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Predicting the National Freight Transport Demand An Application of Multivariate Autoregressive Moving Average Time Series Analysis

By

Dr. Saadia H. Montasser

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I. Introduction

The development of many sectors of the economy is often hampered by the insufficiency of the exisitng facilities of physical distribution. Ridding of such a problem is expected to yield a returns to different sectors far more than the cost involved in the additional facilities. To estimate future needs for each mode of freight transport (waterways, railroad, and motor carriers), it is logical to start by predicting total needs of freight transport, then break down the total to find the size of demand on each mode.

This study concerns itself with the first part, i.e. with developing a proper model which could be used to predict total annual needs of freight transport services.

It is doubtful that analysis of time series of past commodity shipments would by itself suffice to predict such future needs. It does not account for external factors which affect the variable being forecast. In a previous study in which the author took part,[1] it had been suggested that Gross National Product at constant prices would be about the most relevant single variable affecting freight transport needs since most domestic product is physically distributed within the economy.

However, direct application of regression analysis to estimate the relationship between the two series (Demand for freight transport as the dependent variable Y, and GDP at constant prices as the independent variable X) would present some dificulties.

On the one hand, the disturbance terms of each series would not be white noise (random) as required by the model. Rather, they would most likely be serially related.

On the other hand, the trend in both series would tend to dominate the regression thus obsocuring the true regression relation making its identification and estimation rather difficult.

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1) This was in an unpublished study by the Institute of National Planning, Cairo, 1973.

Often in practical studies, these problems are ignored and regression analysis is applied to time series as are. The results would be unbiased estimates of the regression parameters but their estimated variances would be biased. This in turn leads to unreiable tests and inacurate interval estimates.

Therefore, for the correct specification of the regression model which will be used for prediction purposes, it is important to rid each series of any of these these difficulties whenever present before applying regression analysis to them. The suggested procedure, known as the Box-Jenkins Approach, will be applied in the following sequence:

a) Checking for the existence of a trend in each series, in which case data should be detrended first.

b) Checking whether disturbances of each series is white noise, if not, transform them into white noise through the specification and application of the correct ARMA model. This is knows as the prewhitening stage.

c) Applying regression analysis to the prewhitened series of X and Y. Modify until reaching the correct MARMA model.

d) Checking the model for adequacy. This stage is known as diagnostic checking stage.

e) Using the estimated MARMA model for prediction.

II. Detrending and Prewhitening

Prewhitening is applied to both the independent variable X and the dependent variable Y. The purpose is to eliminate the trend within each series leaving what could be called white noise, therefore allowing the application of regression analysis to estimate the regression of the prewhitened y series on the prewhitened X series.

Starting with the X series, the general class of autoregressive moving average (ARMA) model expressing the way an X value is related to its own past values and to current and past error terms is:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p}$$

$$+ u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2} \ldots - \theta_q u_{t-q} \tag{1}$$

Equation (1) should be applied to stationary time series to find out the degree of the ARMA model, i.e. to determine p and q.

If the data are not stationary, they should be made so by first removing the trend in them.

Therefore, the very first step is to find out whether the X series is stationary. Since stationarity exists when the data are horizontal or flutuate around a constant mean, autocorrelations are used to detect the presence of stationarity. Autocorrelation for time lag (k) is equal to

$$r_k = \frac{\sum_{t=1}^{n-k} (X_t-\bar{X})\ (X_{t+k}-\bar{X})}{\sum_{t=1}^{n} (x_t-x)^2}$$

If autocorrelation drops rapidly to zero, the data are stationary and the ARMA model could be applied to them. Otherwise, if autocorrelations drop slowly to zero and several of them are significantly different from zero, it would be a sign of an existing trend within the series, i.e. the data would be nonstationary. Table 1 shows the autocorrelations for the X series for various time lags.[1)]

Table 1
Autocorrelations of Original X Series

Time lag	Autocorrelation
1	0.865
2	0.748
3	0.641
4	0.537
5	0.447
6	0.358
7	0.283
8	0.211
9	0.137
10	0.063

1) The Data used are:
X = GDP at 1954 prices million £ for 1953-1980/1981
Y = Total volume of Freight transport in million ton/kelometer for 1953-1980/1981.

