARISED.
1850 REM. WE ARE GOING TO REPEAT WITH ALL COLUMNS.
1860 FOR J=1 TO N1-1
1870 IF J=J1 THEN 1990
1880 FOR I=2 TO M1
1890 IF R(I)=1E+38 THEN 1940
1900 IF A(I,J1) <=0 THEN 1940
1910 R(I)=A(I,J)/A(1,J1)
1920 S=R(I)
1930 I1=I
1940 NEXT I
1950 REM. . .
1960 GOSUB 1480
1970 REM. . .
1980 IF T=1 THEN 2050
1990 NEXT J
2000 PRINT ON(1)
2010 PRINT ON(1) " NO LEAVING BASIC VARIABLE. . . . . ."
2030 STOP
2040 REM. . .
2050 RETURN
2060 END
```

CASE NO. 1

ITERATION NO. 0

	1	2	3	4	5	
0	-3	-5	0	0	0	0
3	1	0	1	0	0	4
4	0	2	0	1	0	12
5	3	2	0	0	1	18

ITERATION NO. 1

	1	2	3	4	5	
0	-3	0	0	2. 5	0	30
3	1	0	1	0	0	4
2	0	1	0	. 5	0	6
5	3	0	0	-1	1	6

ITERATION NO. 2

	1	2	3	4	5	
0	0	0	0	1. 5	1	36
3	0	0	1	. 33333333	-. 33333333	2
2	0	1	0	. 5	0	6
1	1	0	0	-. 33333333	. 33333333	2

Z= 36

X(3)= 2

X(2)= 6

X(1)= 2

WHICH IS THE OPTIMAL SOLUTION

CASE NO. 2

ITERATION NO. 0

	1	2	3	4	5	
0	-3	-2	0	0	0	0
3	1	0	1	0	0	4
4	0	1	0	1	0	6
5	3	2	0	0	1	18

ITERATION NO. 1

	1	2	3	4	5	
0	0	-2	3	0	0	12
1	1	0	1	0	0	4
4	0	1	0	1	0	6
5	0	2	-3	0	1	6

ITERATION NO. 2

	1	2	3	4	5	
0	0	0	0	0	1	18
1	1	0	1	0	0	4
4	0	0	1.5	1	-.5	3
2	0	1	-1.5	0	.5	3

Z= 18

X(1)= 4

X(4)= 3

X(2)= 3

WHICH IS THE OPTIMAL SOLUTION

MULTIPLE SOLUTION

Section 2:

The Original Simplex Algorithm for Solving Lp Models in Compact form (OSA/Compact)

Consider again the following illustrative example:

$$\text{Max.} \quad Z = 3x_1 + 5x_2$$

s.t.o.

$$x_1 \leq 4$$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

&

$$x_1, x_2 \geq 0$$

To save computer storage,the unit vectors which correspond to the slack variables in the Extended table will be omitted and instead, we place the subscripts of the basic variables in one vector to the left of the table and the ordinary columns will be labeled at the top of the table with the subscripts of the corresponding Non-basic variables in an.other vector. For ex, the Initital Compact Simplex Table for the above ex will appear as follows:

	1	2		
Z	-3	(-5)	0	
3	1	0	4	
4	0	2	12	$\frac{12}{2} = 6$ ← min
5	3	2	18	$\frac{18}{2} = 9$

To determine the next table directly from the proceeding one, the following 6 steps constitute the original simplex Algorithm in compact form:

1- Determine the pivot-element in the same way as for the original simplex Algorithm in Extended form.

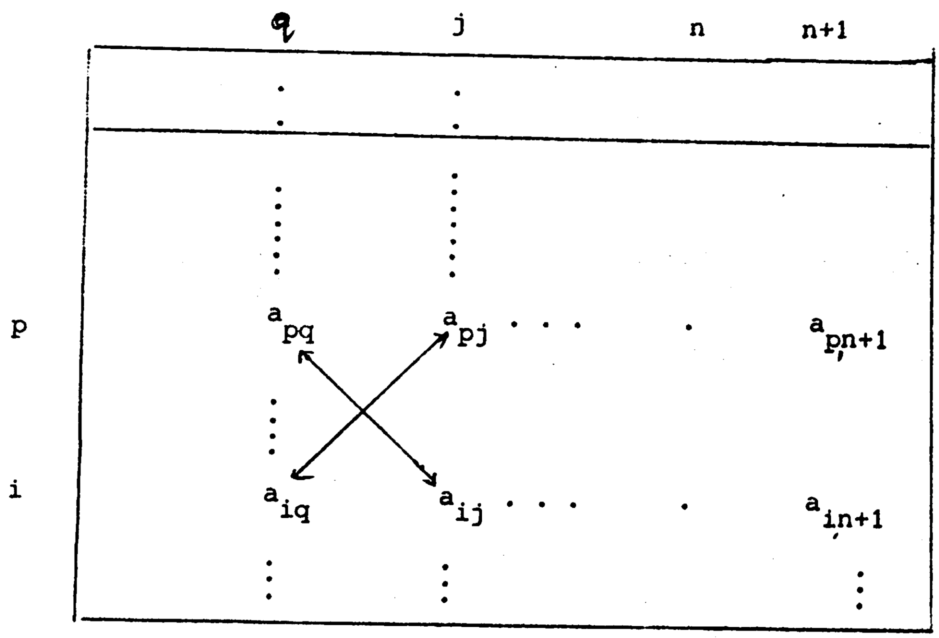
2- New pivot-row (except the pivot-element itself)

$$= \frac{\text{Old pivot - row}}{\text{Pivot - element}}$$

3- New pivot-column (except the pivot - element)

$$\frac{\text{Old pivot - column}}{\text{Pivot - element}}$$

4- For the remaining entries of the table (except the elements in both the pivot-row and the pivote column):



⋮	⋮	⋮
$1/a_{pq}$ . . .	a_{pj}/a_{pq}	a_{pn+1}/a_{pq}
⋮	⋮	⋮
$-a_{iq}/a_{pq}$	$a_{ij} - \dfrac{a_{iq}\,a_{pj}}{a_{pq}}$ . . .	$a_{i,n+1} - \dfrac{a_{iq}\,a_{pn+1}}{a_{pq}}$
⋮	⋮	⋮

$$\underset{new}{(a_{ij})} = \underset{old}{(a_{ij})} - \frac{a_{iq} \cdot a_{pj}}{a_{pq}}$$

New row = old row - pivot-column coeff" x new pivot-row;

"Pivot-column coeff" is the element in this row that is in the pivot - column.

5- New pivot - element = $\dfrac{1}{\text{Old pivot - element}}$

6- Inter-change the subscripts of the LBV and the EBV.

Let us apply this algorithm for the example:

K (j)

Iteration 0

	1	2	
Z	-1	-5	0
3	1	0	4
4	0	2	12
5	3	2	18

L (I)

Iteration 1

	1	4	
Z	-3	5/2	30
3	1	0	4
2	0	(1/2)	6 ←
5	3	-1	6

Iteration 2

	5	4	
Z	1	3/2	36
3	-1/3	1/3	2
2	0	1/2	6
1	1/3	-1/3	2

A proof that the 6 steps of this algorithm hold is accomplished by compacting the tables of the Original simplex Algorithm in extended form and then comparing these tables with those of the algorithm in compact form. For example, let the extended table takes the following form where x_s is the EBV, x_u is the LBV, and a_{pq} is the pivot - element:

eq.no.	B.Vs	coeffs of							r.h.s
		...	x_s	...	x_j	...	x_u	...	
	⋮								
p	x_u		a_{pq}		a_{pj}		1		b_p
	⋮								
i			a_{iq}		a_{ij}		0		b_i
.			.		.		.		.

(1)

The pivoting operation on this extended table produces the following table where X_s in now a Basic variable:

eq. no.	B. Vs	coeffs of			r.h.s.
		... x_s ...	x_j ...	x_u ...	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p	x_s	1	$\frac{a_{pj}}{a_{pq}}$	$\frac{1}{a_{pq}}$	$\frac{b_p}{a_{pq}}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i		0	@	£	$
	.	.	.	.	.

(2)

$$@ = a_{ij} - a_{iq} \frac{a_{pj}}{a_{pq}}$$

$$\$ = b_i - a_{ip} \frac{b_p}{a_{pq}}$$

$$£ = - \frac{a_{iq}}{a_{pq}}$$

Now, compact the extended table (1) by dropping all columns of unit vectors which associated with the slack-variables, we have:

eq. no.	B.Vs	coeff. of					r.h.s.
		...	x_s	...	x_j	...	
	⋮						
p	x_u		a_{pq}		a_{pj}		b_p
i	a_{iq}		a_{iq}		a_{ij}		b_i

(3)

Also, compact table (2) and reorder column x_s with that of x_u and drop all other columns of unit vectors, we have:

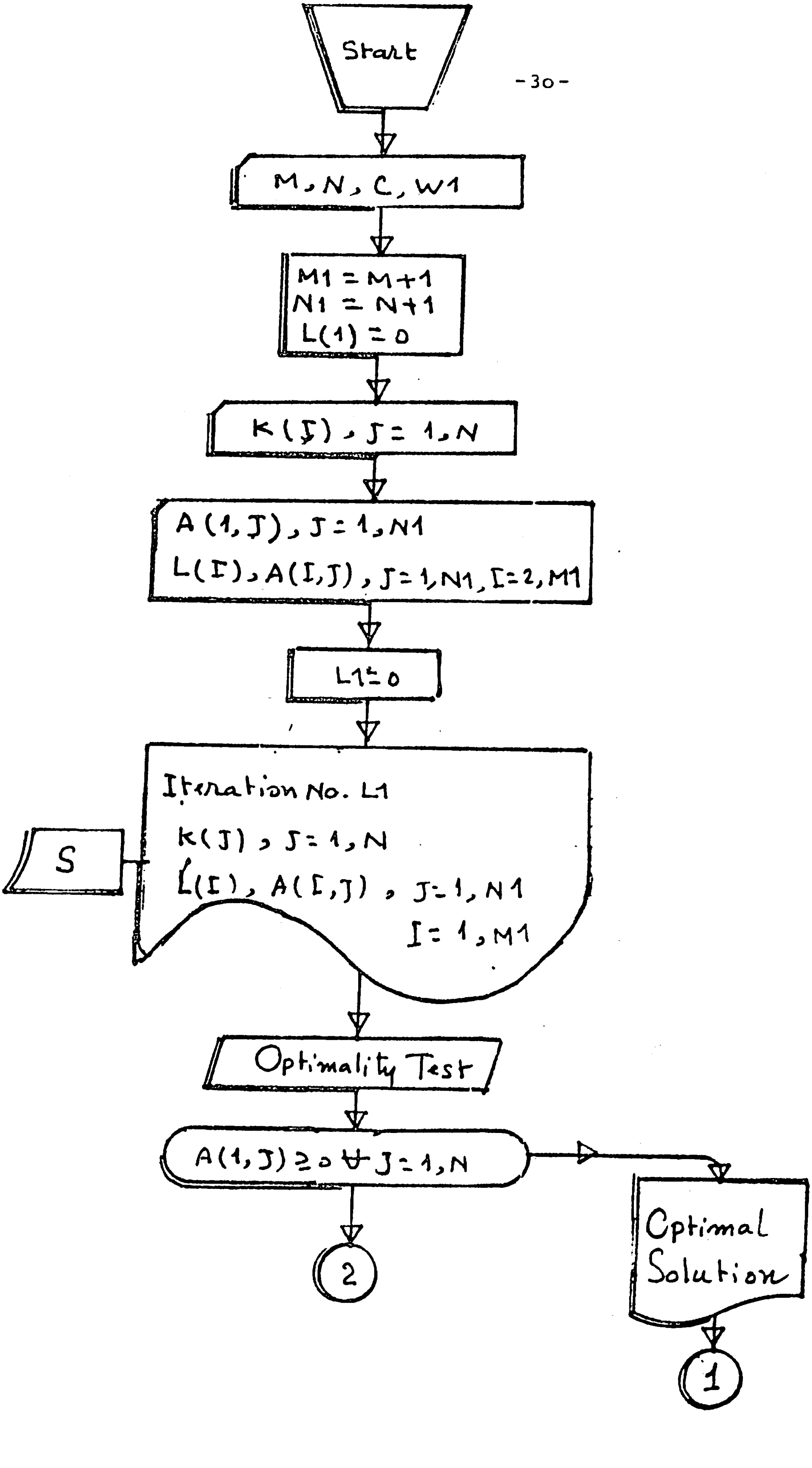
eq. no.	B.Vs	coeff. of								r.h.s
		...	x_s	...	x_j	...	x_u	...		
	⋮									
p	x_u		a_{pq}		a_{pj}		1			b_p
i			a_{iq}		a_{ij}		0			b_i

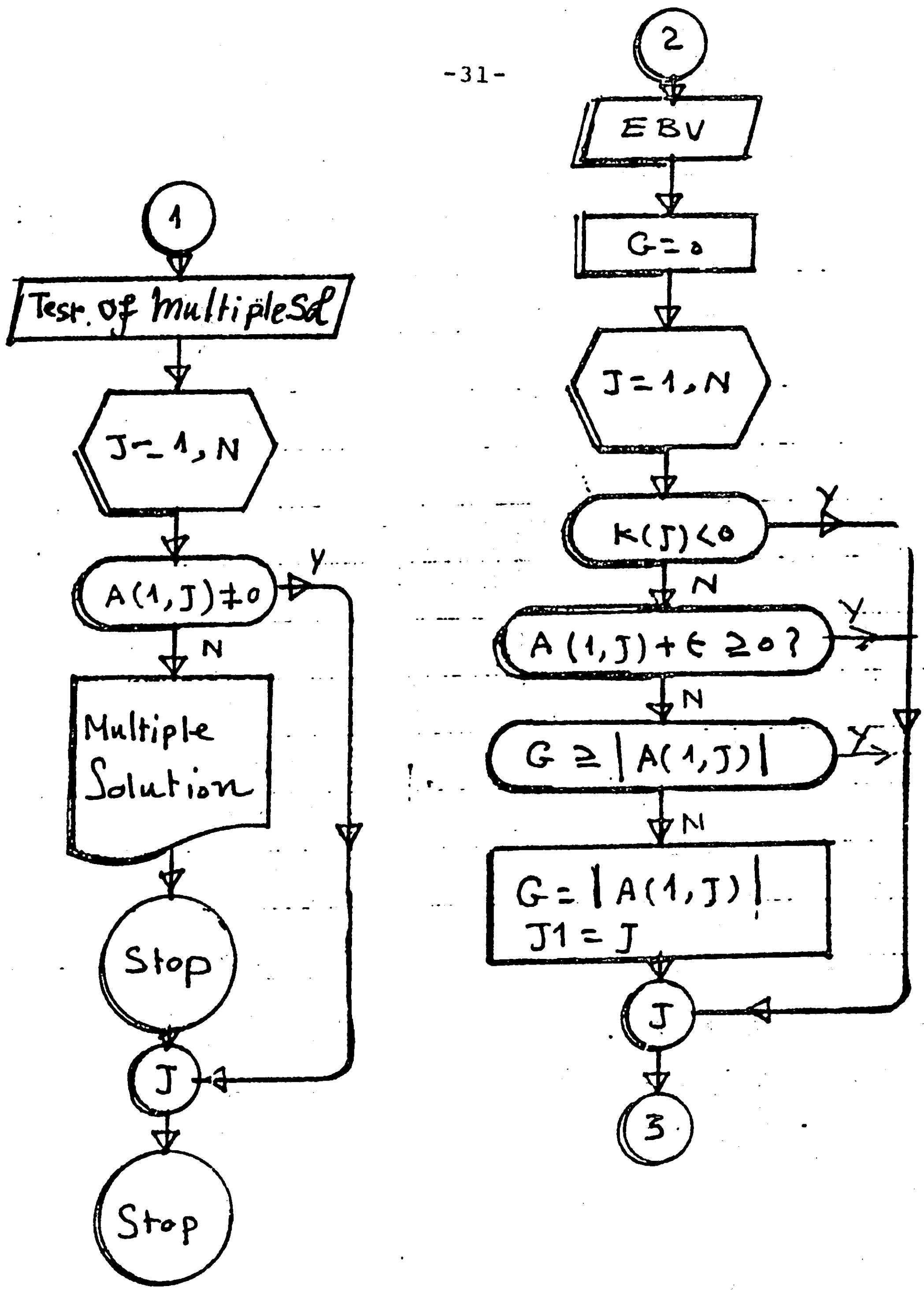
(4)

The 6 step algorithm now is clearly evident by comparing the compact tables (3) and (4)

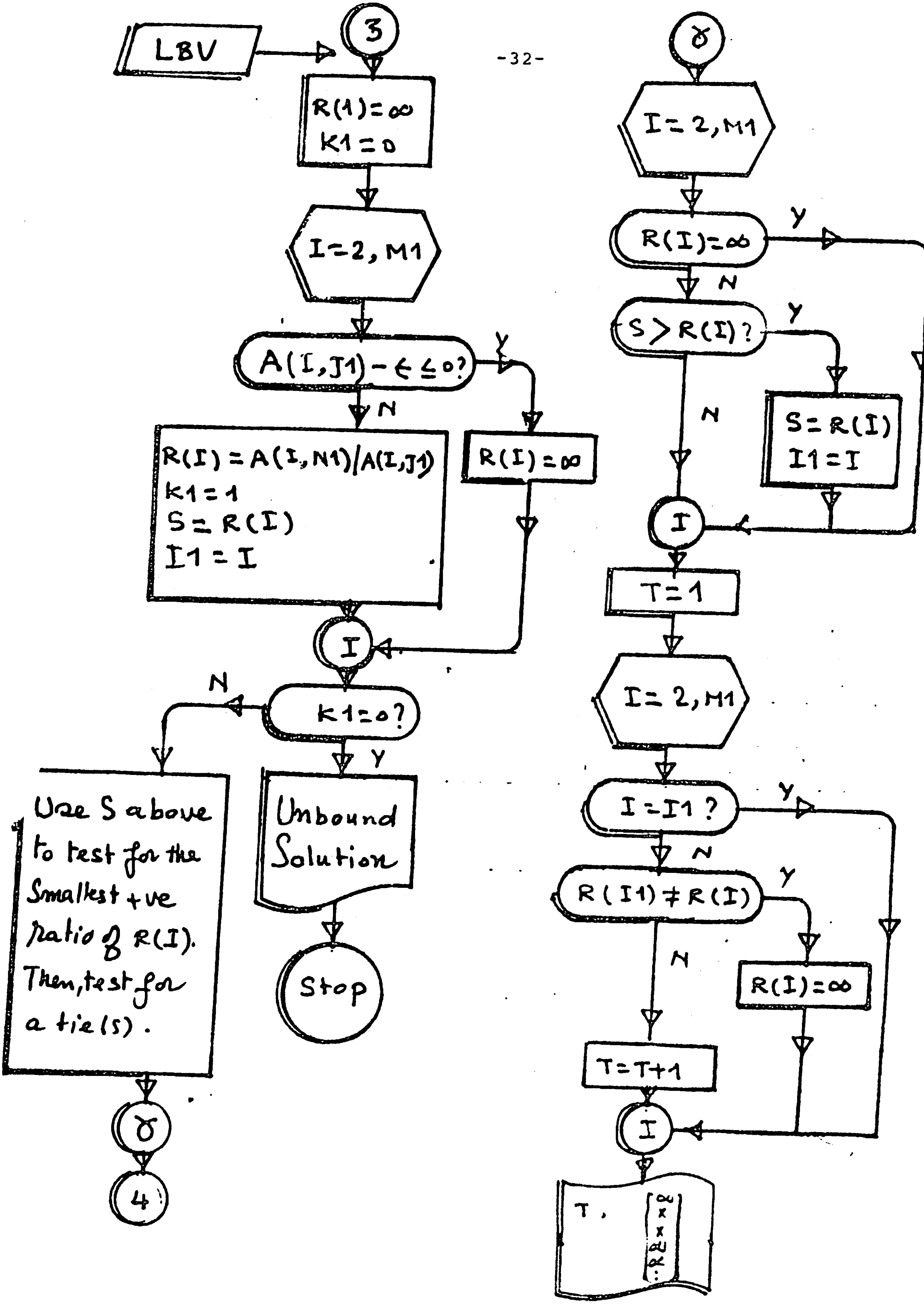
<u>Note that:</u>

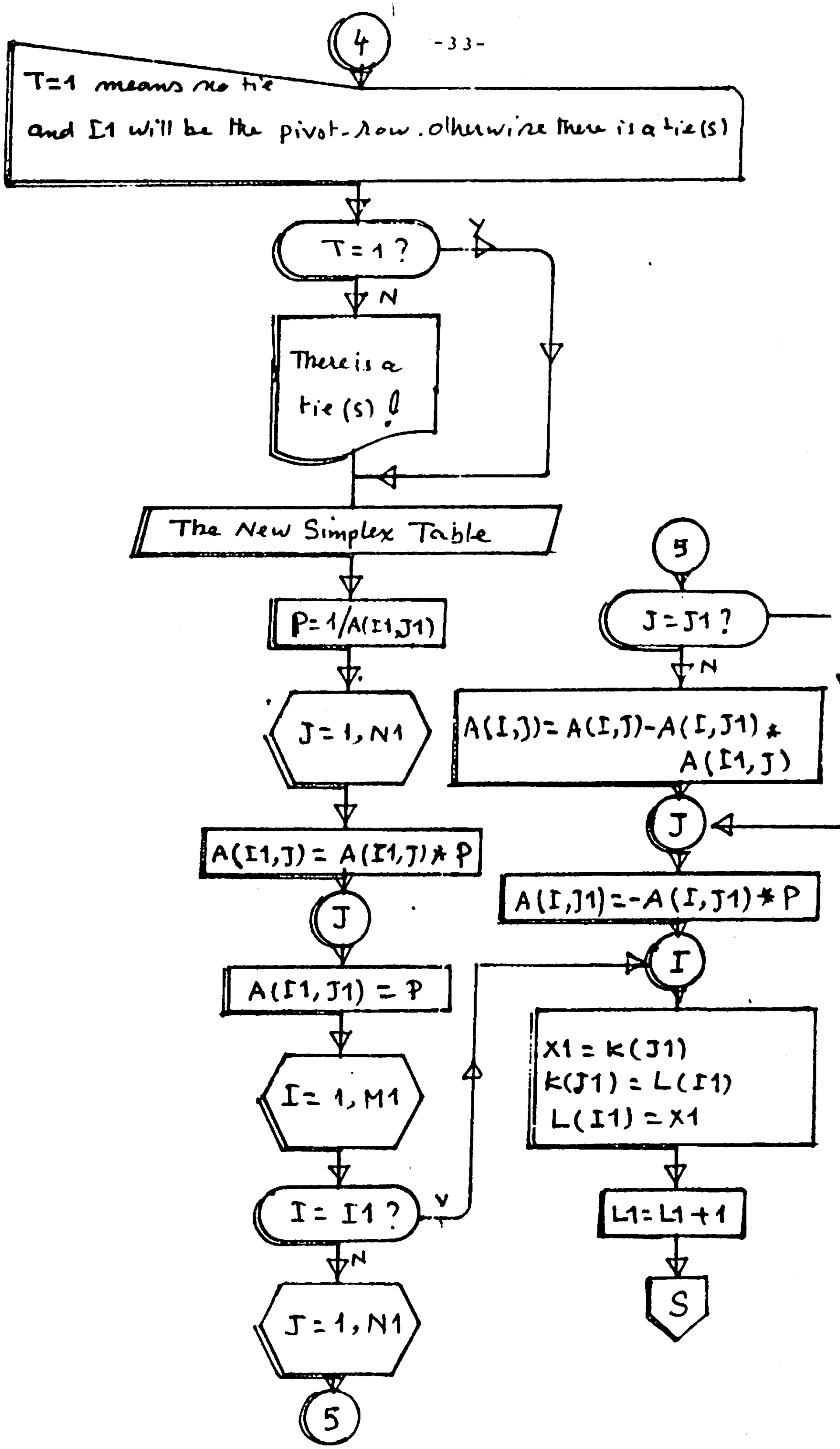
The simplex Algorithm for selecting the pivot-element applied as well to the compact table as it did to the extended table. The only difference is that the pivoting operation is now carried out by the compact table pivoting algorithm.


Start
M, N, C, W1
M1 = M+1
N1 = N+1
L(1) = 0
K(J), J = 1, N
A(1, J), J = 1, N1
L(I), A(I,J), J = 1, N1, I = 2, M1
L1 = 0
Iteration No. L1
K(J), J = 1, N
L(I), A(I,J), J = 1, N1
I = 1, M1
S
Optimality Test
A(1, J) ≥ 0 ∀ J = 1, N
2
Optimal Solution
1



1
Test of MultipleSol
J = 1, N
A(1,J) ≠ 0
Y
N
Multiple Solution
Stop
J
Stop
2
EBV
G = 0
J = 1, N
K(J) < 0
Y
N
A(1,J) + ε ≥ 0 ?
Y
N
G ≥ |A(1,J)|
Y
N
G = |A(1,J)|
J1 = J
J
3


LBV
3
R(1)=∞
K1=0
I=2, M1
A(I,J1)−ϵ≤0?
Y
N
R(I)=A(I,N1)/A(I,J1)
K1=1
S=R(I)
I1=I
R(I)=∞
I
K1=0?
N
Y
Use S above to test for the Smallest +ve ratio of R(I). Then, test for a tie(s).
Unbound Solution
Stop
γ
4
γ
I=2, M1
R(I)=∞
Y
N
S>R(I)?
Y
N
S=R(I)
I1=I
I
T=1
I=2, M1
I=I1?
Y
N
R(I1)≠R(I)
Y
N
R(I)=∞
T=T+1
I
T,


4
T=1 means no tie
and I1 will be the pivot-row. otherwise there is a tie(s)
T = 1 ?
Y
N
There is a tie (s) !
The New Simplex Table
P= 1/A(I1,J1)
J = 1, N1
A(I1,J) = A(I1,J) * P
J
A(I1, J1) = P
I = 1, M1
I = I1 ?
Y
N
J = 1, N1
5
5
J = J1 ?
N
A(I,J)= A(I,J)-A(I,J1) * A(I1,J)
J
A(I,J1)=-A(I,J1) * P
I
X1 = K(J1)
K(J1) = L(I1)
L(I1) = X1
L1= L1 + 1
S

```
60REM  JUNE 1981.
70REM REFERENCES: C.S.WOLFE,LP WITH FORTRAN &
80REM  F S.HILLIER ,OR
90REM
100REM
110REM
120DIMA(20,30),L(20),K(30),R(30)
125OPEN"PR1:",2,1
130PRINT
140PRINT
150PRINT
160PRINT"ENTER M,N,CASE NO.,& PRINTER CODE"
170INPUTM,N,C,W1
180M1=M+1
190N1=N+1
200PRINT"ENTER THE SUBSCRIPTS OF NON-BASIC VS"
210FORJ=1TON
220INPUTK(J)
230NEXTJ
240PRINT"ENTER S.TABLE WITH SUBSCRIPTS OF B.VS"
250FORI=1TOM1
260INPUTL(I)
270FORJ=1TON1
280INPUTA(I,J)
290NEXTJ
300NEXTI
```

```
10REM
20REM
30REM
40L1=0
350T=1
360FORJ=1TON
370PRINTON(2)TAB(T),K(J),
380T=T+10
390NEXTJ
410PRINTON(2)
420FORI=1TOM1
425T=1
430PRINTON(2)TAB(T),L(I),
450FORJ=1TON1
455T=T+10
460PRINTON(2)TAB(T),A(I,J),
480NEXTJ
490PRINTON(2)
500NEXTI
510REM
520REM
530REM
540FORJ=1TON
550IFA(I,J)<0THEN850
560NEXTJ
570REM
580REM   OPTIMAL SOLUTION
```

```
590REM
600PRINTON(2)
610PRINTON(2)USING"      Z= @@@@@@@@.@@@",A(1,N1)
630PRINTON(2)
640FORI=2TON1
650PRINTON(2)USING"X ( @@ ) =@@@@@@.@@@",I,A(I,N1)
670PRINTON(2)
680NEXTI
690PRINTON(2)
710PRINTON(2)"-----WHICH IS THE OPTIMAL SOLUTION-----"
720REM
730REM   TEST OF MULTIPLE SOLUTION.
740REM
750FORJ=1TON
760IFA(1,J)<>0THEN81[illegible]
770PRINTON(2)
790PRINTON(2)"---AND MULTIPLE SOLUTIONS----------"
800GOTO160
810NEXTJ
820GOTO160
830REM
840REM   EBV
850REM
860G=0
870FORJ=1TON
880IFR(J)<0THEN970
890REM
```

```
895REM THE ABOVE TEST IS USED ONLY IN CASE OF TWO-PHASE ALGORITHM
900REM IF K(J)=-VE,THIS COL IS SKIPPED AND NEVER USED AGAIN IN
910REM CHOOSING A PIVOT-ELEMENT ANYHOW. IN OUR CASE-- K(J) ALWAYS +VE
920REM BUT NO HARM TO INCLUDE THIS TEST HERE .
930IF(A(1,J)+.1E-5)>=0THEN970
940IFG>=(-A(1,J))THEN970
950G=-A(1,J)
960J1=J
970NEXTJ
980REM
990REM   LBV & DEGENERACY TEST...
1000REM
1010R(1)=.1E31
1020I1=-1
1030FORI=2TON1
1040IF(A(I,J1)-.1E-5)<=0THEN1100
1050R(I)=A(I,N1)/A(I,J1)
1060REM
1070S=R(I)
1080I1=I
1090GOTO1110
1100R(I)=.1E39
1110NEXTI
1120REM
1130IFI1<>1THEN1190
1140PRINTON(2)
1160PRINTON(2)"      UNBOUND SOLUTION      "
```

```
1170GOTO160
1180REM USE THE ABOVE S TO TEST FOR THE SMALLEST +VE RATIO
1190FORI=2TOM1
1200IFR(I)=.1E37THEN1240
1210IFS<R(I)THEN1240
1220S=R(I)
1230I1=I
1240NEXTI
1250REM  .TEST IF THERE IS ONLY ONE SMALLST VALUE...
1260T=1
1270FORI=2TOM1
1280IFI=I1THEN1360
1290IFR(I1)<>R(I)THEN1350
1300REM
1310REM   THERE IS ATIE
1320REM
1330T=T+1
1340GOTO1360
1350R(I)=.1E3
1360NEXTI
1370REM
1380REM
1390REM
1400REM
1410IFT=1THEN1590
1420REM
1430REM   THERE IS ATIE(S)
```

```
1440REM
1450PRINTON(2)
1460PRINTON(2)USING"THER IS A @@@ TIE(S)''",T
1480PRINTON(2)
1490FORI=2TOM1
1500IFR(I)=.1E39THEN1530
1510PRINTON(2)I
1530NEXTI
1540REM.
1550REM CONTINUE AND MODIFY THE CONSTANTS.IN CASE OF CYCING...
1560REM
1570REM  .PIVOTIHS-OPERATION%NEW S.TABLE...
1580REM.
1590P=1/A(I1,J1)
1600FORJ=1TON1
1610A(I1,J)=A(I1,J)*P
1620NEXTJ
1630A(I1,J1)=P
1640FORI=1TOM1
1650IFI=I1THEN1710
1660FORJ=1TON1
1670IFJ=J1THEN1690
1680A(I,J)=A(I,J)-A(I,J1)*A(I1,J)
1690NEXTJ
1700A(I,J1)=-A(I,J1)*P
1710NEXTI
1720REM
```