Since all autos up to the fifth time lag are significant[1), and the decline in them is rather slow, there exists a trend in the X series which will be removed by differencing.

Taking first differences instead of the original X data. The autos for various time lags are as shown in table 2.

Table 2
Autocorrelations of Differenced X Series

Time lag	Autocorrelation
1	.39
2	.305
3	-.136
4	.037
5	.028
6	.318
7	.045
8	.106
9	-.104
10	.054

The values of the table indicates that the new series of first differences has much lower autos which decline exponentially.

To determine the proper order of the ARMA model we examine the autos and partial autos for various time lags. Table 3 shows the partial

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1) Standard error of the autocorrelation coefficient is equal to

$$1/\sqrt{n-k}$$

autos for the differenced series.

Table 3

Partial Autocorrelations of Differenced X Series

Time lag	Partial Autocorrelation
1	.39
2	.227
3	-.354
4	.30
5	.256
6	.54
7	-.258
8	.40
9	.336

Notice that the first auto, the first and sixth partials are significantly different from zero. This suggests an AR of up to (6), and a MA of (1). However, we will start with the simplest model, an ARMA (1,1) and proceed with the remaining steps to find out whether such a model is adequate or should be modified.

An ARMA (1,1) model is in the form:

$$(X_t - X_{t-1}) = \phi_1 (X_{t-1} - X_{t-2}) + u_t - \theta_1 u_{t-1} \qquad (2)$$

where u_t is white noise such that

$$E(u_t) = 0, \quad E\, u_t^2 = \sigma^2, \quad \text{and} \quad E\, u_t\, u_{t-1} = 0$$

Initial estimates of ϕ_1 and θ_1 should be obtained. These are obtained using the autocorrelation coefficients.

Let us first use $x_t = X_t - X_{t-1}$

and $x_{t-1} = X_{t-1} - X_{t-2}$

Therefore, eq (2) becomes

$$x_t = \phi_1 x_{t-1} + u_t - \theta_1 u_{t-1} \quad (3)$$

The variances and covariances of the mixed ARMA process would then be

$$\gamma_o = E(x_t^2) = E(\phi_1 x_{t-1} + u_t - \theta_1 u_{t-1})^2 \quad (4)$$

$$= \phi_1^2 \gamma_o - 2\phi_1\theta_1 E(x_{t-1} u_{t-1}) + \sigma^2 + \theta_1^2 \sigma^2 \quad (5)$$

$$= \phi_1^2 \gamma_o - 2\phi_1 \theta_1 \sigma^2 + \sigma^2 + \theta_1^2 \sigma^2 \quad (6)$$

since $E(x_{t-1} u_{t-1}) = \sigma^2$

Therefore,

$$\gamma_o(1-\phi_1^2) = \sigma^2(1 + \theta_1^2 - 2\phi_1 \theta_1) \quad (7)$$

and, the variance γ_o is

$$\gamma_o = \frac{(1 + \theta_1^2 2\phi_1 \theta_1)}{1-\phi_1^2} \sigma^2 \quad (8)$$

Likewise, the covariance γ_1 is

$$\gamma_1 = E\,(x_{t-1}\, x_t) = E\left[x_{t-1}\,(\phi_1\, x_{t-1} + u_t - \theta_1\, u_{t-1}\right]$$

$$= \phi_1\, \gamma_0 - \theta_1 \sigma^2 \tag{9}$$

$$= \phi_1 \quad \frac{1 + \theta_1^2 - 2\phi_1\,\theta_1}{1-\phi_1^2}\ \sigma^2 - \theta_1 \sigma^2 \tag{10}$$

$$= \frac{\phi_1 + \phi_1\,\theta_1^2 - 2\phi_1^2\,\theta_1 - (1-\phi_1^2)\,\theta_1}{1-\phi_1^2}\ \sigma^2 \tag{11}$$

$$= \frac{\phi_1 + \phi_1\,\theta_1^2 - 2\,\phi_1^2\,\theta_1 - \theta_1 + \phi_1^2\,\theta_1}{1-\phi_1^2}\ \sigma^2 \tag{12}$$

$$= \frac{\phi_1 + \phi_1\,\theta_1^2 - \phi_1^2\,\theta_1 - \theta_1}{1-\phi_1^2}\ \sigma^2 \tag{13}$$

$$= \frac{(\phi_1 - \theta_1) - \phi_1\,\theta_1\,(\phi_1 - \theta_1)}{1-\phi_1^2}\ \sigma^2 \tag{14}$$

Thus,

$$\gamma_1 = \frac{(\phi_1 - \theta_1)\quad(1 - \phi_1\,\theta_1)}{1-\phi_1^2}\ \sigma^2 \tag{15}$$

ˈ≀ewise γ_2 is

$$\gamma_2 = E\,(x_{t-2}\, x_t)$$

$$= E\left[x_{t-2}\ (\phi_1\, x_{t-1} + u_t - \theta_1\, u_{t-1})\right] \qquad (16)$$

$$= E\left[x_{t-2}\ (\phi_1\ (\phi_1\, x_{t-2} + u_{t-1} - \theta_1\, u_{t-2}) + u_t - \theta_1 u_{t-1}\right] \qquad (17)$$

$$= \phi_1^2\, \gamma_\circ - \theta_1 \phi_1 \sigma^2 \qquad (18)$$

Thus,

$$\gamma_2 = \phi_1\ (\phi_1 \gamma_\circ - \theta_1 \sigma^2) = \phi_1 \gamma_1 \qquad (19)$$

in the same fashion

$$\gamma_k = \phi_1\, \gamma_{k-1} \qquad \text{for} \quad k \geqslant 2 \qquad (20)$$

where k is the number of time lags.

Thus, the autocorrelation functions would be

$$r_1 = \frac{\gamma_1}{\gamma_\circ} = \frac{(\phi_1 - \theta_1)\ (1 - \phi_1\, \theta_1)}{1 - \theta_1^2 - 2\,\phi_1\,\theta_1} \qquad (21)$$

and

$$r_2 = \frac{\gamma_2}{\gamma_\circ} = \frac{\phi_1\, \gamma_1}{\gamma_\circ} = \phi_1\, r_1 \qquad (22)$$

Therefore, $\phi_1 = \dfrac{r_2}{r_1}$ (23)

Next, the autocorrelation estimates for various time lags are used to obtain initial estimates for ϕ_1 and θ_1.

Starting with ϕ_1,

$$\phi_1 = \frac{r_2}{r_1} = \frac{.305}{.39} = .781$$

Substituting this value of ϕ_1 in eq. (21) we can solve the resulting function, which is nonlinear, to get an initial estimate for θ_1

$$.39 = \frac{(.781 - \theta_1)\ (1 - .781\ \theta_1)}{1 - \theta_1^2 - 2 \times .781 \quad 1}$$

or

$$-.39\ \theta_1^2 + \theta_1 - .39 = 0$$

Thus,

$$\theta_1 = \frac{-1 \pm \sqrt{1 - 4(.39)^2}}{-2 \times .39} \qquad (24)$$

$$\theta_1 = 0.479 \quad \text{or} \quad \theta_1 = 2.085$$

The first value $\theta_1 =$ °479 meets the constraint for stationarity (otherwise, with the other value of θ_1 the series would be explosive). Therefore, the initial values of the parameters are

$$\phi_1 = .781 \qquad \theta_1 = .479$$

Final parameter estimates are obtained using Marquardt's method for solving nonlinear equations. This method is often preferred for its practical advantages over the other methods.[1)]