```
1730X1=V(J1)
1740V(J1)=L(I1)
1750L(I1)=X1
1760REM
1770L1=L1+1
1780REM.
1790REM...OUTPUT OF THE NEW SIMPLEX TABLE...
1800REM.
1810PRINTON(2)
1820PRINTON(2)USING"ITERATION NO. @@@",L1
1840PRINTON(2)
1850REM...... ...
1860GOTO350
1870END
```

1	2		
0	-25	20	0
3	2	0	600
4	4	2	1200
5	2	3	1200

ITERATION NO. 1

4	2		
0	6.25	32.5	7500
3	-.5	-1	0
1	.25	.5	300
5	-.5	2	600

Z= 7500.000

X (2) = 0.000

X (3) = 300.000

------WHICH IS THE OPTIMAL SOLUTION------

Section 3:

The Revised Simplex (shadow prices) Algorithm for solving Lp Models (RSA)

Introduction:

The Algorithm here may appear to be unjustified since the OSA is much more simple from the theoritical point of view. Also numerically both algorithms appear to be identical since both of them pass through the same set of feasible basic solutions. But in fact, there are many advantages of the algorithm here which sometimes called the simplex-Multipliers or the shadow prices algorithm.

1) One good advantage is that the original data of the model will not be changed through the pivoting operations of the algorithm so that it will not be truncated by rounding errors.

2) A very useful advange of this algorithm is that the shadow-prices are available (these are not computed by the OSA); these beside their practical meaning can be used for any post-optimization studies such as the effect of modifying the constraints of the model on its optimal solution, or any other sensitivity analysis on the model.

3) In large-scale mathematical programming models where the techniques of Decomposition have to be applied, we cannot procede without the matrix-form, not only because the large amount

of data to be handled, but also because the shadow prices provided by the master program will be needed to determine the preference functions for the different Sub-models.

Now, the general form of the Lp model in matrix form is:

$$\begin{aligned} \text{Max.} \quad & Z = CX \\ \text{S.to:} \quad & AX \leq b \qquad (1) \\ & X \geq 0 \end{aligned}$$

$C = [c_j] \ \forall \ j = 1, 2, \ldots, n$

$X = [x_j] \ \forall \ j = 1, 2, \ldots, n$

$b = [b_i] \ \forall \ i = 1, 2, \ldots, n$

$A = [a_{ij}] \ \forall \ i, j.$

The constraints in equation form take the following matrix form:

$$[\,A, I\,] \cdot \begin{pmatrix} X \\ X_s \end{pmatrix} = b; \quad \begin{pmatrix} X \\ X_s \end{pmatrix} \geq 0 \qquad (2)$$

; $X_s = [x_{n+i}] \ \forall \ i = 1, 2, \ldots, m,$

$I =$ mxm unit - matrix, &

$0 =$ (n+m) - zero - vector.

If we consider the objective function as a constraint, the whole model in equality form can take the following form:

$$\begin{bmatrix} 1 & -C & 0 \\ 0 & A & I \end{bmatrix} \cdot \begin{pmatrix} Z \\ X \\ X_s \end{pmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \qquad (3)$$

We recall that the Ordinary Simplex Algorithm (OSA hereafter) is to obtain one basic feasible solution after another until the optimal solution is reached. One of the key features of the RSA involves the way in which it obtains basic feasible solutions. A basic solution is the solution of the m equations:

$$\begin{bmatrix} A, & I \end{bmatrix} \cdot \begin{bmatrix} X \\ X_s \end{bmatrix} = b$$

in which n of the (n+m) elements of $\begin{bmatrix} X \\ X_s \end{bmatrix}$, the non-basic variables, are set equal to zero.

Eliminating these n variables by equating them to zero leaves a set of m equations in m unknowns, the basic variables,. This set of equations can be denoted by:

$$BX_B = b \qquad (4)$$

Where X_B = the set of basic variables

$$X_B = \begin{bmatrix} X_{B1} \\ X_{B2} \\ \vdots \\ X_{Bm} \end{bmatrix}$$

X_B is obtained by eliminating the non basic variables from $\begin{bmatrix} X \\ X_s \end{bmatrix}$

B = the Basis Matrix

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix}$$

B is obtained by eliminating the columns corresponding to coefficients of non-basic variables from $[A, I]$.

From (4), we have:

$$X_B = B^{-1}b \qquad (5)$$

Letting C_B be the vector obtained by eliminating the coefficients of non-basic variables from $[C, O]$ and reordering the elements to match X_B, the value of the objective Function for this basic solution is:

$$Z = C_B X_B = C_B B^{-1}b \qquad (6)$$

For example, consider the following example:

Max. $Z = 3x_1 + 5x_2$

S. to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

We have:

$$C = [3 \quad 5], \quad [A, I] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad X_5 = \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

The complete set of the successive simplex tables for this example (as presented in Hiller, Operations Research) due to the solution by the OSA is shown as follows:

Iteration	B.Vs	Coefficients of					R.H.S.
		x_1	x_2	x_3	x_4	x_5	
0	Z	-3	-5	0	0	0	0
	x_3	1	0	1	0	0	4
	x_4	0	2	0	1	0	12
	x_5	3	2	0	0	1	18
1	Z	-3	0	0	5/2	0	30
	x_3	1	0	1	0	0	4
	x_2	0	1	0	1/2	0	6
	x_5	3	0	0	-1	1	4

cont./

	Z	0	0	0	3/2	1	36
2	x_3	0	0	1	1/3	-1/3	2
	x_2	0	1	0	1/2	0	6
	x_1	1	0	0	-1/3	1/3	2

We notice that the sequence of basic feasible solutions obtained by the simplex Algorithm (Ordinary or Revised) using the matrix notation are the following:

For Iteration 0, we have:

$$X_B = \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B^{-1}, \text{ So } \quad X_B = \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

$$C_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad \text{so } Z = C_B X_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0$$

For Iteration 1, we have:

$$X_B = \begin{bmatrix} X_3 \\ X_2 \\ X_5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

So

$$X_B = \begin{bmatrix} X_3 \\ X_2 \\ X_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

$$C_B = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix}, \quad \text{So } Z = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30$$

For iteration 2, we have:

$$X_B = \begin{bmatrix} X_3 \\ X_2 \\ X_1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

So

$$X_B = \begin{bmatrix} X_3 \\ X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

$$X_B = [0 \quad 5 \quad 3], \text{ so } Z = [0 \quad 5 \quad 3] \cdot \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36$$

Now, for any basic feasible solution, we notice that:

$$X_B = B^{-1}b \qquad \& \qquad Z = C_B B^{-1}b$$

So the R.H.S. of (3) has become:

$$\begin{bmatrix} Z \\ X_B \end{bmatrix} = \begin{bmatrix} 1 & C_B \cdot B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} C_B B^{-1}b \\ B^{-1}b \end{bmatrix} \quad (7)$$

Premultiplying both sides of (3) by $\begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix}$, we have:

$$\begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 1 & -C & 0 \\ 0 & A & 1 \end{bmatrix} = \begin{bmatrix} 1 & C_B B^{-1}A - C & C_B B^{-1} \\ 0 & B^{-1}A & B^{-1} \end{bmatrix} \quad (8)$$

The desired matrix form of the set of equations (3) for any iteration is:

$$\begin{bmatrix} 1 & C_B^{B-1}A - C & C_B^{B-1} \\ 0 & B^{-1}A & B^{-1} \end{bmatrix} \cdot \begin{bmatrix} Z \\ X \\ X_s \end{bmatrix} = \begin{bmatrix} C_B^{B-1}b \\ B^{-1}b \end{bmatrix}$$

Consider, for example, the final iteration in our example. We see that:

$$C_B = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

So $B^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C_B^{B-1}A - C = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $C_B^{B-1} = [0 \;\; 3/2 \;\; 1]$

and since $X_B = B^{-1}b$ and $Z = C_B^{B-1}b$ have already been found above, we have the following set of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3/2 & 1 \\ 0 & 0 & 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} Z \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \\ 6 \\ 2 \end{bmatrix}$$

Which is the same as shown in the final simplex table above. We can also see that:

i. The m x m Basis-matrix B is obtained by iliminating the columns corresponding to the coefficients of Non-basic variables from [A, I]

ii. X_B= the set of Basic variables. Their values can be obtained from $X_B = B^{-1}b$

iii. C_B= the vector obtained by eliminating the coefficients of Non-basic variables from [C, o] and reordering them to match X_B.

iv. Z, the value of the objective function, can be obtained from:

$$Z = C_B X_B = C_B B^{-1} b$$

We notice that:

1) Only B^{-1} needs to be calculated to calculate all the other numbers in the simplex table from the original model parameters: A, B, & C.

2) Any one of the numbers in this table, except for $Z = C_B \cdot B^{-1} b$, can be obtained by performing only a part of matrix multiplications. Therefore, the required numbers to perform any iteration of the algorithm here can be obtained as needed without expending the computation effort to obtained all the numbers.

3) The columns of the original matrix A could be stored on magnetic tape, durm, or disc instead of being stored on the whole matrix in the main storage of the conputer. This is because only one column of A is required at a time for computing the relevant parts necessary at each iteration.

4) A representation of the inverse matrix called the "Product-Form" of the Inverse" is used to minimize the amount of computation and to derive certain other benefits. Anyhow, the auther intends in a further paper to introduce this very useful feature of the product-form of the inverse since there was no time to include it hers.

Outline of the Algorithm:

1. The Initialization step (Iteration 0):

Beside the Original parameters of the model (A, b, c), the following informations are needed:

a, the Basis matrix at this iteration is $B= I=B^{-1}$ (MxM unit matrix)

b, The subscripts of the Basic variables at this iteration are:

$$V(i) = N+i \quad \forall \; i = 1, 2, \ldots, M.$$

c, The set of Basic variables $X_B = X\ (V(i)) = b_i\ \forall_i = 1, 2, \ldots, M$

d, The value of the Objective Function $Z = C_B\ X_B = C\ (V(i))*\ X(V(i))$

(to distinguish bet the Basis matrix B from the right hand side b from the computer programming point of view), we refer to B by E and b by B hereafter.)

2. The Stopping Rule:

To test for optimality, we have to check the coefficients of the non-basic variables in Z-row. If one of them is negative, the current solution is not optimal and we have to continue with the iterative step. Otherwise, the current solution is the optimal solution. The vector of these non-basic variables coefficients is computed from the following matrix multiplications:

$C_B^{B-1}\ A - C = C\ (V(i))\ {*E*}A_{ij} - C_j$ for only j not in the Basis subscripts i.e., for only $j \neq V(i)$.

3. The Iterative step:

i) Determine the Entering Basic Variable (EBV): to determine EBV, compute the greatest-ve coefficient of the vector defind above (the vector of coefficients of the non-basic variables in Z-row). X (J1) will be the EBV and J1 will be the pivot-column in the pivoting operation.

ii) Determine the leaving Basic variable (LBV): to determine the LBV, the other coefficients of X (J1) in the rest of equations i.e., the coefficients of the pivot-column comes from the relevant part of the matrix multiplication:

$$B^{-1}A = E^{*}A_i,\ J_1 = P_i\ \forall_i = 1, 2, \ldots, M$$

The LBV and therefore the pivot-row can be found such that:

$$\frac{X(V\ (i))}{P_i} = \min.\ \forall\ i = 1, 2, \ldots\ m\ \&\ P_i > 0$$

i.e.,

$$\frac{X\ (I1)}{P\ (I1)} = \min_{i} \left(\frac{X\ (V\ (i))}{P\ (i)} \right)$$

$$P\ (i) > 0$$

X (I1) is the LBV & I1 is the Pivot-row number.

here, we note the following:

a) If no EBV is found, the current solution is optimal, so stop. Otherwise find LBV,

b) If no LBV is found, the solution is Unbound. Otherwise, continue.

iii) Determine the new Basis Matrix B and therefore new B^{-1}. The matrix B (and therefore B^{-1}) changes very little from one iteration to the next. It is much more efficient to derive the new B^{-1} (denote it by B^{-1}_{new}) from the B^{-1} at the preceding iteration (denote it by B^{-1}_{old}). For the initial basic feasible solution, $B=I=B^{-1}$. For the current iteration, the method for computing B^{-1} is based directly upon the interpretation of the elements of B^{-1} and the procedure used by the ordinary simplex Algorithm to obtain the new set of equations from the preceding set.

To describe the method formally, let:

- X_K = the EBV,

a_{ik} = the coefficient of X_K in the current equation i ; i = 1, 2, ... M.

These coefficients are calculated already above.

r = the number of the equation containing the LBV.

Therefore,

$$(B^{-1}_{new})_{ij} = \begin{cases} (B^{-1}_{old})_{ij} - \dfrac{a_{ik}}{a_{rk}} (B^{-1}_{old})_{rj} \; ; & i \neq r \\ \dfrac{1}{a_{rk}} (B^{-1}_{old})_{rj} ; & i = r \end{cases}$$

This is equivalent to the following steps in the computer program:

i) $E_{I1,\, j} = E_{I1,\, j} / P(I1) \;\forall\; j = 1, 2, \ldots, M$

(note that the vector P is used here for the pivot-row)

ii) $E_{ij} = E_{ij} - P_i * E_{I1,\, j} \;\forall\; i = 1, 2, \ldots, M \;\; \& \; i \neq I1$

$j = 1, 2, \ldots, N$

- Compute X_B as follows:

$$X_B = B^{-1}b$$

i.e. $X(V(i)) = E_{ij} * B(V(i))$

- The value of the objective Function is:

$$Z = C_B X_B = C(V(i)) * X(V(i))$$

3. Go to the stopping Rule. Check for the optijal solution. Repeat the same process until the final optimal solution is reached or no basic feasible solution is found, or the solution is unbound.

Illustrative Example:

Consider again the previous example:

Max. $Z = 3x_1 + 5x_2$

S.to: $x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

& $x_1, x_2 \geq 0$

For all the coming iterations, we have:

$$C = [3 \quad 5], \; A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

Initialization steop:

$$X_B = \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B^{-1}, \quad C_B = [0 \quad 0 \quad 0]$$

So $X_B = B^{-1}b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$, & $Z = C_B X_B = 0$

Stopping Rule:

To test if the current solution is optimal or not, we have to check the coefficients of the non-basic variables in Z-row. To do this, compute only the relevant parts of the following matrix multiplications:

$C_B{}^{B-1}A - C$ i.e. C (V(i)) *E*$A_{ij} - C_j$ for only j not in the Basis.

i.e., for only $j \neq V(i)$.

$C_B{}^{B-1}A-C = [0 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} - [3 \quad 5]$; x_1, x_2 non-basic variables = $[-3 \quad -5]$.

both coefficients of x_1 and x_2 are -ve and x_2 has the greatest -ve coefficient, therefore:

the current solution is not optimal and x_2 is the Entering basic variable.

Iteration 1:

i, EBV : X_2

ii, LBV: to determine the LBV, we compute the relevant part of the following matrix multiplication to obtain the other coefficients of x_2 i.e. the elements of the pivot-column.

$$B^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} . & 2 \\ . & 2 \\ . & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = P(i)$$

Using this resulted vector and the vector of X_B computed from the previous iteration. The L.B.V. is computed such that:

$$\frac{X_{Bi}}{P_i} = \min \ ; \quad P_i > 0$$

i.e. $\frac{X(V(I1))}{P(I1)} = \min_i \left(\frac{X(V(i))}{P(i)}\right)$; $\forall i = 1, 2, \ldots m$, $P(i) > 0$

Now

$$X_B = \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \quad \& \quad P_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \frac{12}{1} \leftarrow \min$$

X_4 is the LBV.

iii) (B^{-1}_{new}) : Applying (iii) of the iterative step outlined above, we find that:

$$(B^{-1}_{new}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{(note that } k = 2 \text{ and } r = 2\text{)}$$

$$X_B = \begin{bmatrix} X_3 \\ X_2 \\ X_5 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

$$Z = C_B X_B = [\,0 \quad 5 \quad 0\,] \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30$$

<u>The stopping Rule</u>: To test if this solution is optimal or not, we perform the relevant parts from the following matrix-multiplications:

$$C_B B^{-1} A - C = C\ (V(i)\ {*}E{*}A_{ij}) - C_j;\ J \neq V\ (i) \text{ i.e. for } j = 1, 4 \text{ only.}$$

(We note that X_4 is out of the mutrix multiplications and it contained in $C_B B^{-1}$ and there is no need to compute its coefficient because it is already a slack variable)

Now

$$\begin{bmatrix} 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & . \\ 0 & . \\ 3 & . \end{bmatrix} - \begin{bmatrix} 3 & . \end{bmatrix} = \begin{bmatrix} -3 & . \end{bmatrix}$$

The non-basic variable X_1 has a coefficient of -3 which means that the current solution is not optimal and X_1 is the new EBV.

<u>Iteration 2</u>

i, EBV : X_1

ii, LBV: $B^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & . \\ 0 & . \\ 3 & . \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ (computed already in the previous iteration)

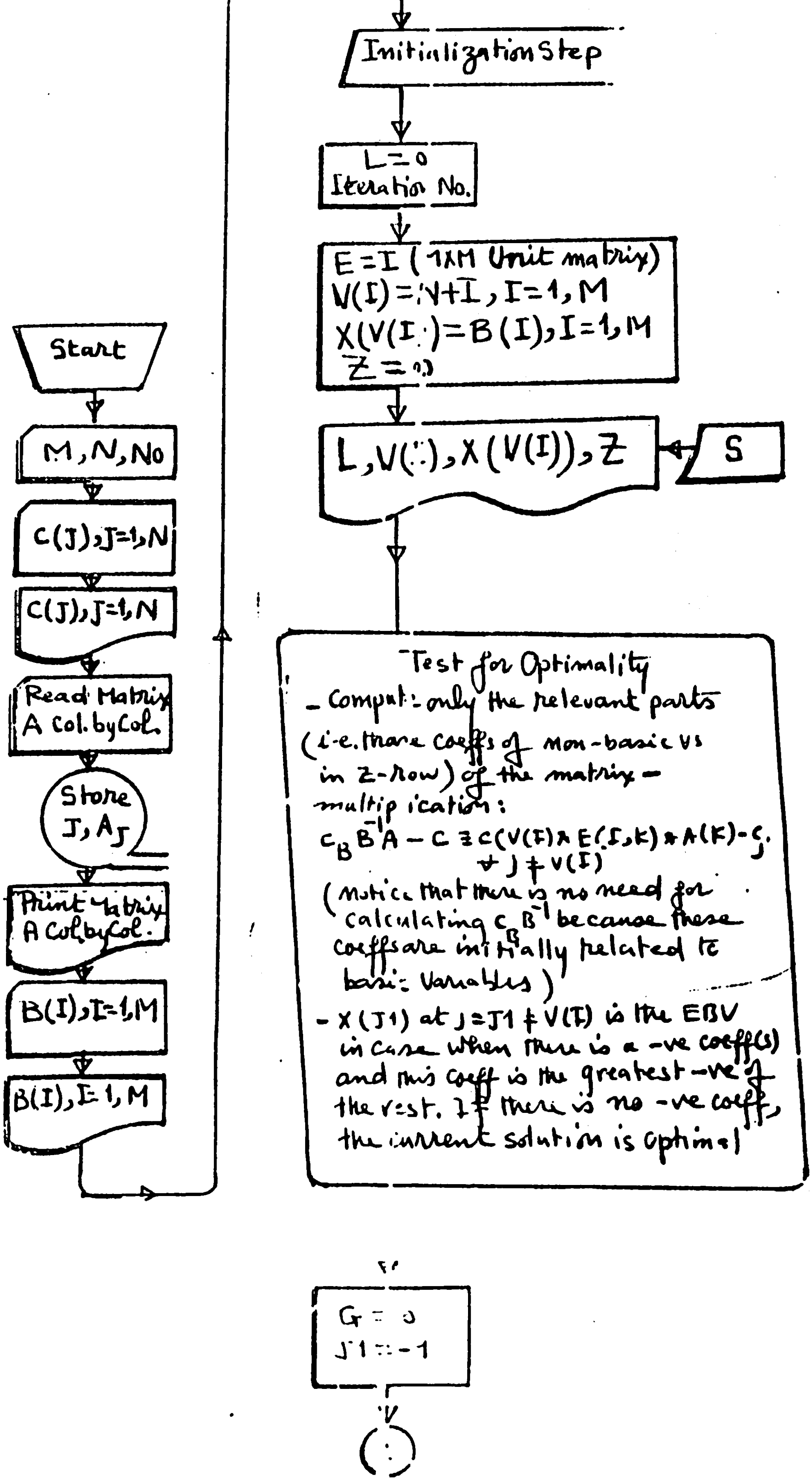
From this vector and the vector of the previous X_B, we find that X_5 is the LBV.

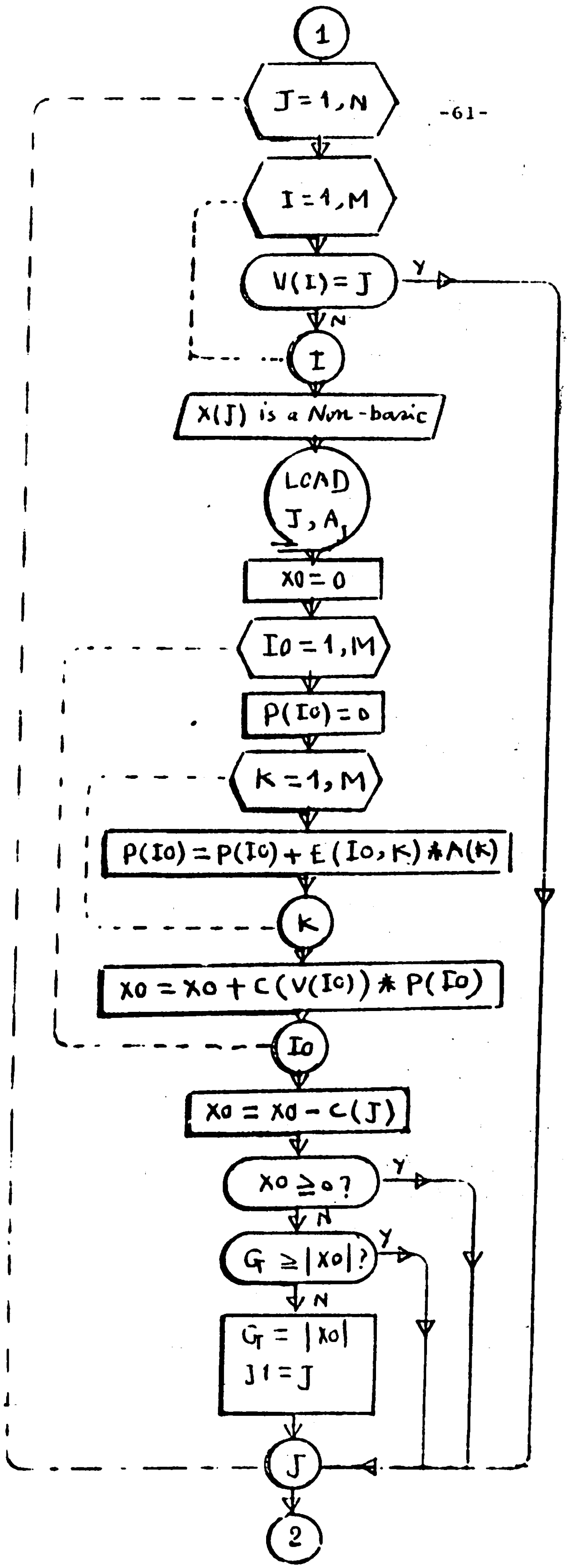
iii, (B^{-1}_{new}) : $B^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix}$

$$X_B = \begin{bmatrix} X_3 \\ X_2 \\ X_5 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$

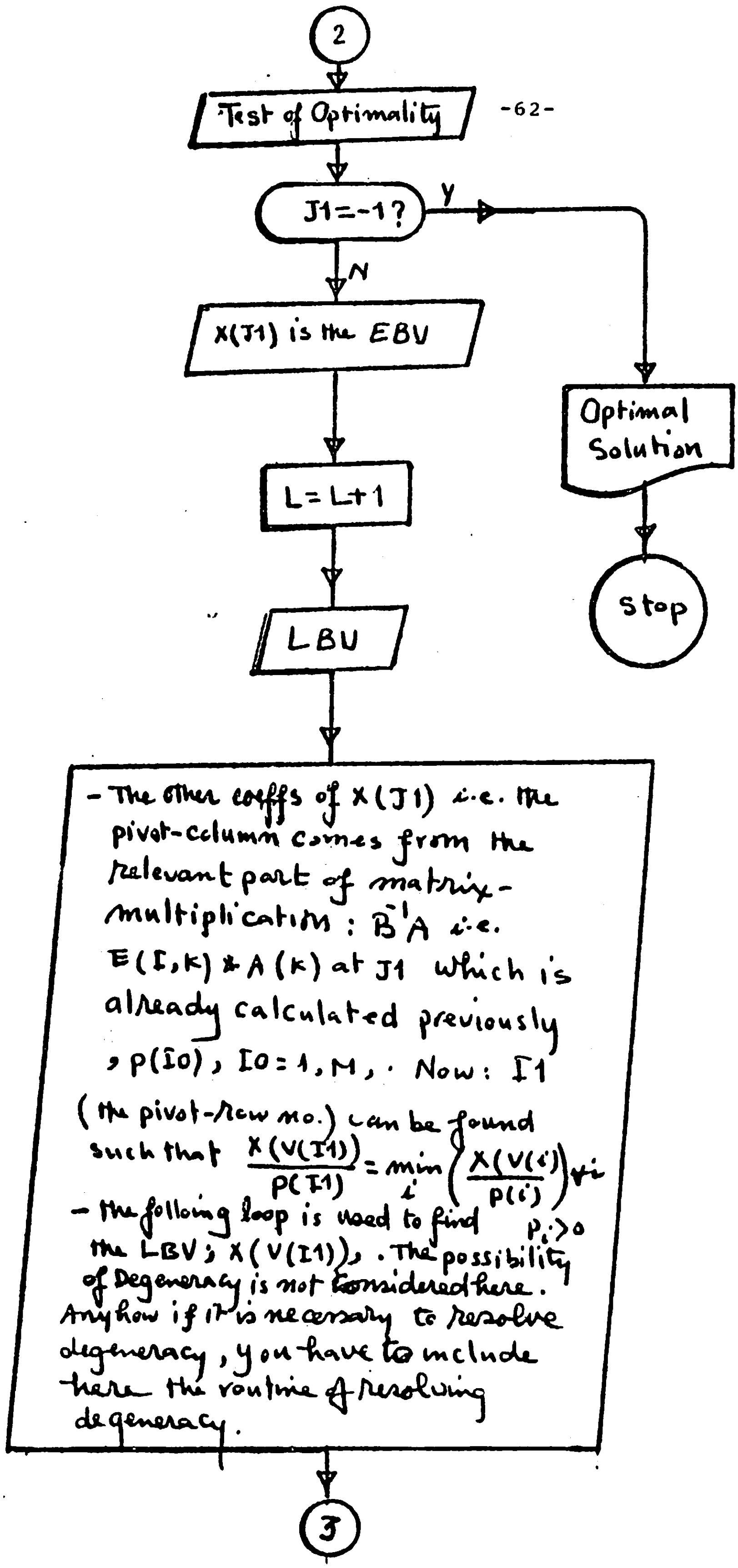
$$Z = C_B \; X_B = \begin{pmatrix} 0 & 5 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} = 36$$

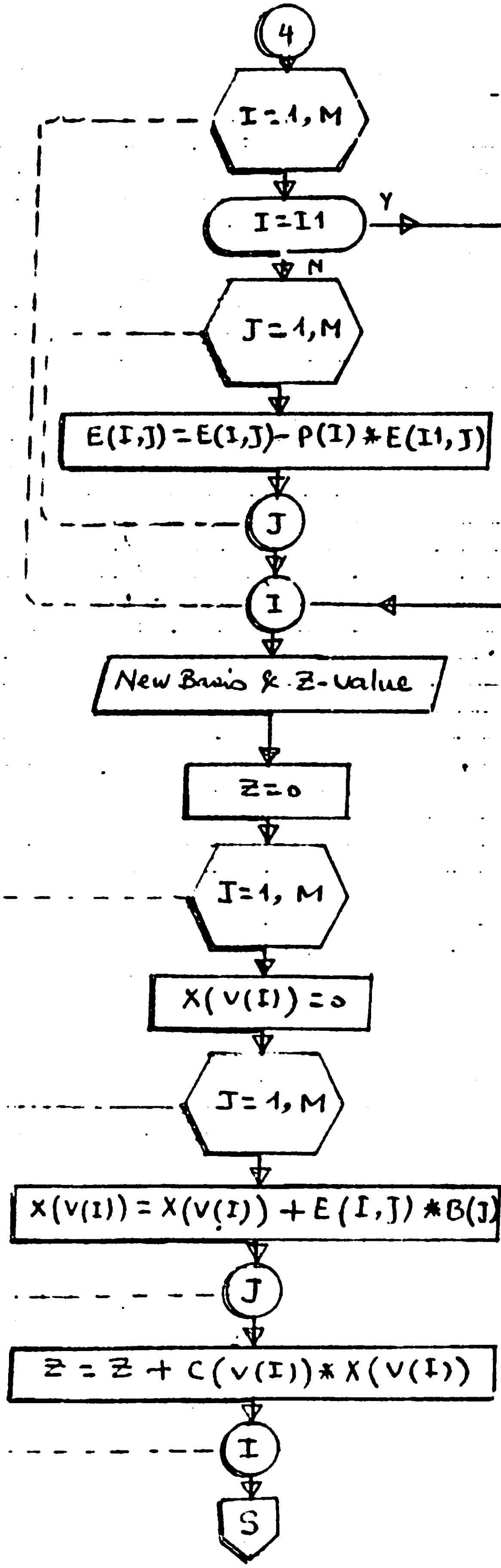
The stopping Rule: Applying the stopping Rule, we find that the current solution is the optimal solution, so stop.

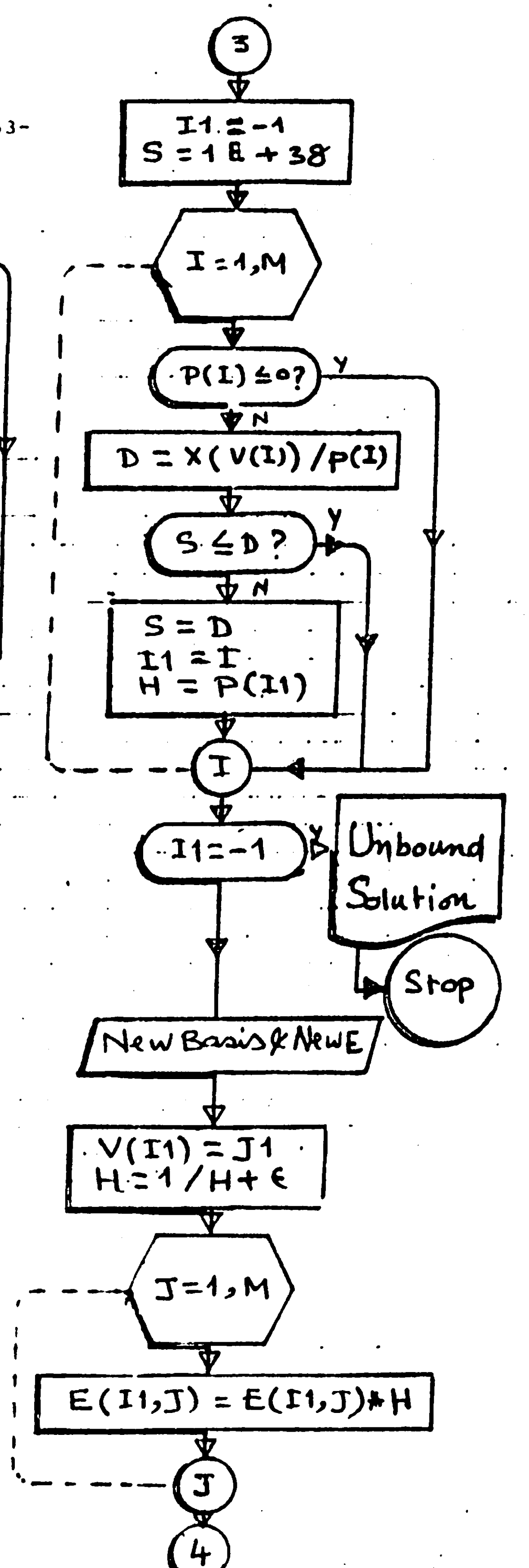

Initialization Step
L = 0
Iteration No.
E = I (1xM Unit matrix)
V(I) = N+I , I=1, M
X(V(I)) = B (I), I=1, M
Z = 0
L, V(I), X (V(I)), Z
S
Start
M, N, No
C(J), J=1, N
C(J), J=1, N
Read Matrix A Col. by Col.
Store J, A_J
Print Matrix A Col by Col.
B(I), I=1, M
B(I), I=1, M
Test for Optimality
- Compute only the relevant parts (i.e. those coeffs of non-basic vs in Z-Row) of the matrix-multiplication:
c_B B^-1 A - C = Σ C(V(I)) * E(I,K) * A(K) - C_J
∀ J ≠ V(I)
(Notice that there is no need for calculating c_B B^-1 because these coeffs are initially related to basic Variables)
- X (J1) at J = J1 ≠ V(I) is the EBV in case when there is a -ve coeff(s) and this coeff is the greatest -ve of the rest. If there is no -ve coeff, the current solution is optimal
G = 0
J1 = -1



1
J = 1, N
I = 1, M
V(I) = J
Y
N
I
X(J) is a Non-basic
LOAD
J, A
X0 = 0
I0 = 1, M
P(I0) = 0
K = 1, M
P(I0) = P(I0) + E(I0, K) * A(K)
K
X0 = X0 + C(V(I0)) * P(I0)
I0
X0 = X0 - C(J)
X0 ≥ 0 ?
Y
N
G ≥ |X0| ?
Y
N
G = |X0|
J1 = J
J
2


2
Test of Optimality
J1=-1?
Y
N
X(J1) is the EBV
Optimal Solution
Stop
L=L+1
LBV
- The other coeffs of X(J1) i.e. the pivot-column comes from the relevant part of matrix-multiplication: B⁻¹A i.e. E(I,K) * A(K) at J1 which is already calculated previously, P(I0), I0=1,M, . Now: I1 (the pivot-row no.) can be found such that X(V(I1))/P(I1) = min_i (X(V(i))/P(i)) ∀i, P_i>0
- the following loop is used to find the LBV; X(V(I1)), . The possibility of Degeneracy is not considered here. Anyhow if it is necessary to resolve degeneracy, you have to include here the routine of resolving degeneracy.
3


4
I=1,M
I=I1
Y
N
J=1,M
E(I,J)=E(I,J)-P(I)*E(I1,J)
J
I
New Basis & Z-value
Z=0
J=1, M
X(V(I))=0
J=1,M
X(V(I))=X(V(I))+E(I,J)*B(J)
J
Z=Z+C(V(I))*X(V(I))
I
S


3
I1=-1
S=1E+38
I=1,M
P(I)≤0?
Y
N
D=X(V(I))/P(I)
S≤D?
Y
N
S=D
I1=I
H=P(I1)
I
I1=-1
Y
Unbound Solution
Stop
New Basis & New E
V(I1)=J1
H=1/H+E
J=1,M
E(I1,J)=E(I1,J)*H
J
4