Using the initial estimates of ϕ_1 and θ_1 on the differenced data x_t we get its corresponding estimate

$$\hat{x}_t = .781 \ x_{t-1} - .479 \ e_{t-1}$$

where e_t is the observed residual value for time t. Thus the mean square error MSE is

$$MSE = \frac{\sum_{t=3}^{28} e_t^2}{26} = \frac{368622}{26} = 14178 \qquad (25)$$

where $e_t = x_t - \hat{x}_t.$

The calculations using initial parameter values are shown in table 1 in the appendix.

Next, to determine the direction of changing the parameter estimates, we introduce a small change, say 1% of original value, once added and once subtracted to the initial value of ϕ_1 while holding θ_1

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1) Other methods for solving nonlinear equations are; linearization of the above nonlinear function around the initial values of the parameters using Taylor series expansion (see 7, p. 482), and steepest descent method (see 2, p. 267).

constant. The predicted values of x_t using these values, called f_{1t} are calculated. The difference between f_{1t} and $\hat{x}_t$ is also found. The resulting values of MSE will determine the appropriate direction of change in ϕ_1.

Interchangeably, ϕ_1 is held constant at its initial value and θ_1 is changed by a small increment in each direction. The predicted values of x_t using these values, called f_{2t} are also calculated. The difference between f_{2t} and $\hat{x}_t$ could also be found. Again, the values of MSE will determine the appropriate direction of change in the value of $_1$. Table 4 shows alternative values of ϕ_1 and θ_1 and their corresponding values of MSE.

Table 4
MSE for alternative Estimates
ϕ_1 and θ_1

Parameter Estimates	MSE
$\phi_1 = .781$, $\theta_1 = .479$	14178
$\phi_1 = .789$, $\theta_1 = .479$	13915
$\phi_1 = .773$, $\theta_1 = .479$	14450
$\phi_1 = .781$, $\theta_1 = .484$	14262
$\phi_1 = .781$, $\theta_1 = .474$	14096

Notice that the value of MSE decreases whenever ϕ_1 is increased or θ_1 is decreased and vice versa. Thus, the search for final parameter estimates should move in the direction of increasing ϕ_1 and decreasing θ_1 to the extent that would make MSE as small as possible. One should be aware not to carry the process to the extent that may cause the series to diverge.

The predicted values of x_t using $\phi_1 = .789$, called f_{1t}, are calculated as follows:

$$f_{1t} = .789 \; x_{t-1} - .479 \; e_{t-1}$$

Likewise, the predicted values of x_t, called f_{2t}, using $\theta_1 = .474$ are calculated using the equation

$$f_{2t} = .781 \; x_{t-1} - .474 \; e_{t-1}$$

The differences between $\hat{x}_t$ and f_{1t} on the one hand, and between $\hat{x}_t$ and f_{2t} on the other hand are used to form the symmetric matrix D_{ij} whose elements are the products and cross products of the differences i.e.

$$D_{11} = \sum_{t=3}^{28} (f_{1t} - \hat{x}_t)(f_{1t} - \hat{x}_t) = 118.02 \qquad (26)$$

$$D_{12} = D_{21} = \sum_{t=3}^{28} (f_{1t} - \hat{x}_t)(f_{2t} - \hat{x}_t) = 40.26 \qquad (27)$$

$$D_{22} \quad \sum_{t=3}^{28} (f_{2t} - \hat{x}_t)(f_{2t} - \hat{x}_t) = 15.46 \qquad (28)$$

The necessary calculations are shown in tables II and III in the appendix.

The D_{ij} matrix thus becomes

$$D_{ij} = \begin{bmatrix} 118.02 & 40.26 \\ 40.26 & 15.46 \end{bmatrix}$$

Next, the D_{ij} values are scaled and standardized by dividing them by the product $P_i P_j$ where P_i is the incremental change in ϕ_1 , P_i =.00781, and P_j is the incremental change in θ_1 , P_j = .00479. The new D_{ij} values become.

$$D_{11} = \frac{118.02}{(.00781)^2} = 1934877.817 \tag{29}$$

$$D_{12} = \frac{40.26}{(.00781)(.00479)} = 1076185.715 \tag{30}$$

$$D_{22} = \frac{15.46}{(.00479)^2} = 673811.585 \tag{31}$$

These last values are then constrained by dividing each of them by $O_i\ O_j$

where

$$O_1 = \sqrt{D_{11}} = 1391 \text{ , and } O_2 = \sqrt{D_{22}} = 820.86 \tag{32}$$

The resulting matrix J_{ij} is

$$J_{ij} = \begin{bmatrix} 1 & .943 \\ .943 & 1 \end{bmatrix}$$

Whose values are scaled between ± 1

Consider next the two values q_1 and q_2

where

$$q_1 = \sum_{t=3}^{28} (x-\hat{x})\ (f_{1t} - \hat{x})/p_1 \qquad (33)$$

$$= \frac{3475}{.00781} = 444982$$

and

$$q_2 = \sum_{t=3}^{28} (x-\hat{x})\ (f_{2t} - \hat{x})\ /p_2 \qquad (34)$$

$$= \frac{1069}{.00479} = 223269$$

Notice that the numerator of q_1 is the cross product of errors when ϕ_1 = .789 holding θ_1 contant. When this is divided by the incremental change in the parameter P_1 it gives a relative measure of how a change in the parameter affects the errors. q_2 could be interpreted likewise with respect to giving a relative measure of how the errors are affected as a result of a small change in θ_1 while holding ϕ_1 constant.

Next, q_1 and q_2 are also scaled dividing them by O_1 and O_2 respectively yielding.

$$h_1 = \frac{q_1}{o_1} = 319.9 \; , \; h_2 = \frac{q_2}{O_2} = 272 \qquad (35)$$

Thus, the J_{ij} matrix times p_i will be equal to h_i i.e. $J_{ij}\, p_i = h_i$. Notice that the J_{ij} matrix includes all possible changes in the parameters while h_i includes changes in the errors from changing one parameter at a time. Instead of using the J_{ij} matrix as derived before, a constant quantity, say .01, is added to its diagonal i.e.

$$J_{ij} = \begin{bmatrix} 1.01 & .943 \\ .943 & 1.01 \end{bmatrix}$$

upon which the new relationship

$J_{ij}\, P_i = h_i$ could be solved for P_i

giving

$$P = J^{-1} h$$

$$= \begin{bmatrix} 1.01 & .943 \\ .943 & 1.01 \end{bmatrix}^{-1} \begin{bmatrix} 319.9 \\ 272 \end{bmatrix} = \begin{bmatrix} 508 \\ -206 \end{bmatrix}$$

The elements of the last vector are scaled as they are divided by 0_1 and 0_2 respectively. The results would give the correction factors to be applied to ϕ_1 and θ_2,

$$P_1 = \frac{508}{1391} = .365$$

$$P_2 = \frac{-206}{820.86} = -.251$$

The correction factors are added to the initial ϕ_1 and θ_1 to the extent that the process does not become divergent. Thus ϕ_1 will be increased only to .995 (less than 1) while θ_1 is decreased to

$$\theta_1 = .479 - .251 = .228$$

Thus, the final parameter estimates of the parameters are

$$\phi_1 = .995 \quad , \quad \theta_1 = .228$$

and the final generating equation is $\hat{x} = .995\ x_{t-1} - .228\ e_{t-1}$ (36)

Appling this equation yields MSE

$$MSE = \frac{289688}{26} = 11142$$

Which is minimum within the constraint of convergency of the series.Estimates using final parameter values are shown in table IV in the appendix.