```
10 REM.  A COMPLETE PROGRAM TO SOLVE A LP MODEL OF THE <= FORM
20 REM.  THE REVISED SIMPLEX METHOD IS USED
30 REM.  THIS PROGRAM DO NOT BOTHER WITH THE THE POSSIBLITY OF CYOLING
40 REM.  INCASE OF DEGENERACY.
50 REM.  PROGRAMMED BY STUDENTS STUDYING THE COURSE 1424
60 REM   SUPERVISED BY DR.ABDALLA EL-DAOUSHY
70 REM.  MAY,1981
80 REM.  ......................
90 REM.  .......................................
100 DIM C(30),A(10,20),B(10),E(10,10),P(10),X(10)
101 OPEN "PR:",1,1
130 PRINT "ENTER M,N,& CASE NO "
140 INPUT M,N,NO
150 PRINT"ENTER THE N COEFFS OF Z-FUNCTION"
160 FOR J=1 TO N
170 INPUT C(J)
180 NEXT J
190 FOR J=N+1 TO N+M
200 C(J)=0
210 NEXT J
220 PRINT "ENTER THE ELEMENTS OF MATRIX A AND R.H.S."
230 FOR I=1 TO M
240 FOR J=1 TO N
250 INPUT A(I,J)
260 NEXT J
270 INPUT B(I)
280 NEXT I
290 REM.  OUPUT OF INPUT DATA FOR PURPOSE OF CHECKING...
300 PRINT ON(1)
310 PRINT ON(1) M,N,NO
330 PRINT ON(1)
340 FOR J=1 TO N
350 PRINT ON(1) C(J),
370 NEXT J
380 PRINT ON(1)
390 FOR I=1 TO M
400 FOR J=1 TO N
410 PRINT ON(1) A(I,J),
420 NEXT J
430 PRINT ON(1)B(I)
450 NEXT I
451 PRINT ON(1):PRINT ON(1)
460 REM
470 REM...INITIALIZATION STEP
480 REM.
490 L=0
500 MAT E=IDN
510 FOR I=1 TO M
520 V(I)=N+I
530 X(V(I))=B(I)
```

```
540 NEXT I
550 Z=0
560 REM.
570 REM...OUTPUT OF ITERATION NO. L.
580 REM.
590 PRINT ON(1)"ITERATION NO. ";L:PRINT ON(1)"----------------"
610 PRINT ON(1)
620 FOR I=1 TO M
630 PRINT ON(1)"X(";V(I),")=",X(V(I))
650 NEXT I
660 PRINT ON(1)
670 PRINT ON(1)"Z=",Z
690 PRINT ON(1)
700 REM.
710 REM. THE COEFFS OF NON-BASIC VARIABLES IN Z-ROW,THE GREATEST -VE
720 REM. ONE OF THEM , AND THE TEST FOR MULTIPLE SOLUTIONS
730 REM.
740 J1=-1
750 G=0
760 REM..........
770 FOR J=1 TO N
780 FOR I=1 TO M
790 IF J=V(I) THEN 940
800 NEXT I
810 XO=0
820 FOR IO=1 TO M
830 P(IO)=0
840 FOR K=1 TO M
850 P(IO)=P(IO)+E(IO,K)*A(K,J)
860 NEXT K
870 XO=XO+C(V(IO))*P(IO)
880 NEXT IO
890 XO=XO-C(J)
900 IF XO >=0 THEN 940
910 IF G>=ABS(XO) THEN 940
920 G=ABS(XO)
930 J1=J
940 NEXT J
950 REM.
960 REM
970 REM.
980 REM...TEST OF OPTIMALITY...
990 REM.
1000 IF J1<>(-1) THEN 1090
1010 REM.
1020 REM...OPTIMAL SOLUTION...
1030 REM.
1040 PRINT ON(1)
1050 PRINT ON(1)TAB(15),"WHICH IS THE OPTIMAL SOLUTION..."
1070 STOP
```

```
1080 REM.
1090 L=L+1
1100 REM...THE EBV IS X(J1)...
1110 REM...  TO FIND THE LBV...
1120 REM.  THE PIVOT-COLUMN...
1130 REM.  THE..........   ON...
1140 FOR IO=1 TO M
1150 P(IO)=0
1160 FOR K=1 TO M
1170 P(IO)=P(IO)+E(IO,K)*A(K,J1)
1180 NEXT K
1190 NEXT IO
1200 REM......
1210 I1=-1
1220 S=1E+38
1230 FOR I=1 TO M
1240 IF P(I) <=0 THEN 1300
1250 D=X(V(I))/P(I)
1260 IF S <=D THEN 1300
1270 S=D
1280 I1=I
1290 H=P(I1)
1300 NEXT I
1310 REM.......
1320 IF I1<>(-1) THEN 1400
1330 PRINT ON(1)
1340 PRINT ON(1)TAB(6),"UNBOUND SOLUTION"
1360 STOP
1370 REM.
1380 REM...THE NEW BASIS, THEIR VALUES , & THE VALUE OF Z...
1390 REM.
1400 V(I1)=J1
1410 H=1/H+0.000005
1420 REM.....
1430 FOR J=1 TO M
1440 E(I1,J)=E(I1,J)*H
1450 NEXT J
1460 FOR I=1 TO M
1470 IF I=I1 THEN 1570
1480 FOR J=1 TO M
1490 REM
1500 E(I,J)=E(I,J)-P(I)*E(I1,J)
1510 REM
1520 REM
1530 NEXT J
1540 REM...................
1550 REM...................
1560 REM...................
1570 NEXT I
1580 REM.......
```

```
1590 REM.......
1600 Z=0
1610 FOR I=1 TO M
1620 X(V(I))=0
1630 FOR J=1 TO M
1640 X(V(I))=X(V(I))+E(I,J)*B(J)
1650 NEXT J
1660 Z=Z+C(V(I))*X(V(I))
1670 NEXT I
1680 REM.
1690 PRINT ON(1)
1700 GOTO 590
1710 END
```

3	2	1
3	5	
1	0	4
0	2	12
3	2	18

ITERATION NO. 0

X(3)=	4
X(4)=	12
X(5)=	18
Z=	0

ITERATION NO. 1

X(3)=	4
X(2)=	6.00006
X(5)=	5.99988
Z=	30.0003

ITERATION NO. 2

X(3)=	2.00001
X(2)=	6.00006
X(1)=	1.99999
Z=	36.00027

WHICH IS THE OPTIMAL SOLUTION...

2	2	2
2	3	
1	2	4
1	1	3

ITERATION NO. 0

X(3)= 4
X(4)= 3

Z= 0

ITERATION NO. 1

X(2)= 2.00002
X(4)= .99998

Z= 6.00006

ITERATION NO. 2

X(2)= 1.0000175
X(1)= 1.999985

Z= 7.0000225

WHICH IS THE OPTIMAL SOLUTION.

3	2	3
3	2	
1	0	4
0	2	12
3	2	18

ITERATION NO. 0

X(3)= 4
X(4)= 12
X(5)= 18

Z= 0

ITERATION NO. 1

X(1)= 4.00002
X(4)= 12
X(5)= 5.99994

Z= 12.00006

ITERATION NO. 2

X(1)= 4.00002
X(4)= 6
X(2)= 3

Z= 18.00006

WHICH IS THE OPTIMAL SOLUTION. . .

SECTION 4:-

THE DUAL SIMPLEX ALGORITHM FOR SOLVING LP MODELS IN EXTENDED FORM (DSA/EXTENDED)

For the dual simplex algorithm, all the coeffs in eq o of the initial simplex table must be non-negative so that the initial solution is super-optimal, but not necessary be feasible since one or more of the basic variables are negative. For this reason, we look for feasibility rather than optimality.

OUTLINE OF THE ALGORITHM:

I. The Initialization Step:-

i) introduce only slack variables as needed to construct the Initial dual simplex table. Notice that all the coeffs of the basic variables in eq o are zero and the coeffs of the non-basic variables are non-negative or zero.

ii) Go to the stopping rule to see if this solution is feasible, and therefore optimal, or not?

II. The Stopping Rule:

The determine if this solution is feasible, and therefore optimal, it is necessary to check if all the basic variables have non-negative values? If so, then this solution is feasible and optimal, so stop; otherwise, go to the iterative step.

III. The Iterative Step:

i) Determine the LBV:

Select the vasic variable with the greatest negative value to leave the basis. Let I1 the pivot-row No.

ii) Determine the EBV:

Select the non-basic variable whose coeff in eq o reaches zero first as an increasing multiple of eq no. I1 is added to eq 0. This is can be done as follows:

1. Consider only the negative coefficients of the non-basic variables in eq no. I1 (the eq containing the LBV).

2. For only the non-basic variables, determine

$$\frac{a_{0,J1}}{a_{I1},J1} = \min_{j} \left(\frac{a_{0,j}}{|a_{I1,j}|} \right); \; a_{I1,j} < 0$$

; J1 refer to the pivot-column no.

3. Resolve degeneracy if necessary.

iii) Determine the New Dual Simplex Table:

1. The pivot-element is $a_{I1,J1}$

2. New pivot-row $= \dfrac{\text{old pivot-row}}{\text{pivot-element}}$.

3. Any other new row

= old row-(pivot-column coeff)X new pivot row.

; (pivot-column coeff) is that element in this row that is in the pivot-column.

4. Enterchange the suffix of the old basic variable, I1, with the suffix of the old non-basic variable, J1,.
5. Notice that all the elements in the pivot-column, except the pivot-element itself, reduces to zero at the end of calculating this new dual simplex table.

Let us consider the dual of the same primal-problem discussed in this memo. to facilitate comparison:

The primal problem was:

$$\max.\ Z = 3x_1 + 5x_2$$

s/to:

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The dual problem is:

$$\min:\quad Y_0 = 4y_1 + 12y_2 + 18y_3$$

s.to:

$$\begin{aligned} y_1 \quad + 3y_3 &\geq 3 \\ 2y_2 + 2y_3 &\geq 5 \end{aligned}$$

&

$$y_1, \quad y_2, y_3 \geq 0$$

Remember that min y_0 = max $(-y_0)$ and rewrite this problem in our standard form, we get:

$$\max(-y_0) = -4y_1 - 12y_2 - 18y_3$$

s.to:

$$-y_1 \qquad - 3y_3 \leq -3$$

$$-2y_2 - 2y_3 \leq -5$$

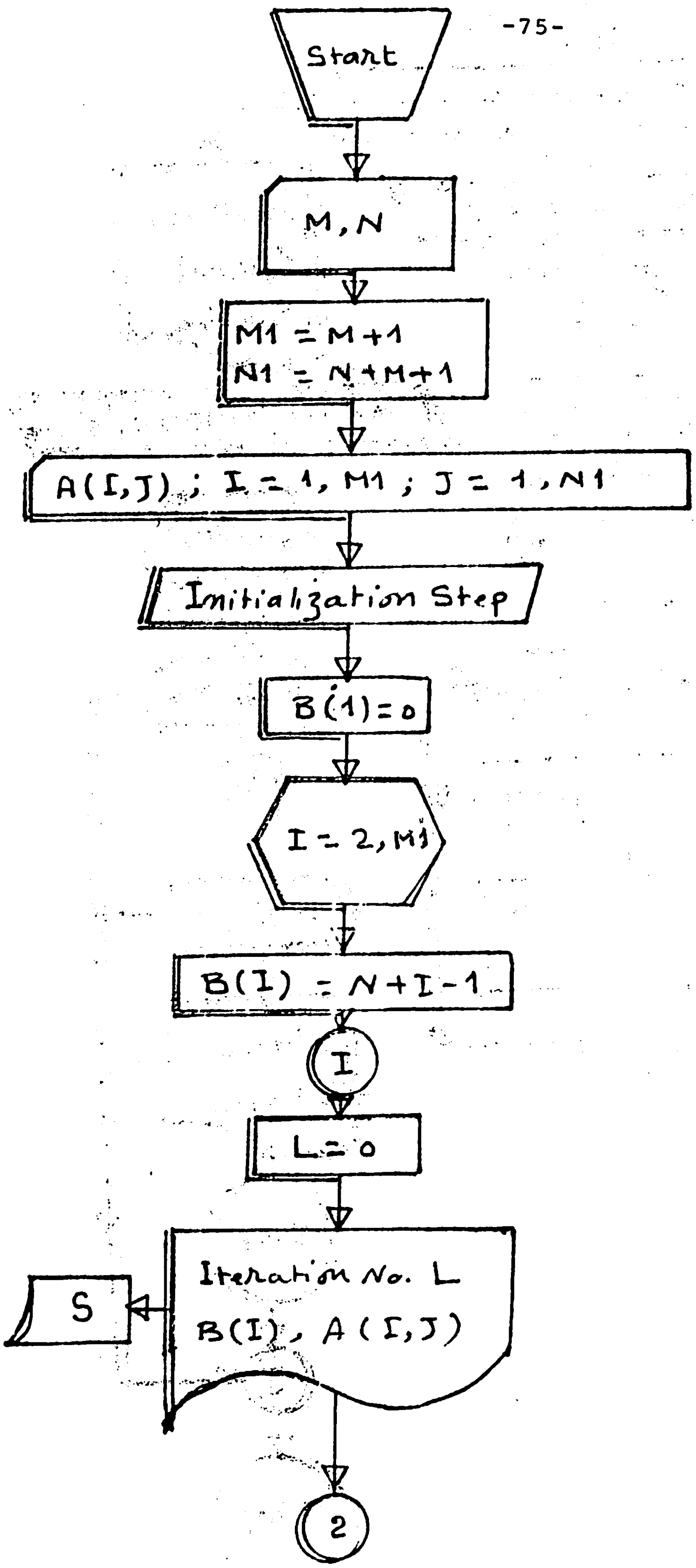
$$\& \quad y_i \geq 0 \; \forall \; i = 1,2,3,.$$

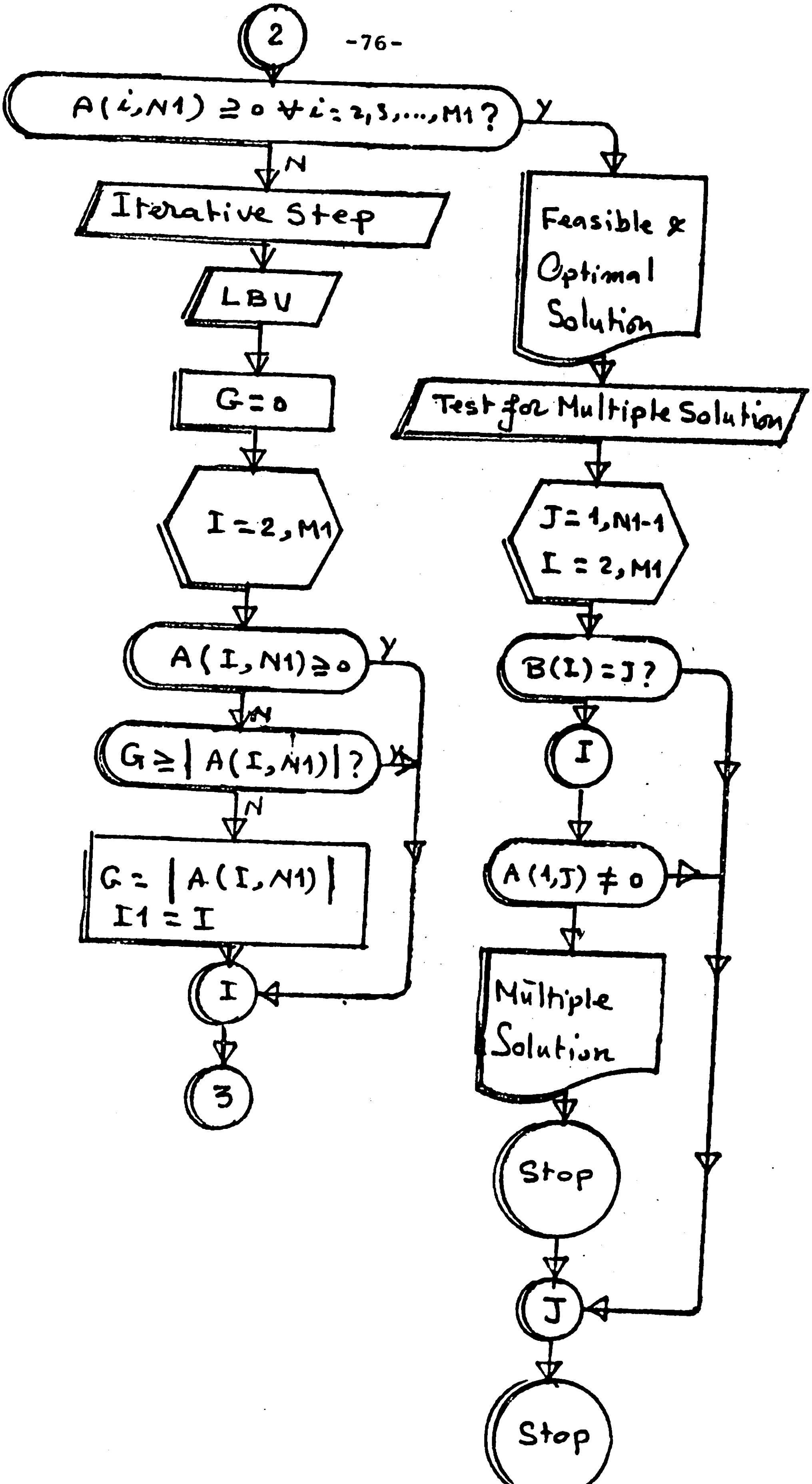
Using the procedure outlined above, the successive dual simplex tables are:

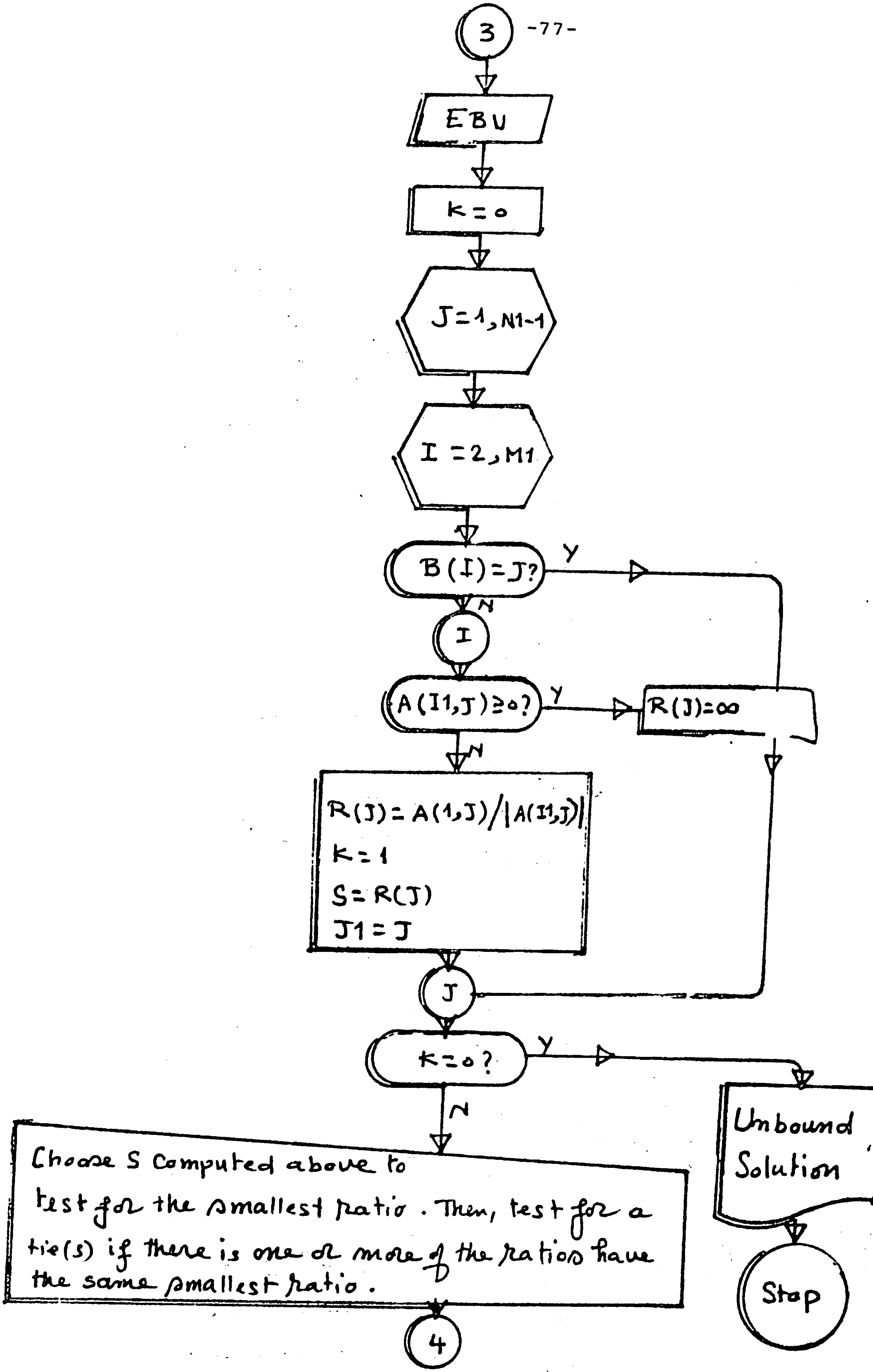
Iteration no.	eq no.	B.V.S.	Coeffs of y_1	y_2	y_3	y_4	y_5	r.h.s
0	0	$(-y_0)$	4	12	18	0	0	0
	1	y_4	-1	0	-3	1	0	-3
	2	y_5	0	-2	-2	0	1	-5
1	0	$(-y_0)$	4	0	6	0	6	-30
	1	y_4	-1	0	-3	1	0	-3
	2	y_2	0	1	1	0	-½	5/2
2	0	$(-y_0)$	2	0	0	2	6	-36
	1	y_3	1/3	0	1	-1/3	0	1
	2	y_2	-1/3	1	0	1/3	-½	3/2

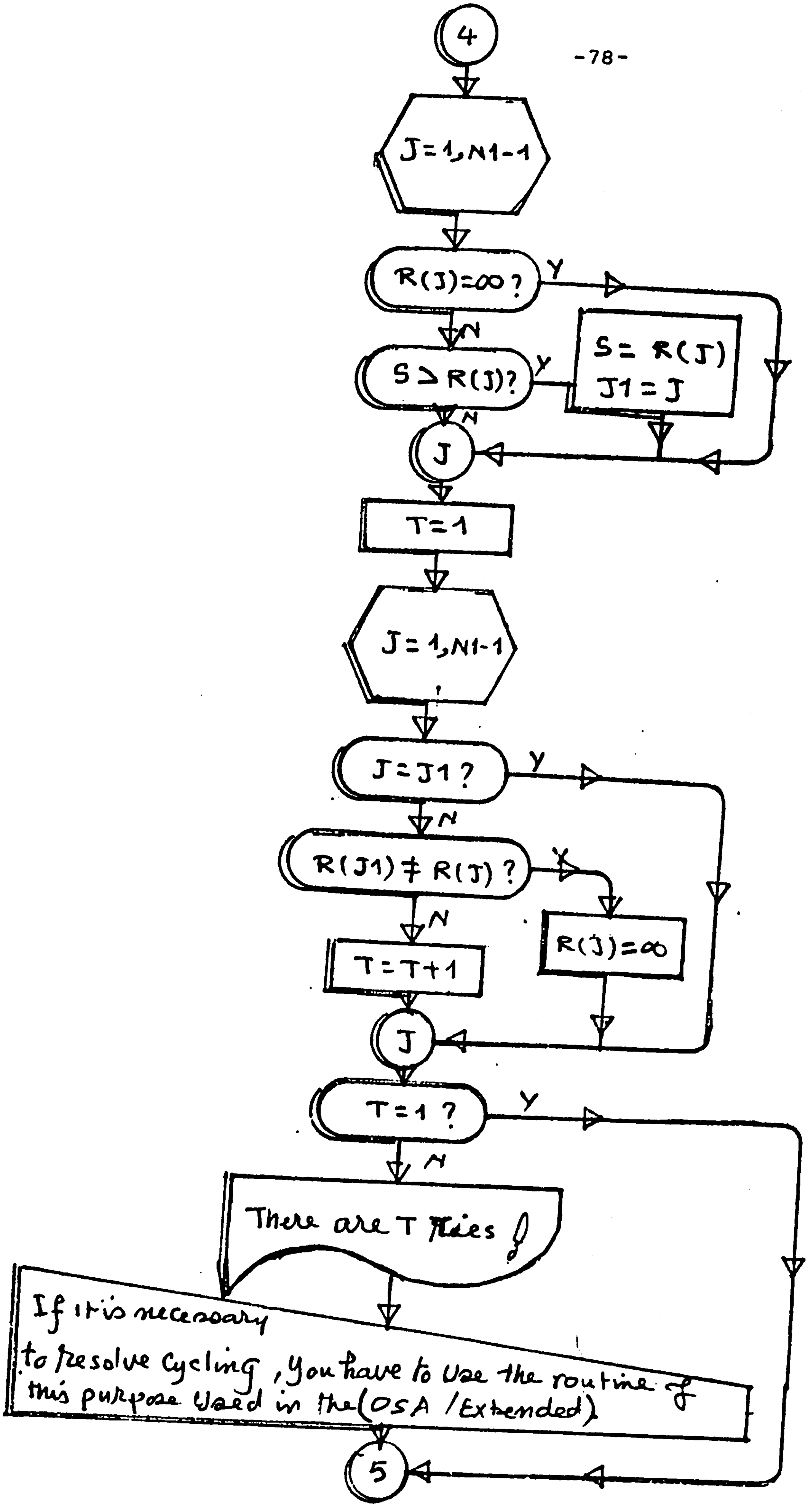
The optimal solution is $(-y_0) = -36$ i.e. $y_0 = 36$

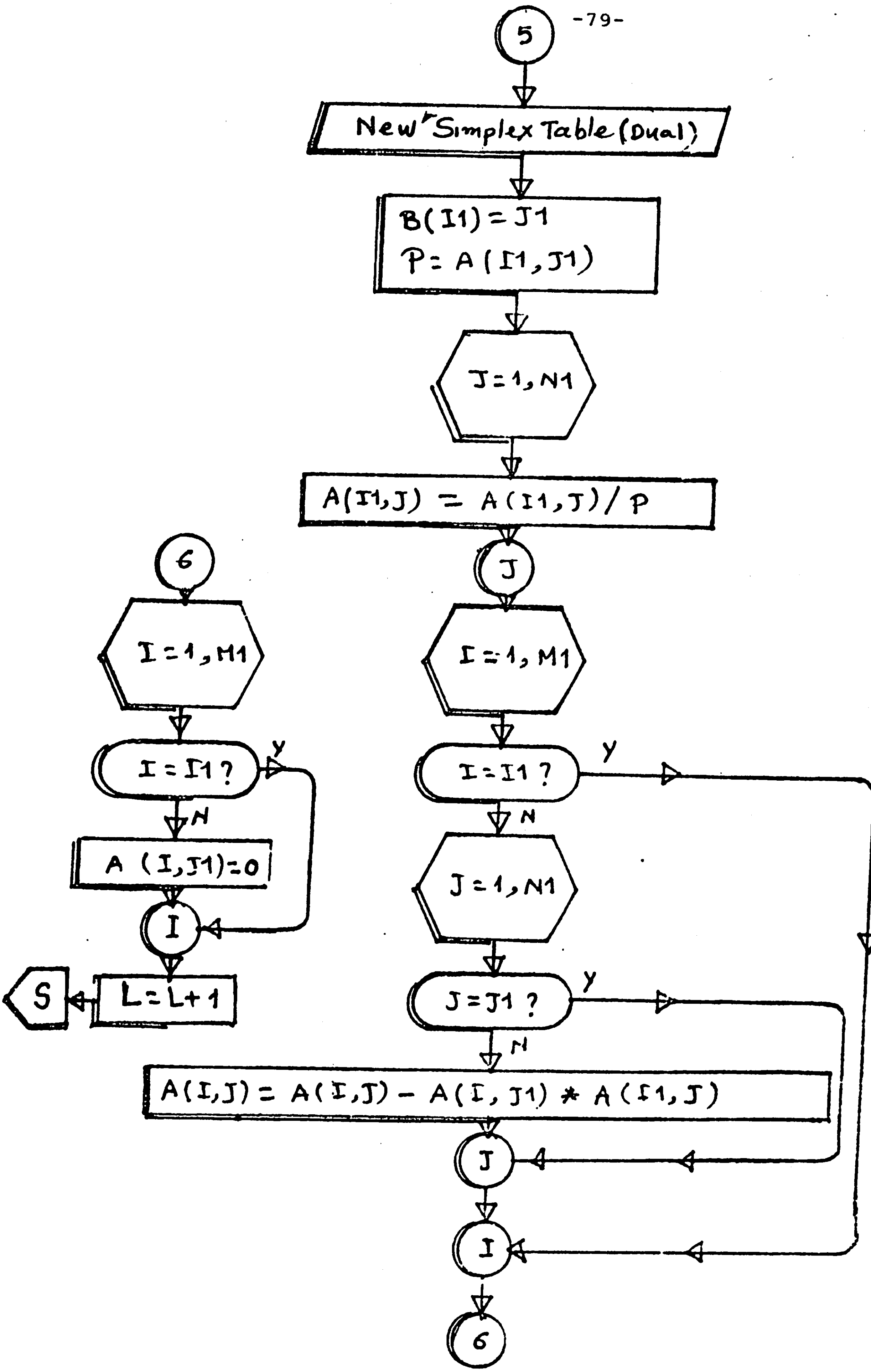
$$y_2 = 3/2, \quad y_3 = 1$$


Start
M, N
M1 = M+1
N1 = N+M+1
A(I,J) ; I = 1, M1 ; J = 1, N1
Initialization Step
B(1)=0
I = 2, M1
B(I) = N+I-1
I
L = 0
Iteration No. L
B(I), A(I,J)
S
2


2
A(i,N1) ≥ 0 ∀ i = 2,3,...,M1 ?
Y
N
Iterative Step
LBV
G = 0
I = 2, M1
A(I,N1) ≥ 0
Y
N
G ≥ |A(I,N1)| ?
Y
N
G = |A(I,N1)|
I1 = I
I
3
Feasible &
Optimal
Solution
Test for Multiple Solution
J = 1, N1-1
I = 2, M1
B(I) = J ?
I
A(1,J) ≠ 0
Multiple
Solution
Stop
J
Stop


3
EBU
K = 0
J = 1, N1-1
I = 2, M1
B(I) = J?
Y
N
I
A(I1,J) ≥ 0?
Y
R(J) = ∞
N
R(J) = A(1,J)/|A(I1,J)|
K = 1
S = R(J)
J1 = J
J
K = 0 ?
Y
N
Choose S computed above to test for the smallest ratio. Then, test for a tie(s) if there is one or more of the ratios have the same smallest ratio.
Unbound Solution
Stop
4


4
J = 1, N1-1
R(J)=∞ ?
Y
N
S ≥ R(J)?
S = R(J)
J1 = J
J
T=1
J = 1, N1-1
J = J1 ?
R(J1) ≠ R(J) ?
R(J)=∞
T=T+1
T = 1 ?
There are T ties
If it is necessary
to resolve cycling, you have to use the routine of
this purpose used in the (OSA / Extended)
5


5
New Simplex Table (Dual)
B(I1) = J1
P = A(I1, J1)
J = 1, N1
A(I1,J) = A(I1,J)/P
J
I = 1, M1
I = I1 ?
y
N
J = 1, N1
J = J1 ?
y
N
A(I,J) = A(I,J) - A(I, J1) * A(I1, J)
J
I
6
6
I = 1, M1
I = I1 ?
y
N
A (I,J1) = 0
I
L = L+1
S