The residuals of the ARMA model using final parameter estimate are calculated as

$$e_t = x_t - \hat{x}_t = x_t - .995\, x_{t-1} + .228\, e_{t-1}$$

Next, the autocorrelations of the residuels, shown in table 5 below, are examined. It is observed that the autos are significant for both K = 3 and K = 6. This could be an indication of a pattern in the data that repeats itself every three time periods especially that for K = 9 the auto is also high though not significant. However, the introduction of higher order ARMA models did not improve the situation much while leading to the loss of some precious degrees of freedom. This is because the total number of observations available is already small for time series analysis.

On the other hand, it is widely known that aggregate indices, such as GDP, tend to hide compositional changes thus biasing their measurements.[1] Thus, if the published GDP statistics do not include some indication of the measurement errors involved, they can't be accounted for and some of this error will be a component of the error term thus obscuring

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1) See (1, P. 140 and 4, P.287).

its randomness. Since information about the range of error in GDP is lacking, we will proceed with the ARMA (1,1) model just estimated.

The residuals in table IV in the appendix will be used as the prewhitened x series.

Table 5

Autocorrelations of residuals

of x

Time lag	Autocorrelation
1	-°24
2	°11
3	-°53
4	-.008
5	-.07
6	°54
7	-.21
8	°177
9	-°33
10	.09

We next proceed by applying the same ARMA equation to the differenced Y series. Table shows the autocorrelations of the residuals of y.

Table 6

Autocorrelations of residuals of y

Time lag	Autocorrelation
1	-.12
2	-.22
3	.11
4	-.19
5	.19
6	.06
7	-.23
8	-.09
9	.03
10	.04

The autocorrelations of the residuals of the differenced y variable are all nonsignificant indicating that all y_e are white noise where

$$y_e = y-\hat{y}$$

III Regression Analysis

Regression analysis could now be applied to the prewhitened X and Y series, i.e. y_e is regressed on x_e. To identify the form of the regression relationship, cross autocorrelation between y_e and x_e are found to determine whether some lagged dependent or independent variables should be included in the relation as explanatory variables. The cross autocorelation for time lag (K) is equal to

$$r_{xy}(k) = \frac{\sum (x_t - \bar{x})\ (Y_{t+k} - \bar{Y})}{\sqrt{\sum (x_t - \bar{x})^2 \quad (Y_t - \bar{Y})^2}} \tag{37}$$

Table 7 shows the cross autos for various time lags

Table 7

Cross Autocorrelations of x_e and y_e series

Time lag	Cross auto
0	.38
1	-.07
2	-.20
3	.10
4	-.03
5	.04
6	.39
7	-.32
8	-.19
9	-.03
10	.18

Considering the degrees of freedom for each time lag, only the cross auto for zero time lag $r_{xy}(0)$ is found to be significant. Thus, multiple autoregressive moving average model, MARMA is of the form.

$$y_{et} = \delta x_{et} + e_t \tag{38}$$

Applying this MARMA model yield

$$\delta = 2.17$$

The residuals of the MARMA model e_t are found where

$$e_t = y_t - \hat{y}_t \tag{39}$$

$$\hat{y}_t = 2.17\ x_t \tag{40}$$

IV Testing the Adequacy of the Model

To find out whether the model is adequate or should be modified, cross autos between the prewhitened x values that is x_{et} and e_t are calculated. If they indicate white noise, the MARMA model is correctly specified. Likewise, the autos of e_t are found, if they indicate randomness, then, the noise model is also correctly specified. If either one is not random, one should go back to the appropriate stage, and reestimate the relevant model. Table 8 shows the autos and the cross autos for various time lags.