```
10 REM.....DSA/EXTENDED.....
20 REM.  A PROGRAM TO SOLVE A LP PROBLEM USING
30 REM.  THE DUAL SIMPLEX ALGORITHM IN EXTENDED FORM.
40 REM.  THIS PROG DO NOT BOTHER WITH THE POSSIBILITY
50 REM.  OF CYCLING IN CASE OF DEGENERACY .
60 REM.  PROGRAMMED BY DR.  ABDALLA EL-DAOUSHY
70 REM.  MAY, 1981.
80 REM...............
90 REM.
100 REM.
110 DIM B(21),A(21,31),R(31)
111 OPEN "PR:",1,1
120 PRINT "ENTER M,N,&CASE NO."
130 INPUT M,N,C
140 M1=M+1
150 N1=N+1
160 PRINT "ENTER THE INITIAL S. TABLEAU"
170 FOR I=1 TO M1
180 FOR J=1 TO N1
190 INPUT A(I,J)
200 NEXT J
210 NEXT I
220 REM  NOTE THAT THE COEFFS OF THE NON-BASIC VARIASLES IN Z-ROW
230 REM  MUST BE NON-VE 90 THAT THE I B.SOLUTION IS SUPER OPTIMAL
240 REM BUT NOT FEASIBLE.....
290 REM...
300 REM.  INITIALIZATION STEP......
310 REM.  BASIS CREATION
320 B(1)=0
330 FOR I=2 TO M1
340 B(I)=N+I-1
350 NEXT I
360 L=0
370 REM.  THE OUTPUT OF ITERATION NO.
380 REM.
390 PRINT ON(1) "ITERATION NO ";L
391 PRINT ON(1) "------------------"
400 PRINT ON(1)
401 PRINT ON(1),TAB(14)," ";
410 FOR J=1 TO N1-1
420 IF J>10 THEN 460
430 PRINT ON(1)J,
450 GOTO 470
460 PRINT ON(1)
470 NEXT J
480 PRINT ON(1)
490 FOR I=1 TO M1
500 PRINT ON(1)B(I),
520 FOR J=1 TO N1
530 IFJ>10 THEN 570
```

```
540 PRINT ON(1)A(I,J),
560 GOTO 580
570 PRINT ON(1)
580 NEXT J
590 PRINT ON(1)
600 NEXT I
610 REM.
620 REM..TEST OF FEASIBILITY.
630 REM.
640 FOR I=2 TO M1
650 IF A(I,N1)<0 THEN 980
660 NEXT I
670 REM
680 REM. FEASIBLE SOLUTION....
690 REM.
700 PRINT ON(1)
710 PRINT ON(1)
720 PRINT ON(1)"Z= ",A(1,N1)
740 FOR I=2 TO M1
750 PRINT ON(1)"X(",B(I),")=",A(I,N1)
770 NEXT I
780 PRINT ON(1)
790 PRINT ON(1)TAB(15),"WHICH IS THE OPTIMAL SOLUTION"
810 REM.
820 REM.TEST OF MULTIPLE SOLUTION.....
830 REM.
840 FOR J=1 TO N1-1
850 FOR I=2 TO M1
860 IF B(I)=J THEN 920
870 NEXT I
880 IF A(1,J)<>0 THEN 920
890 PRINT ON(1)TAB(13),"AND MULTIPLE SOLUTION........,"
910 STOP
920 NEXT J
930 STOP
940 REM
950 REM. THE ITERATIVE STEP....
960 REM
970 REM..TO FIND THE LBV...
980 G=0
990 FOR I=2 TO M1
1000 IF A(I,N1)>=0 THEN 1040
1010 IF G >= ABS(A(I,N1)) THEN 1040
1020 G=ABS(A(I,N1))
1030 I1=I
1040 NEXT I
1050 REM.
1060 REM.. I1 IS THE PIVOT-ROW NO. AND X(I1) IS THE LBV.
1070 REM.
1080 REM.. TO FIND THE EBV..
```

```
1090 K=0
1100 FOR J=1 TO N1-1
1110 FOR I=2 TO M1
1120 IF B(I)=J THEN 1200
1130 NEXT I
1140 IF A(I1,J)>=0 THEN 1200
1150 R(J)=A(1,J)/ABS(A(I1,J))
1160 K=1
1170 S=R(J)
1180 J1=J
1190 GOTO 1210
1200 R(J)=1E+38
1210 NEXT J
1220 REM.. TEST OF UNBOUND SOLUTION...
1230 IF K<>0 THEN 1290
1240 PRINT ON(1)"UNBOUND SOLUTION ...."
1260 STOP
1270 REM. USE S TO TEST FOR THE SMALLEST RATIO AND THEN TEST IF THERE
1280 REM. IS ONE OR MORE RATIOS HAVE THE SAME SMALLEST RALIO (TIE(S))
1290 FOR J=1 TO N1-1
1300 IF R(J)=1E+38 THEN 1340
1310 IF S<R(J) THEN 1340
1320 S=R(J)
1330 J1=J
1340 NEXT J
1350 REM. J1 IS THE PIVOT-COLUMN NO. AND X(J1 )IS THE EBV..
1360 REM.. TO TEST IF THERE IS A TIE(S)...
1370 T=1
1380 FOR J=1 TO N1-1
1390 IF J=J1 THEN 1440
1400 IF R(J1)<>R(J) THEN  1430
1410 T=T+1
1420 GOTO 1440
1430 R(J)=1E+38
1440 NEXT J
1450 IF T=1 THEN 1540
1460 PRINT ON(1)
1470 PRINT ON(1)"THERE ARE",T," TIE(S)"
1490 REM.
1500 REM..IF YOU WANT TO RESOLVE DEGENERACY, YOU HAVE TO INCLUDE
1510 REM.THE SUBPROGRAM OF THIS PURPOSE. OTHERWISE, CONTINUE...
1520 STOP
1530 REM . THE NEW SIMPLEX TABLEAU...
1540 B(I1)=J1
1550 P=A(I1,J1)
1560 FOR J=1 TO N1
1570 A(I1,J)=A(I1,J)/P
1580 NEXT J
1590 REM...
1600 FOR I=1 TO M
```

```
1610 IF I=I1 THEN 1660
1620 FOR J=1 TO N1
1630 IF J=J1 THEN 1650
1640 A(I,J)=A(I,J)-A(I,J1)*A(I1,J)
1650 NEXT J
1660 NEXT I
1670 REM....
1680 FOR I=1 TO M
1690 IF I=I1 THEN 1710
1700 A(I,J1)=0
1710 NEXT I
1720 REM....
1730 L=L+1
1740 GOTO 380
1750 END
```

ITERATION NO. 0

	1	2	3	
0	4	12	18	0
4	-1	0	-3	-3
5	0	-2	-2	-5

ITERATION NO. 1

	1	2	3	
0	4	0	6	-30
4	-1	0	-3	-3
2	0	1	1	2.5

ITERATION NO. 2

	1	2	3	
0	2	0	0	-36
3	.33333333	0	1	1
2	0	1	1	2.5

Z=	-36		
X(	3	)=	1
X(	2	)=	2.5

WHICH IS THE OPTIMAL SOLUTION

SECTION 5:

THE DUAL SIMPLEX ALGORITHM FOR SOLVING LP MODELS IN COMPACT FORM (DSA/COMPACT).

Outline of the Algorithm:

I. The initialization Step:

i) Start with a compact table which is optimal but not necessarly be feasible.

ii) Go to the stopping rule to see if this solution is feasible, and therefore optimal, or not?

II. The Stopping Rule:

To determine if this solution is feasible, and therefore optimal, it necessary to check if all the basic variables have non-negative values in the v.h.s. column? If, so stop: Otherwise, go to the Iterative step.

III. The Iterative Step:

i) Determine the L.B.V

Choose the pivot-row I1 by picking the greatest-ve value in the r.h.s. column.

ii) Determine the EBV:

i) Using only the negative entries in the pivot-row, compute the ratios of the entries in the first row (eq. o) to the corresponding entries in the pivot-row. Note that all of these ratios are negative.

2) Determine the pivot-column, J1, by picking the smallest absolute value of the ratios computed above.

iii) Determine the New dual simplex table

1. New pivot-row (except the pivot-element itself)

$$= \frac{\text{old pivot-row}}{\text{pivot-element}}$$

2. New pivot-column (except the pivot-element itself)

$$= \frac{\text{old pivot-column}}{\text{pivot-element}}$$

3. For the remaining entries (except the pivot-element)

New row = old row - "pivot-column Coeff" x New pivot-row; "pivot-column coeff" is the element in this row that is in the pivot-column.

4. New pivot element $= \dfrac{1}{\text{old pivot - element}}$

5. Interchange the subscripts of the LBV and EBV.

Example:

$$\text{Max } (-y_0) = -4y_1 - 12y_2 - 18y_3$$

s.to:

$$y_1 \qquad - 3y_2 \leq -3$$

$$-2y_1 - 2y_2 \qquad \leq -5$$

& $y_i \geq 0 \quad \forall i$

The successive dual simplex tables in compact form are:

Iteration 0

	1	2	3	
$-y_0$	4	12	18	0
4	-1	0	-3	-3
5	0	-2	-2	-5

M = 2
N = 3

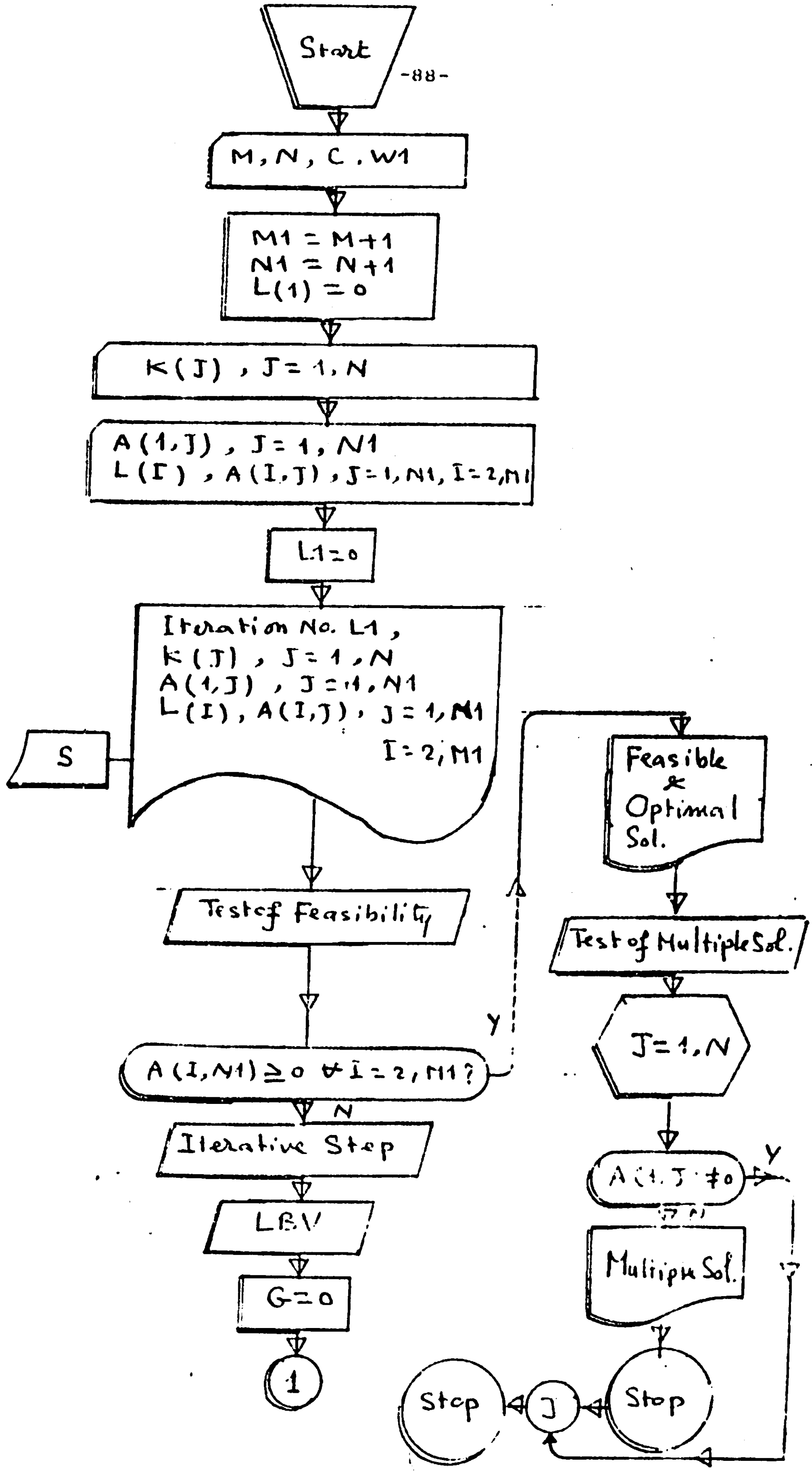
Iteration 1

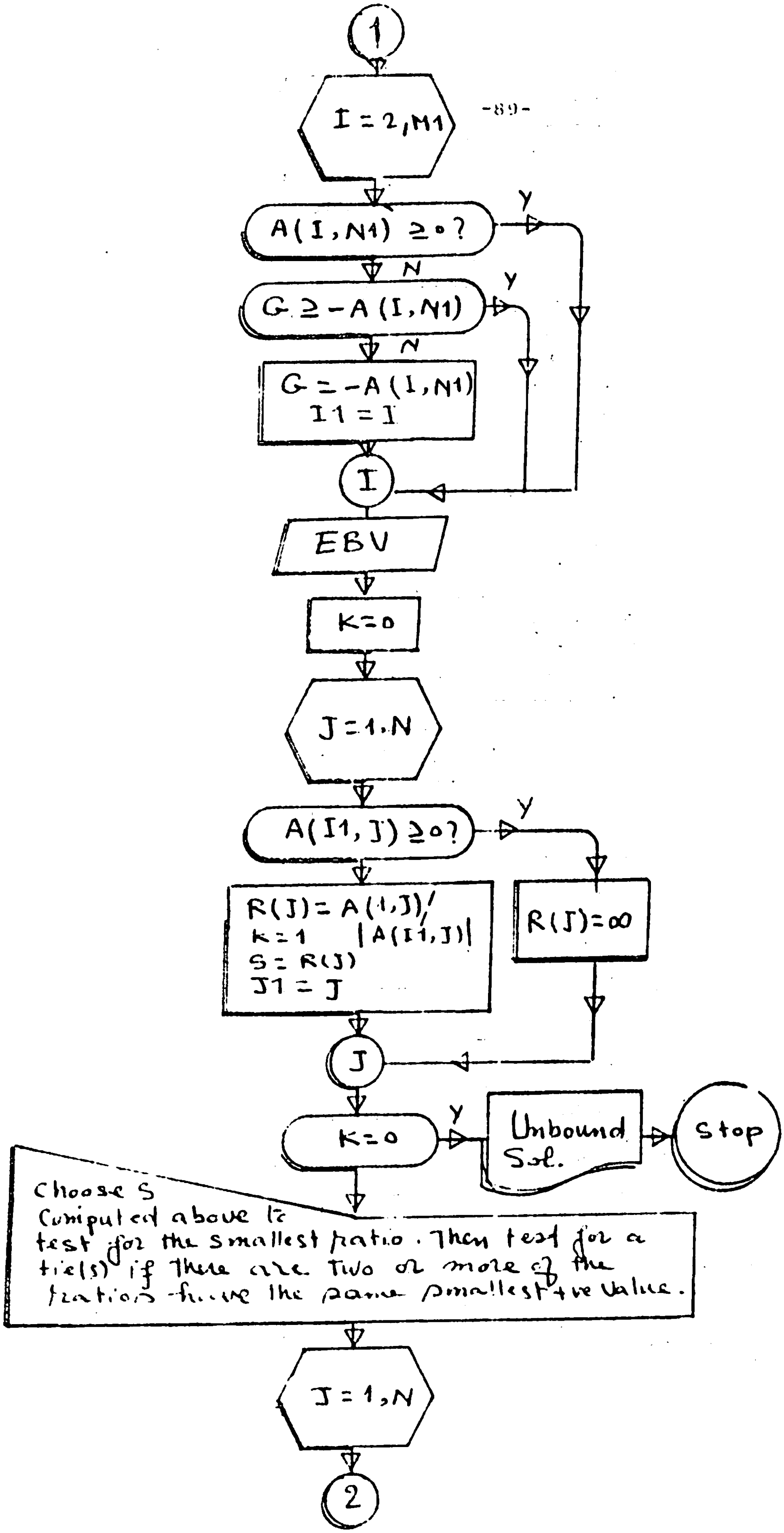
	1	5	3	
$-y_0$	4	6	6	-30
4	-1	0	-3	- 3
2	0	-½	1	5/2

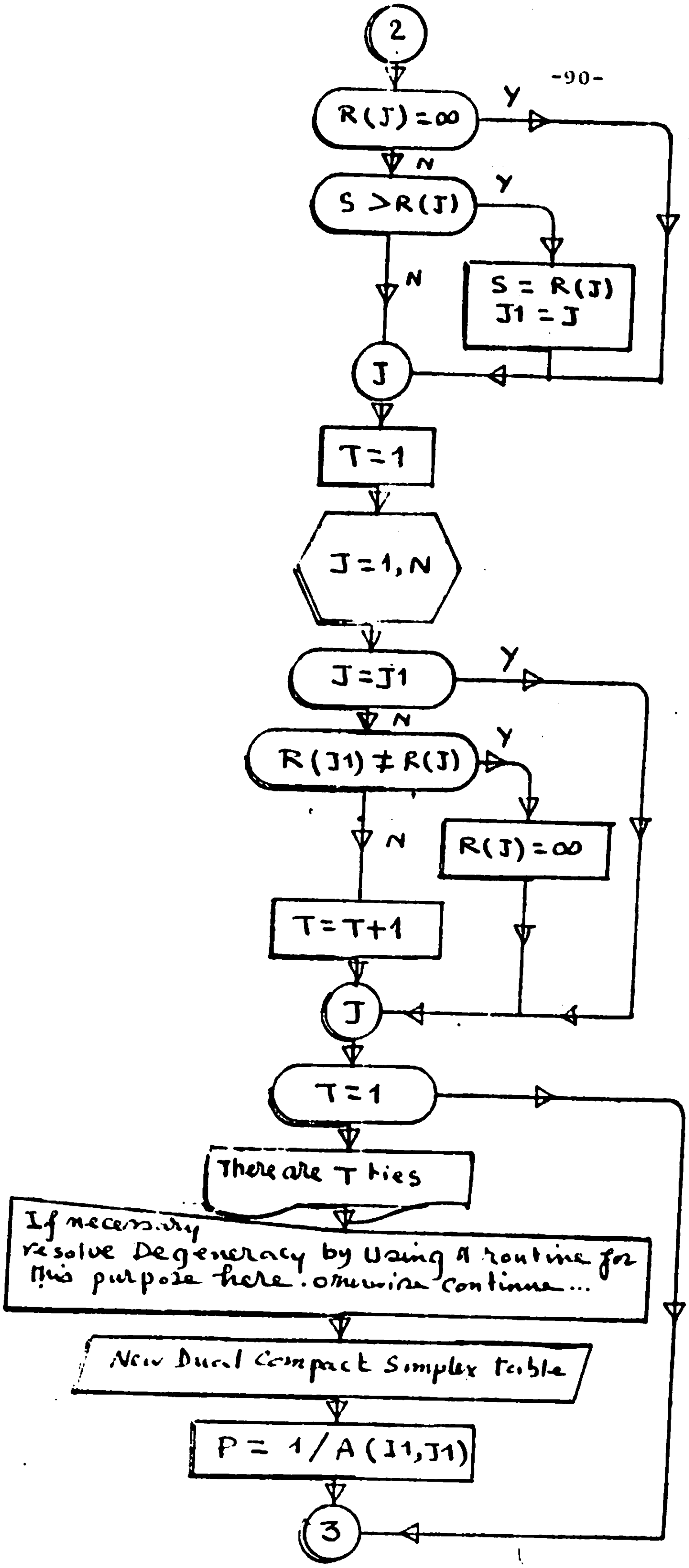
	1	5	4	
$-y_0$	2	6	2	-36
3	1/3	0	-1/3	3/2
2	-1/3	-½	1/3	3/2

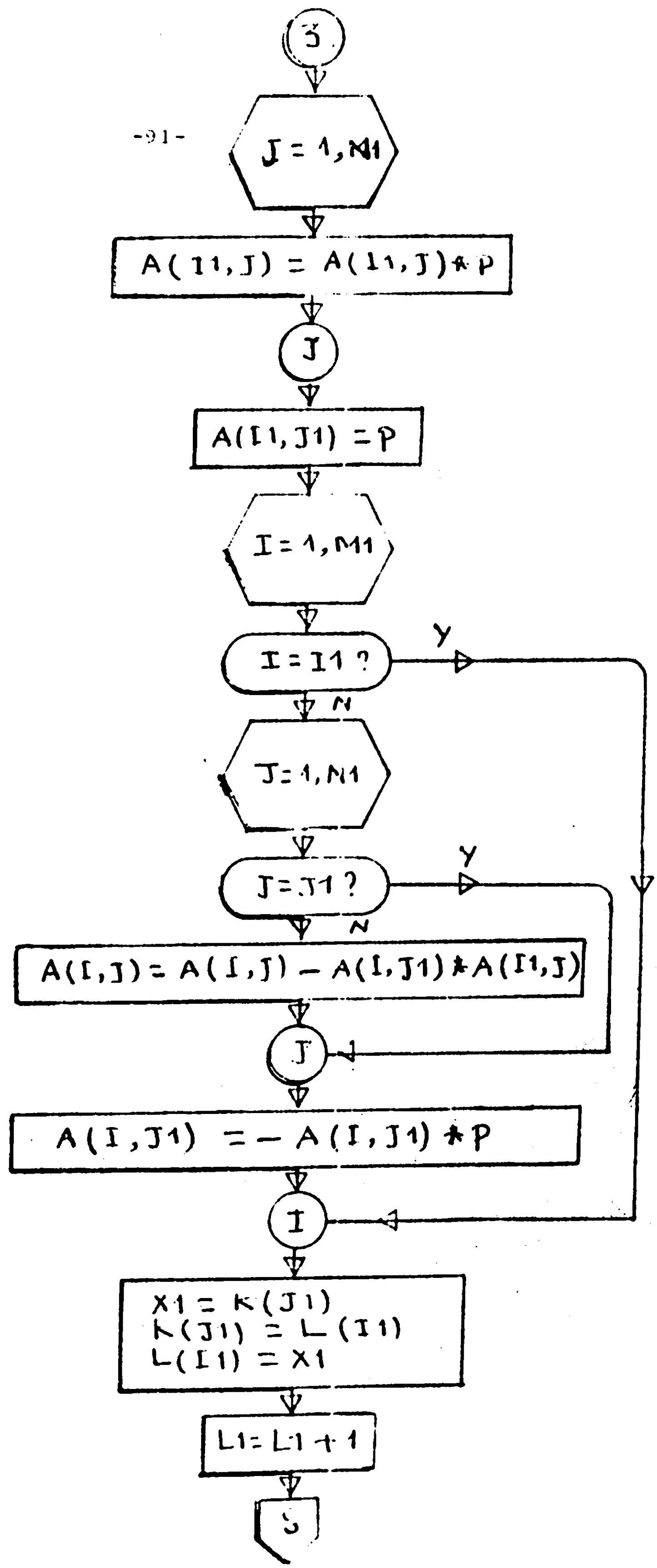
The optimal and feasible solution is:

$$y_2 = 3/2, \quad y_3 = 1 \quad \& \quad y_0 = 36$$


Start
M, N, C, W1
M1 = M+1
N1 = N+1
L(1) = 0
K(J), J = 1, N
A(1, J), J = 1, N1
L(I), A(I, J), J = 1, N1, I = 2, M1
L1 = 0
Iteration No. L1,
K(J), J = 1, N
A(1, J), J = 1, N1
L(I), A(I, J), J = 1, M1
I = 2, M1
S
Test of Feasibility
A(I, N1) ≥ 0 ∀ I = 2, M1 ?
Y
N
Iterative Step
LBV
G = 0
1
Feasible & Optimal Sol.
Test of Multiple Sol.
J = 1, N
A(1, J) ≠ 0
Y
Multiple Sol.
Stop
J
Stop


1
I = 2,M1
A(I,N1) ≥ 0 ?
Y
N
G ≥ -A(I,N1)
Y
N
G = -A(I,N1)
I1 = I
I
EBV
K=0
J=1,N
A(I1,J) ≥ 0?
Y
R(J)= A(1,J)/|A(I1,J)|
k=1
S= R(J)
J1 = J
R(J)=∞
J
K=0
Y
Unbound Sol.
Stop
Choose S Computed above to test for the smallest ratio. Then test for a tie(s) if there are two or more of the ratios have the same smallest +ve value.
J = 1,N
2


2
R(J)=∞
Y
N
S > R(J)
Y
N
S = R(J)
J1 = J
J
T=1
J=1,N
J=J1
Y
N
R(J1) ≠ R(J)
Y
N
R(J)=∞
T=T+1
J
T=1
There are T ties
If necessary
resolve Degeneracy by Using a routine for this purpose here. otherwise continue...
New Dual Compact Simplex table
P = 1/A(J1,J1)
3



3
J = 1, M1
A(I1, J) = A(I1, J) * P
J
A(I1, J1) = P
I = 1, M1
I = I1 ?
Y
N
J = 1, M1
J = J1 ?
Y
N
A(I,J) = A(I,J) – A(I,J1) * A(I1,J)
J
A(I, J1) = – A(I, J1) * P
I
X1 = K(J1)
K(J1) = L(I1)
L(I1) = X1
L1 = L1 + 1
5

```
10 REM......[illegible]
20 REM. A PROGRAM TO SOLVE LP MODELS BY [illegible] IN
30 REM IN COMPACT FORM.
40 REM. THIS PROGRAM DOES NOT WORK WITH THE [illegible]
50 REM. IN CASE OF DEGENERACY.
60 REM. PROGRAMMED BY ABDALLA [illegible]
[illegible]
80 REM. ...............
90 REM.
[illegible]
110 PRINT "ENTER [illegible] NO.,"PRINTER LIN[illegible]"
[illegible]
130 REM. M=NO. OF CONSTRAINTS
140 REM. N=NO. OF ORDINARY VARIABLES.
150 M1=M+1
160 N1=N+1
170 PRINT "[illegible]"
180 FOR J=1 TO N
190 INPUT E(J)
200 NEXT J
210 PRINT "ENTER INITIAL [illegible]"
220 FOR I=1 TO M1
230 INPUT E(I)
240 FOR J=1 TO N1
250 INPUT A(I,J)
260 NEXT J
270 NEXT I
280 REM.
290 REM.OUTPUT OF ITERATION NO.I1.....
300 REM.
310 I1=0
320 WRITE (N1,*)
330 WRITE (N1,340)I1
340 FORMAT "ITERATION NO. ",F3.0,/14"-"
350 WRITE (N1,*)
360 WRITE (N1,370)E(1)
370 FORMAT 7X,F2.0
380 FOR J=2 TO N
390 WRITE (N1,400)E(J)
400 FORMAT 9X,F2.0
410 NEXT J
420 WRITE (N1,*)
430 FOR I=1 TO M1
440 WRITE (N1,450)E(I)
450 FORMAT F2.0
460 FOR J=1 TO N1
470 WRITE (N1,480)A(I,J)
480 FORMAT 7F10.3
490 NEXT J
500 WRITE (N1,*)
```

```
510 NEXT I
520 REM.
530 REM...TEST OF FEASIBILITY...
540 REM.
550 FOR I=2 TO M1
560 IF A[I,N1]<0 THEN 860
570 NEXT I
580 REM.
590 REM... FEASIBLE AND THERFORE OPTIMAL..
600 REM.
610 WRITE (W1,*)
620 WRITE (W1,630)A[1,N1]
630 FORMAT 5X,"Z = ",F12.3
640 FOR I=2 TO M1
650 WRITE (W1,660)L[I],A[I,N1]
660 FORMAT "X ( ",F2.0," ) =",F10.3
670 WRITE (W1,*)
680 NEXT I
690 WRITE (W1,*)
700 WRITE (W1,710)
710 FORMAT 15"-","WHICH IS FEASIBLE AND OPTIMAL",15"-"
720 REM.
730 REM...TEST OF MULTIPLE SOLUTION...
740 REM.
750 FOR J=1 TO N
760 IF A[1,J]#0 THEN 810
770 WRITE (W1,*)
780 WRITE (W1,790)
790 FORMAT /,13"-","AND MULTIPLE SOLUTIONS",10"-"
800 GOTO 110
810 NEXT J
820 GOTO 110
830 REM.
840 REM...ITERATIVE STEP...
850 REM.
860 REM..LBV...
870 G=0
880 FOR I=2 TO M1
890 IF A[I,N1] >= 0 THEN 930
900 IF G >= -A[I,N1] THEN 930
910 G=-A[I,N1]
920 I1=I
930 NEXT I
940 REM.
950 REM..I1 IS THE PIVOT-ROW & L(I1) IS THE SUBSC. OF THE LBV..
960 REM.
970 REM...EBV...
980 K=0
990 FOR J=1 TO N
1000 IF (A[I1,J]+1E-06) >= 0 THEN 1060
```

```
1010 R[J]=A[I,[illegible]]/A[I,J]
1020 K=1
1030 S=R[J]
1040 J1=J
1055 GOTO 1070
1060 R[J]=1E+38
1070 NEXT J
1080 REM.
1090 REM...TEST OF UNBOUND SOLUTION...
1100 REM.
1110 IF K=0 THEN 1190
1120 WRITE (W1,1130)
1130 FORMAT "..... UNBOUND SOLUTION ....."
1140 GOTO 110
1150 REM.
1160 REM..USE S ABOVE TO TEST FOR THE SMALLEST RATIO
1170 REM. AND THEN TEST IF THERE IS A TIE...
1180 REM.
1190 FOR J=1 TO N
1200 IF R[J]=1E+38 THEN 1240
1210 IF S<R[J] THEN 1240
1220 S=R[J]
1230 J1=J
1240 NEXT J
1250 REM..J1 IS THE PIVOT-COL & R[J1] IS THE SUBSC. OF THE KEY..
1260 REM...TEST IF THERE IS A TIE(S)...
1270 T=1
1280 FOR J=1 TO N
1290 IF J=J1 THEN 1340
1300 IF R[J1]#R[J] THEN 1330
1310 T=T+1
1320 GOTO 1340
1330 R[J]=1E+38
1340 NEXT J
1350 IF T=1 THEN 1510
1360 WRITE (W1,*)
1370 WRITE (W1,1380)T
1380 FORMAT "THERE ARE ",F2.0," TIE(S) AT THE COLS :"
1390 WRITE (W1,*)
1400 FOR J=1 TO N
1410 IF R[J]=1E+38 THEN 1440
1420 WRITE (W1,1430)J
1430 FORMAT 10X,F2.0
1440 NEXT J
1450 REM.
1460 REM...IF YOU WANT TO RESOLVE CYCLING DUE TO DEGENERACY
1470 REM. YOU HAVE TO INCLUDE ROUTINE HERE FOR THIS PURPOSE. .
1480 REM.
1490 REM...THE NEW DUAL S.TABLE IN COMPACT FORM...
1500 REM.
```