Table 8

Autocorrelations of MARMA Residuals and their cross autos with x_{et}

Time lag	Autocorrelation of e_t	Time lag	Cross Auto of e_t and X_{et}
		0	.02
1	.24	1	.01
2	-.18	2	.11
3	-.07	3	-.14
4	-.04	4	-.23
5	-.05	5	-.01
6	-.07	6	-.13
7	-.25	7	.20
8	-.16	8	-.03
9	-.04	9	.13
10	-.01	10	-.01

Since all values are not significantly different from zero, randomness is indicated. The MARMA model and the noise modèl are both adequate and could be used for forecasting.

V <u>Forecasting.</u>

GDP Figures for the two years 1981/82 and 1982/83 were obtained, deflated and differenced. Thus the first differences of GDP, 1954 = 100, were

$$x_{81/82} = 532 \quad , \quad x_{82/83} = 376$$

Therefore, the estimated y values are

$$\hat{y}_{81/82} = 532 \times 2.17 = 1154$$

$$\hat{y}_{82/83} = 376 \times 2.17 = 816$$

Next, these estimates are dedifferenced back to original data by sucessive addition to last year's value in the original series, therefore, the estimated demand for freight transport for the two years are

$$\hat{Y}_{81/82} = 13993 + 1154 = 15147$$

$$\hat{Y}_{82/83} = 15147 + 816 = 15963$$

The actual values of total freight transport were 16505 and 16992 million ton/Kelometer for the two years respectively. It is worth indicating that the prediction process could continue beyond 1982/83, the last year for which data on the independent variable x are available. In this case the independent variable will have to be forecast first using its generating process in eq. (36). Then, these predicted values are used to forecast the values of the dependent variable, demand for freight transport. It is expected that forecasts, using a MARMA model will be more reliable, as compared with traditional methods'forecasts., This is so since the estimated variances will be unbiased as a result of using a regression model which has random residuals.

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APPENDIX

Table I

Estimates Using initial parameter values

Observation No.	x	$\hat{x}$	e	e^2
2	52.40	0.00	0.00	0.00
3	45.50	40.72	4.58	20.74
4	24.00	33.34	7.34	87.31
5	27.00	23.22	9.78	14.29
6	121.00	19.28	101.72	10347.73
7	109.00	45.78	63.22	3997.36
8	66.00	54.84	11.16	124.45
9	45.00	46.20	-1.20	1.45
10	116.00	35.72	80.28	6444.72
11	125.00	52.14	72.86	5308.23
12	285.00	62.73	222.27	49405.64
13	82.00	116.12	-34.12	1163.89
14	7.00	80.38	-73.38	5385.14
15	-51.00	40.62	-91.62	8393.80
16	138.00	4.05	133.95	17941.56
17	139.00	43.62	95.38	9097.76
18	336.00	62.87	273.13	74599.56
19	104.00	131.59	-27.59	761.05
20	101.00	74.44	6.56	43.06
21	94.00	75.74	18.26	333.50
22	89.00	64.67	24.33	592.12
23	123.00	57.85	65.15	4244.10
24	328.00	64.86	263.14	69243.87
25	199.00	130.12	68.88	4744.07
26	352.00	122.43	229.57	52703.84
27	226.00	164.75	61.05	3727.52
28	347.00	147.26	199.74	39895.51
				368622.44

Table II

Calculations Introducing a Small Change in ϕ_1

Observation No. No	x	F_1	e	e^2
2	52.40	0.00	0.00	0.00
3	45.50	41.34	4.16	17.28
4	24.00	33.91	-9.91	98.18
5	27.00	23.68	3.32	11.01
6	121.00	19.71	101.29	10258.90
7	109.00	46.95	62.05	3849.84
8	66.00	56.28	9.72	94.47
9	45.00	47.42	-2.42	5.85
10	116.00	36.66	79.34	6294.30
11	125.00	53.52	71.48	5109.14
12	285.00	64.39	220.61	48670.12
13	82.00	119.19	-37.19	1383.20
14	7.00	82.51	-75.51	5702.16
15	-51.00	41.69	-92.69	8592.09
16	138.00	4.16	133.84	17912.82
17	139.00	44.77	94.23	8878.68
18	336.00	64.54	271.46	73692.50
19	104.00	135.07	-31.07	965.53
20	101.00	96.94	4.06	16.48
21	94.00	77.74	16.26	264.25
22	89.00	66.38	22.62	511.69
23	123.00	59.39	63.61	4046.77
24	328.00	66.58	261.42	68342.69
25	199.00	133.57	65.43	4281.12
26	352.00	125.67	226.33	51225.31
27	226.00	169.32	56.8	3213.09
28	347.00	151.16	195.84	38352.39
				361789.81