```
1510 P=1/A[I1,J1]
1520 FOR J=1 TO N1
1530 A[I1,J]=A[I1,J]*P
1540 NEXT J
1550 REM.
1560 A[I1,J1]=P
1570 REM.
1580 FOR I=1 TO M1
1590 IF I=I1 THEN 1690
1600 FOR J=1 TO N1
1610 IF J=J1 THEN 1650
1620 REM.
1630 A[I,J]=A[I,J]-A[I,J1]*A[I1,J]
1640 REM.
1650 NEXT J
1660 REM.
1670 A[I,J1]=-A[I,J1]*P
1680 REM.
1690 NEXT I
1700 REM.
1710 X1=L[J1]
1720 L[J1]=L[I1]
1730 L[I1]=X1
1740 REM.
1750 L1=L1+1
1760 GOTO 330
1770 REM.
1780 END
```

ENTER M,N,CASE NO [illegible]PRINTER CODE
ENTER SUBSCRIPTS OF INITIAL NON BASIC VS.
ENTER INITIAL S.TABLE WITH SUBSCRIPTS OF B.V.S.

ITERATION NO. 0

	1	2	3	[illegible]
0	4.000	12.000	10.000	0.000
4	-1.000	0.000	-3.000	-2.000
5	0.000	-2.000	-2.000	-5.000

ITERATION NO. 1

	1	5	[illegible]	
0	4.000	6.000	6.000	-30.000
4	-1.000	0.000	-3.000	-3.000
2	0.000	-0.500	1.000	2.500

ITERATION NO. 2

	1	5		
[illegible]	2.000	2.000	2.000	-36.000
3	0.333	0.000	-0.333	1.000
2	0.330	-0.500	0.333	1.500

Z = -36.000
X (3) = 1.000

X (2) = 1.500

---------------------- WHICH IS FEASIBLE AND OPTIMAL
ENTER M,N,CASE NO.S&PRINTER CODE

SECTION 6:

The Primal-Dual Algorithm For Solving Mixed System LP Models (PDA/Mixed) in Compact form

INTRODUCTION:

By a combination of the OSA and the DSA, artificial variables may be avoided completely. Avoiding artifical variables reduces the size of the table and usually makes a marked reduction in the number of iteration necessary to optimize the model.

Example: This ex is due to C.S. Wolfe (2)

	1	2	
z	-2	-1	0
3	1	7	63
4	-3	-1	-9
5	-3	-2	-15
6	-5	-6	-30
7	-1	-4	- 8
8	8	3	80

This table is neither optimal nor feasible. This difficulty can be handled by a combination of the two algorithms mentioned above. All possible pivot-elements should be considered by both the OSA and DSA.

In our example, the DSA produces no pivot-element since all the elements in eq o are all-ve. Applying the OSA, the first column gives us the pivot-column and the 7th row gives us the pivot-row. i.e., the pivot-element is the element circled above

After pivoting operation, we get the following table:

	8	2	
Z	.25	-.25	20
3	-.125	6.615	53
4	.375	.125	21
5	.375	.875	15
6	.625	4.125	20
7	.125	-3.625	2
1	.125	.375	10

This table is <u>feasible</u> but not optimal.

The next pivoting operation is determined by picking the pivot-element circled above. This leads to the following final table:

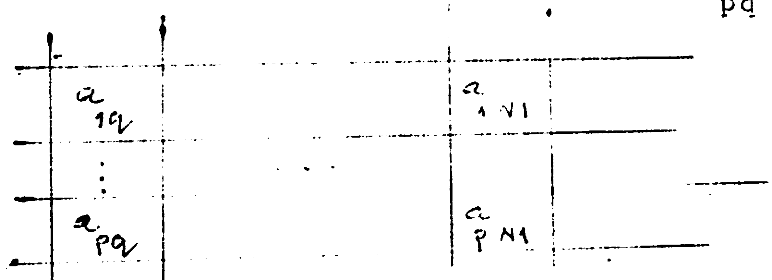
	8	3	
Z	0.245	0.038	22
2	-. 019	.151	8
4	.377	-.019	20
5	.358	.132	22
6	.547	.623	53
7	.057	.547	31
1	.132	-.057	7

The same result could be found by using the artificial variables which requires 7 iterations instead of the 2 iterations required here. Moreover, the table size has been reduced from 8 x 3 to 7 x 3.

Anyhow, the main problem arises when the table is neither Feasiable nor optimal and the two algorithms, mentioned above, produce different pivot-elements. In this case, a choice may be made by answering the questions: Which Pivot-element makes the greater favourable change in the objective value? A Pivot-element chosen by the OSA increases the value of the objective function and advance toward optimality, while a pivot-element by the DSA decreases the value of the objective function and advances towards Feasibility.

We know that the value of the objective function after the pivoting operation with a_{pq} as a pivot-element is:

$$a'_{1,N1} = a_{1,N1} - \left(a_{1q} \cdot \frac{a_{p,N1}}{a_{pq}} \right)$$



This means that the net change in the objective function value is the absolute value of the value between brackets.

Let us refer to the Increase in Z - value, due to a pivot-element chosen by OSA, by I_p

$$\text{i.e.} \qquad I_p = \left| a_{1q} \cdot \frac{a_{p,N1}}{a_{pq}} \right|$$

The corresponding Decrease in z-value due to a pivot-element chosen by DSA is D_q

$$\text{i.e.} \qquad D_q = \left| a_{PN1} \cdot \frac{a_{1q}}{a_{pq}} \right|$$

Now, when the two pivot-elements are possible, the choice may be made by picking the pivot-element corresponding to the larger of I_p and D_q. Thus, the next table will make the largest possible change in the computation procedure either towards optimality or towards Feasibility. In most cases, this choice should result in the fewest number of pivoting operations necessary to reach the final table.

Let us solve the following illustrative exemple:

Max. $Z = x_1 + x_2 - 2x_3 - 4x_4$

S.to:

$$x_1 - 2x_2 + 4x_3 - 3x_4 = 10$$

$$2x_1 + 3x_2 - x_3 + 5x_4 \leq 15$$

$$3x_1 - x_2 + 2x_3 + 3x_4 \geq 12$$

$$x_j \geq 0, \; j = 1,2,3,4.$$

Firstly, the equality constraint is replaced by two inequalities and we use only slack variables so that the Initial Simplex table in compact form appears as follows:

	1	2	3	4	
Z	-1	-1	2	4	0
5	1	-2	4	-3	10
6	-1	2	-4	3	-10
7	(2)	3	-1	5	15
8	-3	1	(-2)	-3	-12

This table is neither Feasible nor Optimal.

- Applying the OSA, we find that:

$$I_p = I_4 = \left| a_{11} \cdot \frac{a_{45}}{a_{41}} \right| = \frac{1 \times 15}{2} = 7.5$$

- Applying the DSA, we find that:

$$D_q = D_3 = \left| a_{55} \cdot \frac{a_{13}}{a_{53}} \right| = \frac{12 \times 2}{2} = 12$$

- ∴ $D_q \geq I_p$, we pick the pivot-element associated with the DSA. i.e. the pivot-element is at (5,3)Ø = -2.

	1	2	8	4	
Z	-4	0	1	1	-12
5	-5	0	2	(-9)	-14
6	(5)	0	-2	9	14
7	3.5	2.5	-.5	6.5	27
3	1,5	-.5	-.5	1.5	6

Also, this table is neither Feasiable nor optimal. Apply both the OSA and DSA, we get:

$$I_3 = 11,2 \qquad D_4 = 1.56$$

$I_3 > D_4$, we pick the pivot-element associated with the OSA to obtain the following table:

	6	2	8	4	
Z	.8	0	-.6	8.2	-.8
5	1	0	0	0	0
1	.2	0	-.4	1.8	2.8
7	-.7	2.5	-9	.2	11.2
3	-.3	-.5	1	-1.2	1.8

This table is Feasible but not Optimal.

A Final table is determined by choosing the pivot-element circled above so that the final table is:

	6	2	7	4	
Z	.333	1.677	.667	8.333	6.667
5	1	0	0	0	0
1	.111	1.111	.444	1.889	7.778
8	-.778	2.778	1.111	.222	12.444
3	-.222	-.778	-.111	-1.222	.556

Note that the slack variables x_5 and x_6 are necessarily both zero since they arose from an equality constraint. This final table appears to be degenerate with a Basic variable =0. However, No further pivoting - operation is possible and so the solution is unique. Slack variables associated with an equality constraint must be zero in the final table and do not represent a true degeneracy in the optimal basic feasible solution. (The solution to this ex can be achieved in 4 iterations by using artificial variables).

Let us summarize this Algorithm:

1. All constraints must be inequalities of the form:

$$\sum_{j \rightarrow} a_{ij} x_j \leq b_i \quad \forall\ i$$

where b_i may be negative, zero, or positive.

2. The objective function is in maximization form.

3. The initial compact simplex table is constructed by using only slack variables. This table may be Infeasible, Not Optimal, or both.

4. Applying the OSA;

   i) For most - ve coeff in z-row, determine J1 (the pivot-column No.).

   ii) I1 (the pivot-row No.) can be determined such as

$$\frac{a_{I1,N1}}{a_{I1,J1}} = \min \left(\frac{a_{i,N1}}{a_{i,J1}} \right) ; \quad a_{i,J1} > 0, \ a_{i,N1} \geq 0$$

   iii) If no pivot-element is available, try with the next most-ve of the remaining coeffs in Z-row to determine J1. Repeat this process until either a pivot-element is found or none is available.

   iv) If No pivot-element is available, go to the DSA.

   v) If a pivot-element is found, store I1 and J1and compute only the increase in Z-value from the formula:

$$P9 = \left| a_{1,j1} \cdot \frac{a_{I1,\ N1}}{a_{I1,\ j1}} \right|$$

Then go to the DSA

5. Apply the DSA:

   i) For the most negative element in the r.h.s., determine the pivot-row no., I2,

   ii) J2, the pivot-column no., can be determined such that:

$$\left(\frac{a_{1,j2}}{a_{I2,J2}} \right) = \min_{j} \left(\frac{a_{1,j}}{|a_{I2,j}|} \right) \quad \begin{matrix} a_{I2,j} < 0 \\ a_{1,j} \geq 0 \end{matrix} \quad \forall j .$$

   iii) If no pivot-element is available, try with the most negative element of the remaining elements in the r.h.s column until either a pivot-element is found or none exists.

   iv) If no pivot-element is available, check if this is the final table (Feasible and Optimal) or there is no basic feasible solution to the model under consideration.

v) In case of available pivot-element due to this DSA, compute the decrease in Z- value from the formula:

$$D9 = \left| a_{1,j2} \cdot a_{I2,N1} \Big/ a_{I2,J2} \right|$$

6. If both steps 4 and 5 each produce a pivot-element, compare P9 with D9 and choose the pivot-element associated with the larger of them.

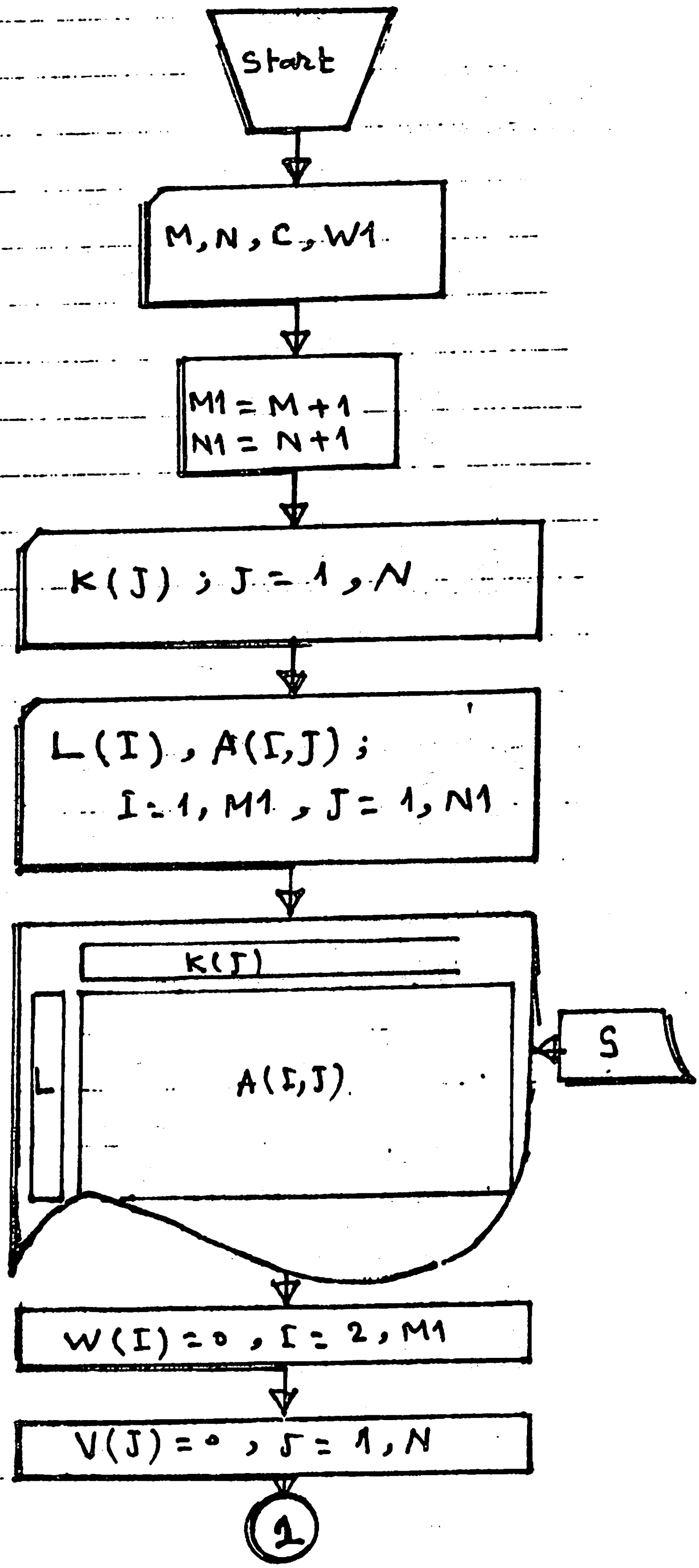
7. Compute the new simplex table by using the pivot-element chosen above.

8. Check for Feasibility and optimality and repeat steps 4 through 7 until the final table is reached or no pivot-element can be found and therefore no basic feasible solution is available.

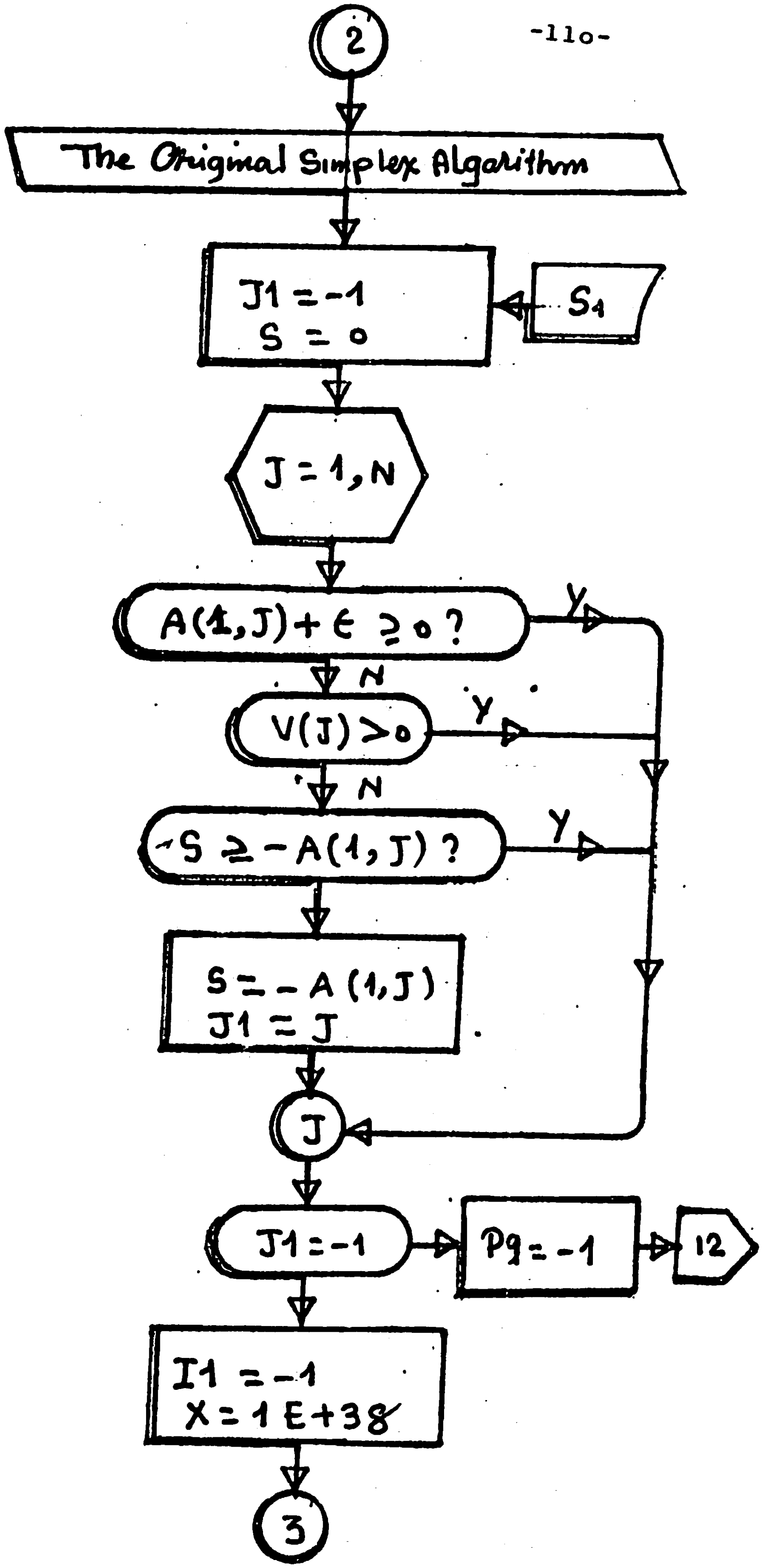
Interpretation of the Final Table:

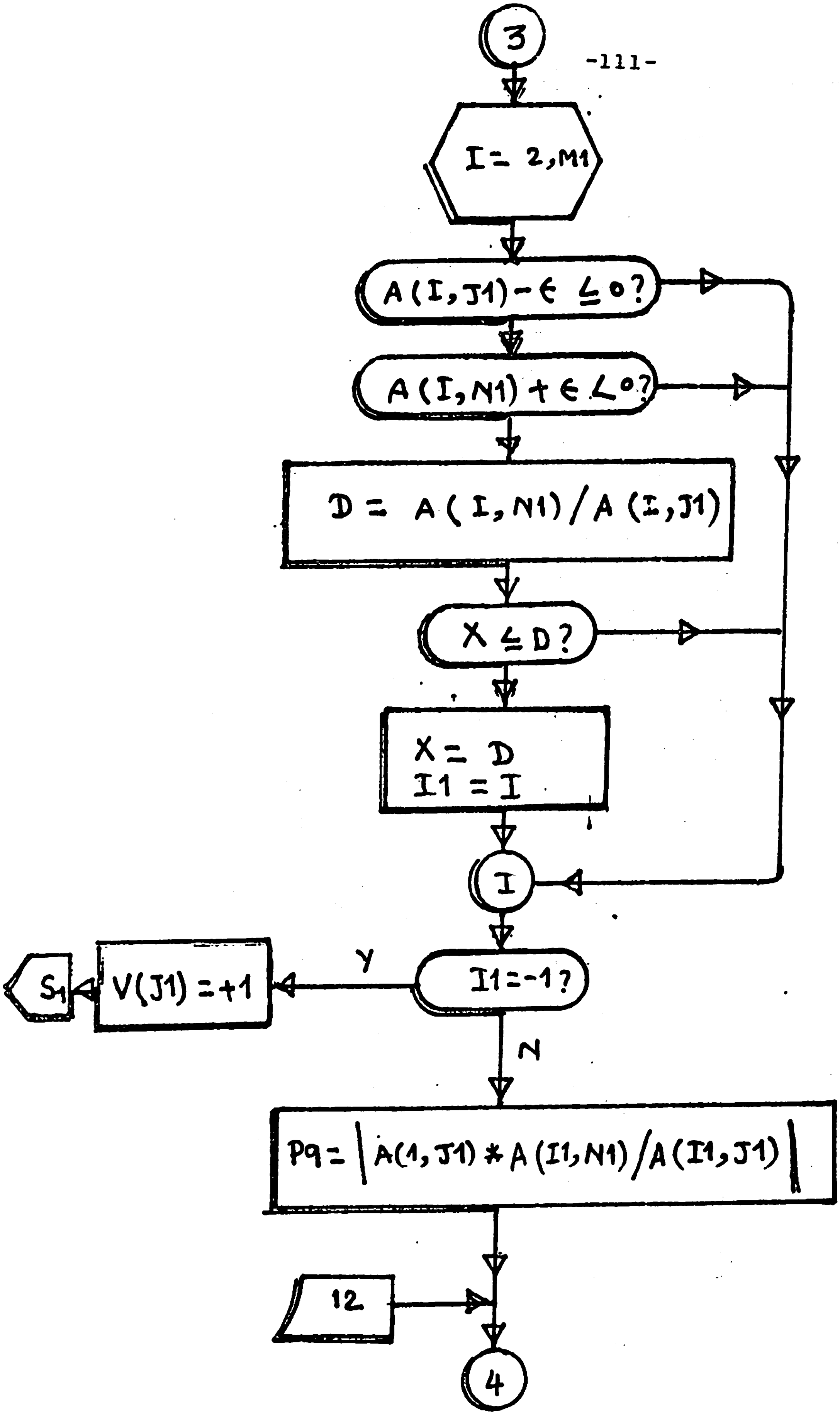
1. The final table may be feasible and optimal. In this case a solution has been found to both the primal and Dual programe of the model.
2. The solution may be degenerate, bot true degeneracy should be distingwished from that arising due to equality constraints. For any equality constraint to be satisfied, its two associated out of the Basic, i.e. already zero, but the other remains in the solution at zero-value. This is not a true degeneracy as was in our example.
3. The final table may be Optimal but not feasible and no pivotelement can be found, then the primal program has no feasible solution and its Dual is unbounded,
4. The final table may be Feasible nut not optimal (since on e or more element in z-row still-ve) due to the non-positive elements of the pivot-column. then, the primal program is unbounded and the Dual program has no feasible solution.
5. The final table may also both Infeasible and not Optimal.
6. One of the above situations will occur so that the original model either has an optimal solution, or it is unbounded, or it is inconsistent, or it is both unbounede and Inconsistent.
7. Care must be taken to distinguish between the Basic and Non-basic Variables in reading of the answers from the final

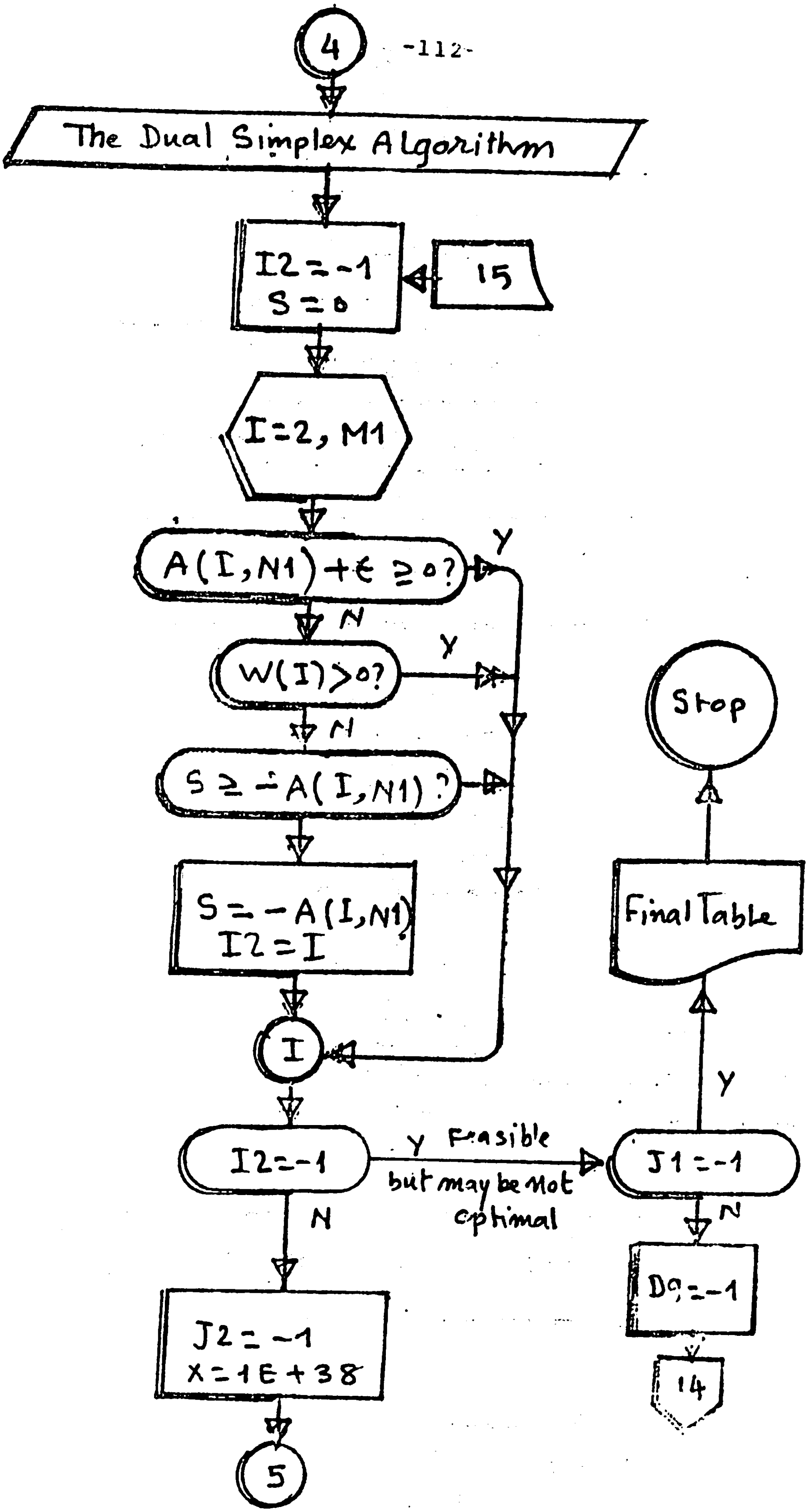
table. For the Primal program, the Basic Variables are denoted by the subscripts L(I) to the left of the table, and their values are found in the r.h.s. column. For the Dual program, the Basic Variables are denoted by the subscripts K(J) across the top of the top ofthe table, and their values are found in the Z-row. In both cases, the Non-basic variables are always zero.

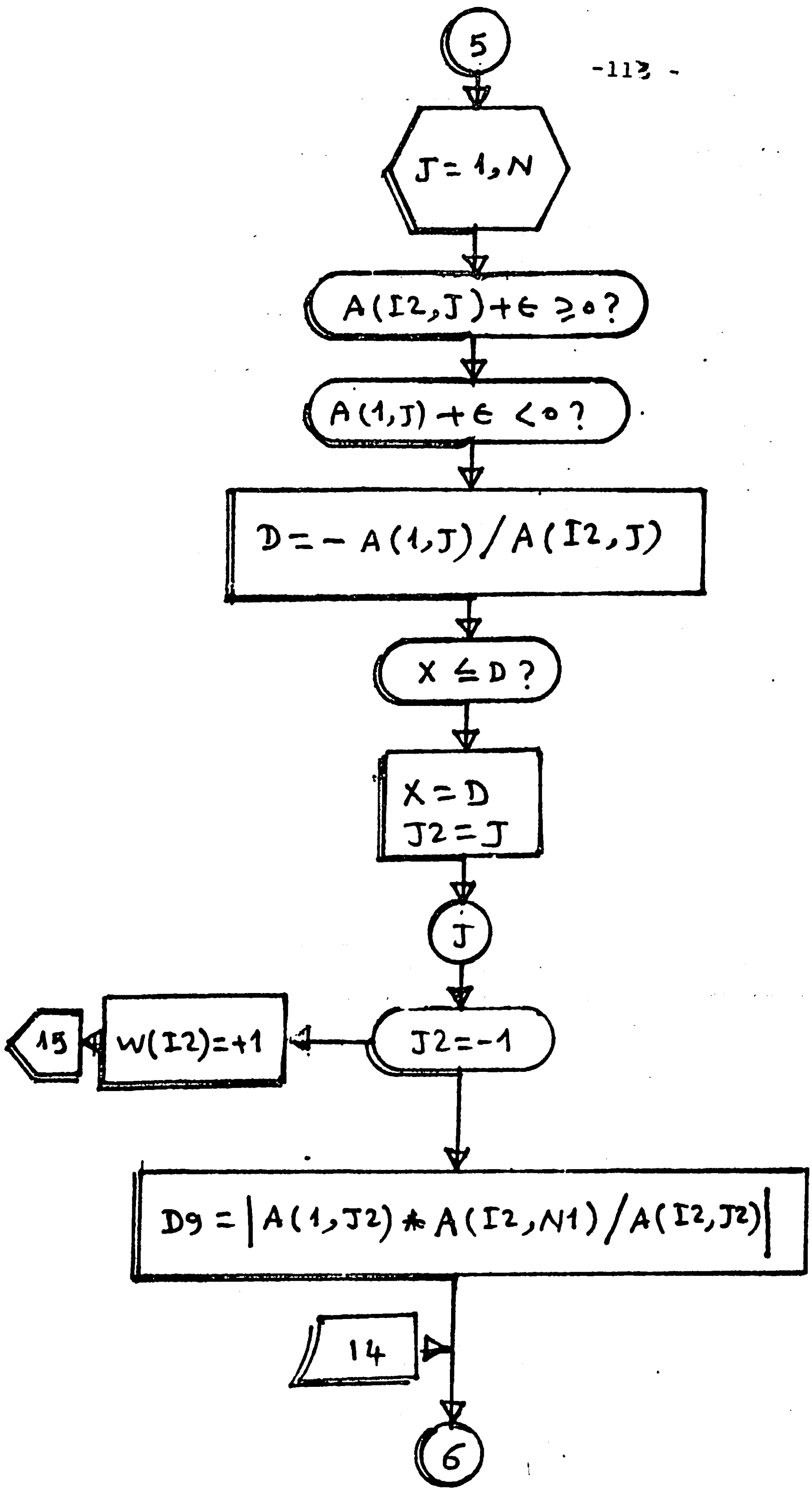


Start
M, N, C, W1
M1 = M + 1
N1 = N + 1
K(J) ; J = 1, N
L(I), A(I,J) ;
I = 1, M1, J = 1, N1
K(J)
L
A(I,J)
S
W(I) = 0, I = 2, M1
V(J) = 0, J = 1, N
1

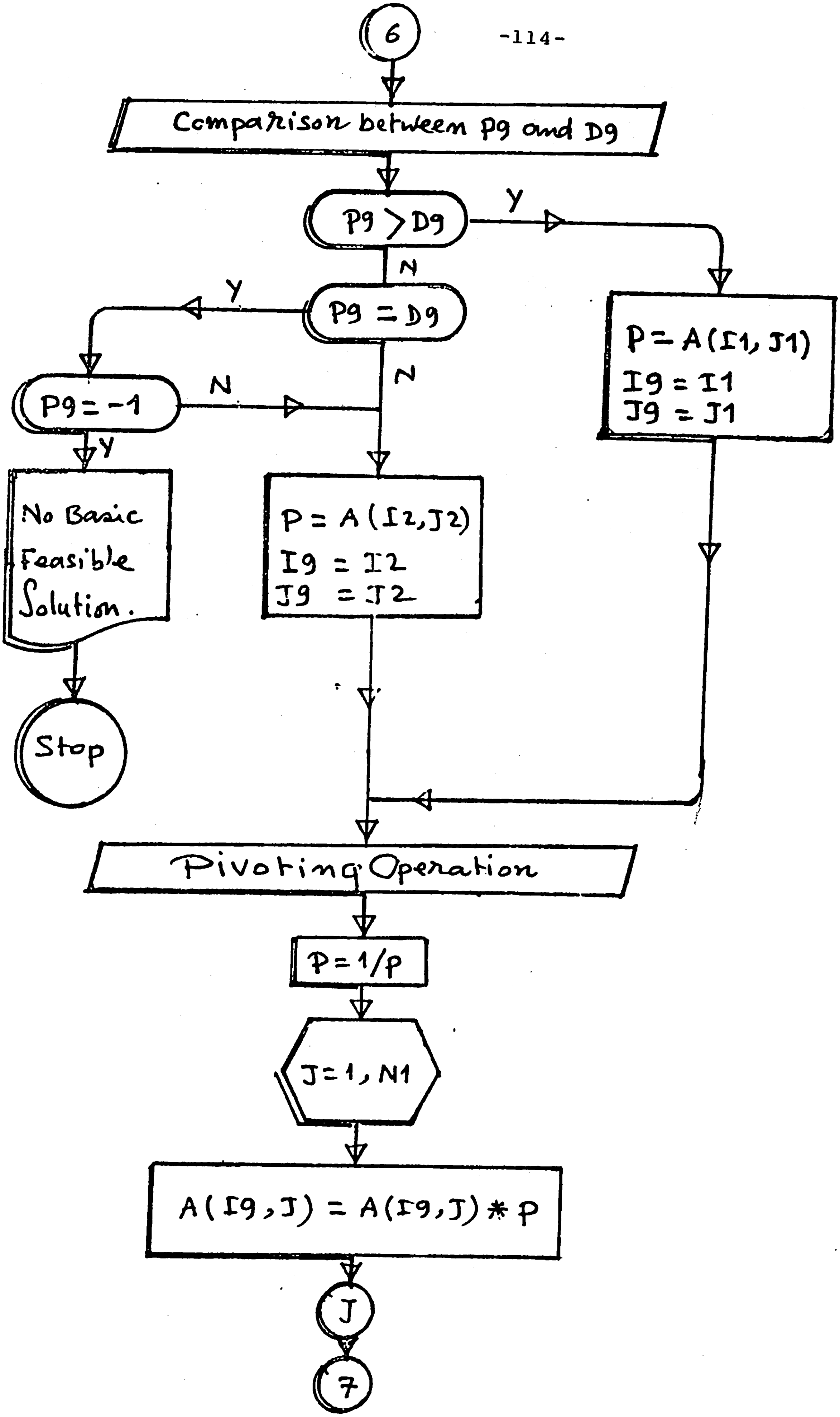

2
The Original Simplex Algorithm
J1 = -1
S = 0
S1
J = 1, N
A(1,J) + ε ≥ 0 ?
Y
N
V(J) > 0
Y
N
S ≥ - A(1, J) ?
Y
S = - A (1,J)
J1 = J
J
J1 = -1
Pq = -1
12
I1 = -1
X = 1 E+38
3

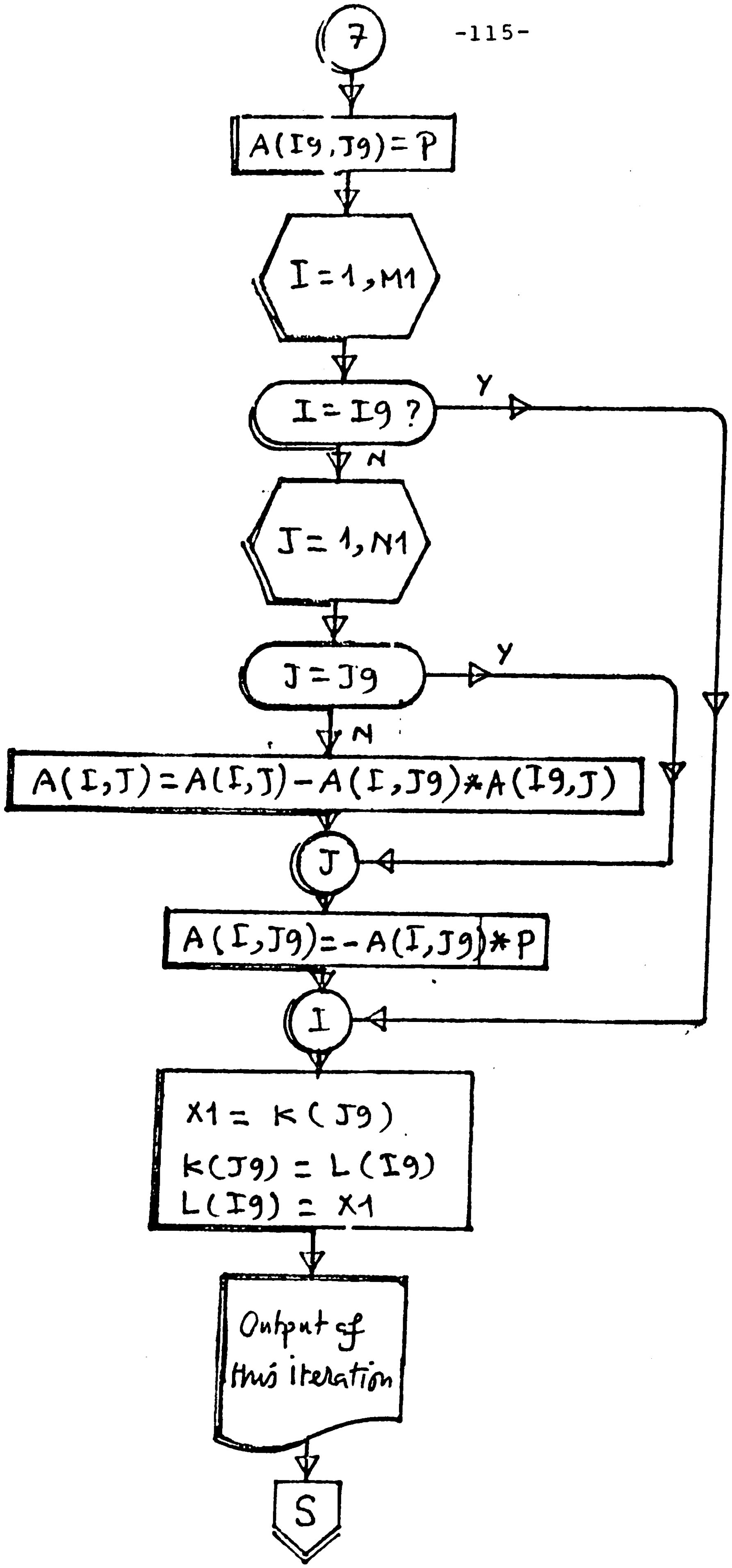

3
I= 2,M1
A(I,J1)-ε ≤ 0?
A(I,N1)+ε < 0?
D= A(I,N1)/A(I,J1)
X ≤ D?
X= D
I1 = I
I
I1=-1?
Y
V(J1) =+1
S1
N
Pq= |A(1,J1) * A(I1,N1)/A(I1,J1)|
12
4


4
The Dual Simplex Algorithm
I2 = -1
S = 0
15
I=2, M1
A(I,N1)+ε ≥ 0?
Y
N
W(I)>0?
Y
N
S ≥ -A(I,N1)?
S = -A(I,N1)
I2 = I
I
I2=-1
Y Feasible
but may be not
optimal
N
J2 = -1
X=1E+38
5
J1 = -1
Y
N
Final Table
Stop
D9 = -1
14



5
J = 1, N
A(I2,J) + ∈ ≥ 0 ?
A(1,J) + ∈ < 0 ?
D = − A(1,J) / A(I2,J)
X ≤ D ?
X = D
J2 = J
J
J2 = −1
W(I2) = +1
15
D9 = |A(1,J2) * A(I2,N1) / A(I2,J2)|
14
6


6
Comparison between P9 and D9
P9 > D9
Y
N
P9 = D9
Y
N
P = A(I1, J1)
I9 = I1
J9 = J1
P9 = -1
N
Y
No Basic Feasible Solution.
P = A(I2,J2)
I9 = I2
J9 = J2
Stop
Pivoting Operation
P = 1/P
J = 1, N1
A(I9,J) = A(I9,J) * P
J
7


7
A(I9,J9)=P
I=1,M1
I=I9 ?
Y
N
J=1,N1
J=J9
Y
N
A(I,J)=A(I,J)-A(I,J9)*A(I9,J)
J
A(I,J9)=-A(I,J9)*P
I
X1= K(J9)
K(J9) = L(I9)
L(I9) = X1
Output of this iteration
S