$\sum (f_{1t} - \hat{x})^2 = 118.02$

Table III

Calculations introducing a small change in θ_1

Observation No.	x	F_2	e	e^2
2	52.40	0.00	0.00	0.00
3	45.50	40.92	4.58	20.94
4	24.00	33.37	-9.37	87.73
5	27.00	23.18	3.82	14.56
6	121.00	19.28	101.72	10347.34
7	109.00	46.28	62.72	3933.19
8	66.00	55.40	10.60	112.32
9	45.00	46.52	-1.52	2.32
10	116,00	35.87	80.13	6421.35
11	125.00	52.61	72.39	5239.90
12	285.00	63.31	221.69	49144.91
13	82.00	117.51	-35.51	1260.65
14	7.00	80.87	-73.87	5457.02
15	-51.00	40.48	-91.48	8368.98
16	138.00	3.53	134.47	18081.76
17	139.00	44.04	94.96	9017.41
18	336.00	63.55	272.45	74230.19
19	104.00	133.27	-29.27	856.95
20	101.00	95.10	5.90	34.81
21	94.00	76.08	17.92	320.97
22	89.00	64.92	24.08	579.75
23	123.00	58.10	64.90	4212.53
24	328.00	65.30	262.70	69012.06
25	199.00	131.65	67.35	4536.36
26	352.00	123.49	228.51	52215.03
27	226.00	166.60	59.40	3528.33
28	347.00	148.35	198.65	39461.62
				366498.94

$\sum(f_{1t}-\hat{x})\ (f_{2t}-\hat{x}) = 40.26$ $\quad \sum(f_{2t}-\hat{x})^2 = 15.46$

$\sum(x-\hat{x})\ (f_{1t}-\hat{x}) = 3475$ $\quad \sum(x-\hat{x})\ (f_{2t}-\hat{x}) = 1069$

Table IV

Estimates using Final Parameter Values

Observation No.	x	$\hat{x}$	e	e^2
2	52.40	0.00	0.00	0.00
3	45.50	52.14	-6.64	44.06
4	24.00	46.79	-22.79	519.20
5	27.00	29.08	- 2.08	4.31
6	121.00	27.34	93.66	8772.54
7	109.00	99.04	9.96	99.20
8	66.00	106.18	-40.18	1614.77
9	45.00	74.83	-29.83	889.95
10	116.00	51.58	64.42	4150.36
11	125.00	100.73	24.27	588.96
12	258.00	118.84	166.16	27608.55
13	82.00	245.69	-163.69	26794.71
14	7.00	118.91	-111.91	12524.19
15	-51.00	32.48	-83.48	6969.05
16	138.00	-31.71	169.71	28801.95
17	139.00	98.62	40.38	1630.88
18	336.00	129.10	206.90	42808.68
19	104.00	287.15	-183.15	33542.55
20	101.00	145.24	-44 .24	1956.94
21	94.00	110.58	-16. 58	274.93
22	89.00	97.31	-8. 31	69.06
23	123.00	90.45	32.55	1059.52
24	328.00	114.96	213.04	45384.53
25	199.00	277.79	-78.79	6207.52
26	352.00	215.97	136.03	18504.54
27	226.00	319.22	-93.22	8690.87
28	347.00	246.13	100.87	10175.71
				289687.56