```
10 REM.
20 REM.
30 REM. A BASIC PROG.  OF PRIMAL-DUAL ALGORITHM FOR SOLVING
40 REM.  A MIXED-SYSTEM LP MODELS. . .
50 REM. PROGRAMMED BY ABDALLA EL-DAOUSHY
60 REM.  JUNE, 1982 . . . .
70 REM.  REFERENCES. C. S. WOLFE(LP WITH FORTRAN)
80 REM.  & F. S. HILLIER (OR) . . .
90 REM.
100 REM.
110 DIM A(20,30),W(20),L(20),K(30),V(30)
111 OPEN "PR:",1,1
120 PRINT "ENTER M,N,CASE NO."
130 INPUT M,N,C
140 REM.   M=NO.  OF CONSTRAINTS,
150 REM.   N=NO. OF ORDINARY VARIABLES.
160 M1=M+1
170 N1=N+1
180 PRINT  "ENTER THE SUBSCRIPTS OF NON-BASIC VS"
190 FOR J=1 TO N1-1
200 INPUT K(J)
210 NEXT J
220 PRINT "ENTER S.  TABLEAU WITH THE SUBSRIPTS OF THE BASIC VS"
230 FOR I=1 TO M1
240 INPUT L(I)
250 FOR J=1 TO N1
260 INPUT A(I,J)
270 NEXT J
280 NEXT I
290 REM.
300 REM.
310 REM.
320 T=0
321 PRINT ON(1)TAB(14)," ",
330 PRINT ON(1)K(1),
350 FOR J=2 TO N1-1
360 PRINT ON(1)K(J),
380 NEXT J
390 PRINT ON(1)
400 FOR I=1 TO M1
410 PRINT ON(1)L(I),
430 FOR J=1 TO N1
440 PRINT ON(1)A(I,J),
460 NEXT J
470 PRINT ON(1)
480 NEXT I
490 REM.  THE FOLLOWING W(I) & V(I) ARE INITIALIZED BY
500 REM. ZEROS AT THE BEGINING OF EVERY ITERATION
510 REM. W(I) WILL BE USED TO SKIP THE I-TH ROW IF THAT ROW HAS
520 REM. THE GREATEST -VE ELEMENT BUT WITHOUT POSSIBLE PIVOT-ELEMENT.
```

```
530 REM.  IN THIS CASE, THE NEXT GREATEST-VE ELEMENT IN THE R.H.S.COL.
540 REM.  WILL BE USED TO DETERMINE A POSSIBLE PIVOT-ELEMENT AND SO ON.
550 REM. THE SAME FOR V(J)....
560 REM. ROW I OR COL J WILL BE SKIPPED WHENEVER W(I) OR V(I) IS  +VE.
570 FOR I=1 TO M1
580 W(I)=0
590 NEXT I
600 FOR J=1 TO N1
610 V(J)=0
620 NEXT J
630 REM.  . . THE ORIGINAL SIMPLEX ALGORITHM. . .
640 REM.
650 REM.  . . EBV/OSA  . . .
660 J1=-1
670 S=0
680 FOR J=1 TO N1-1
690 IF  (A(1,J)+1E-06) >=0 THEN 740
700 IF V(J)>0 THEN 740
710 IF S >= (-A(1,J)) THEN 740
720 S=-A(1,J)
730 J1=J
740 NEXT J
750 IF J1<>(-1) THEN 790
760 P9=-1
770 GOTO 1000
780 REM.  J1=-1 MEANS THE SOL.  IS OPTIMAL BUT MAY BY INFEASIBLE.
790 I1=-1
800 X=1E+38
810 FOR I=2 TO M1
820 IF (A(I,J1)-1E-06) <=0 THEN 880
830 IF (A(I,N1)+1E-06)<0 THEN 860
840 D=A(I,N1)/A(I,J1)
850 IF X <= D THEN 880
860 X=D
870 I1=I
880 NEXT I
890 REM.
900 REM.
910 REM.
920 IF I1<>(-1) THEN 960
930 V(J1)=1
940 GOTO 660
950 REM.
960 P9=ABS(A(1,J1)*A(I1,N1)/A(I1,J1))
970 REM.
980 REM.  . . THE DUAL SIMPLEX ALGORITHM. .
990 REM.
1000 I2=-1
1010 S=0
1020 FOR 1=2 TO M1
```

```
1030 IF (A(I,N1)+1E-06) >=0 THEN 1080
1040 IF W(I)>0 THEN 1080
1050 IF S>=(-A(I,N1)) THEN 1080
1060 S=-A(I,N1)
1070 I2=I
1080 NEXT I
1090 IF I2<>(-1) THEN 1290
1100 REM.
1110 REM.  I2=-1 IN CASE OF FEASIBLE SOL. BUT MAY BE NOT OPTIMAL.
1120 REM.
1130 IF J1=(-1) THEN 1160
1140 D9=-1
1150 GO TO 1470
1160 PRINT ON(1)
1170 PRINT ON(1)TAB(21),"...  THIS IS THE FINAL TABLEAU ..."
1190 PRINT ON(1)
1200 FOR I=1 TO M1
1210 PRINT ON(1)"X (",L(I),") =",A(I,N1)
1230 PRINT ON(1)
1240 NEXT I
1250 GOTO 120
1260 REM.
1270 REM.
1280 REM.
1290 J2=-1
1300 X=1E+38
1310 FOR J=1 TO N1-1
1320 IF (A(I2,J)+1E-06) >=0 THEN 1380
1330 IF (A(1,J)+1E-06)<0 THEN 1380
1340 D=-A(1,J)/A(I2,J)
1350 IF X <=D THEN 1380
1360 X=D
1370 J2=J
1380 NEXT J
1390 IF J2<>(-1) THEN 1430
1400 W(I2)=1
1410 GOTO 1000
1420 REM
1430 D9=ABS(A(1,J2)*A(I2,N1)/A(I2,J2))
1440 REM.
1450 REM.
1460 REM.
1470 IF P9>D9 THEN 1600
1480 IF P9<>D9 THEN 1550
1490 IF P9<>(-1) THEN 1550
1500 PRINT ON(1)
1510 PRINT ON(1)"-----NO BASIC FEASIBLE SOLUTION-----"
1530 GOTO 120
1540 REM.
1550 P=A(I2,J2)
```

```
1560 I9=I2
1570 J9=J2
1580 GOTO 1660
1590 REM.
1600 P=A(I1,J1)
1610 I9=I1
1620 J9=J1
1630 REM.
1640 REM. . . . PIVOTING-OPERATION . .
1650 REM.
1660 P=1/P
1670 FOR J=1 TO N1
1680 A(I9,J)=A(I9,J)*P
1690 NEXT J
1700 REM.
1710 A(I9,J9)=P
1720 REM.
1730 FOR I=1 TO M1
1740 IF I=I9 THEN 1820
1750 FOR J=1 TO N1
1760 IF J=J9 THEN 1780
1770 A(I,J)=A(I,J)-A(I,J9)*A(I9,J)
1780 NEXT J
1790 REM.
1800 A(I,J9)=-A(I,J9)*P
1810 REM.
1820 NEXT I
1830 REM.
1840 REM. . . EXCHANGING THE SUBSCRIPTS OF EBV & LBC. . .
1850 REM.
1860 X1=K(J9)
1870 K(J9)=L(I9)
1880 L(I9 )=X1
1890 REM.
1900 REM. . . OUTPUT OF THE NEW SIMPLEX TABLEAU. . .
1910 REM.
1920 T=T+1
1930 PRINT ON(1)
1940 PRINT ON(1)"ITERATION NO. ";T
1950 PRINT ON(1)"=============="
1960 PRINT ON(1)
1970 REM.
1980 GOTO 321
1990 END
```

	1	2	
0	-2	-1	0
3	1	7	63
4	-3	-1	-9
5	-3	-2	-15
6	-5	-6	-30
7	-1	-4	-8
8	8	3	80

ITERATION NO. 1
==============

	8	2	
0	.25	-.25	20
3	-.125	6.625	53
4	.375	.125	21
5	.375	-.875	15
6	.625	-4.125	20
7	.125	-3.625	2
1	.125	.375	10

ITERATION NO. 2
==============

	8	3	
0	.24528302	.37735849E-1	22
2	-.18867925E-1	.1509434	8
4	.37735849	-.18867925E-1	20
5	.35849057	.13207547	22
6	.54716981	.62264151	53
7	.56603774E-1	.54716981	31
1	.13207547	-.56603774E-1	7

... THIS IS THE FINAL TABLEAU ...

X (0) = 22

X (2) = 8

X (4) = 20

X (5) = 22

X (6) = 53

X (7) = 31

X (1) = 7

	1	2	3	4	
0	-1	-1	2	4	0
5	1	-2	4	-3	10
6	-1	2	-4	3	-10
7	2	3	-1	5	15
8	-3	1	-2	-3	-12

ITERATION NO. 1
===============

	1	2	8	4	
0	-4	0	1	1	-12
5	-5	0	2	-9	-14
6	5	0	-2	9	14
7	3 5	2. 5	-. 5	6. 5	21
3	1. 5	-. 5	-. 5	1. 5	6

ITERATION NO 2
==============

	6	2	8	4	
0	8	0	-. 6	8. 2	-. 8
5	1	0	. 2220446E-15	-. 88817842E-15	-. 88817842E-15
1	2	0	-. 4	1. 8	2. 8
7	- 7	2. 5	. 9	. 2	11. 2
3	-. 3	-. 5	. 1	-1. 2	1. 8

ITERATION NO 3
==============

	6	2	7	4	
0	33333333	1. 6666667	. 66666667	8. 3333333	6. 6666667
5	1	-. 61679057E-15	-. 24671623E-15	-. 93752167E-15	-. 36514002E-14
1	- 11111111	1. 1111111	. 44444444	1. 8888889	7. 7777778
8	- 77777778	2. 7777778	1. 1111111	. 22222222	12. 444444
3	- 22222222	-. 77777778	-. 11111111	-1. 2222222	. 55555556

... THIS IS THE FINAL TABLEAU ...

X (0) = 6. 6666667

X (5) = -. 36514002E-14

X (1) = 7. 7777778

X (8) = 12. 444444

X (3) = 55555556

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