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THE ECONOMETRIC APPROACH TO STUDYING BUSINESS CYCLES: THE GENERAL MODEL AND SPECTRAL ANALYSIS

BY

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PREFACE

In the name of Allah, most Gracious, most Merciful. Praise be to Allah, the Cherisher of the worlds. May His blessings and peace be on his prophet and messenger Muhammad, the lord of the faithful.

Studying business cycles can be purely theoretical. In fact, there are many theoretical explanations to economic fluctuations ranging from under-consumption to multiplier-accelerator interaction. However, the scientific approach hardly suggests deduction alone to be the only criterion for evaluating scientific principles.

Business cycles can also be studied in a pure empirical way. The work of Burns and Mitchell is a good example. Empiricists do not depend on economic theory nor do they relate their own analysis to any theoretical structure. They take all available data as the first step. Then, they proceed with some smoothing processes to discover economic fluctuations.

A third school stands between the theoreticians and the empiricists. This school follows the econometric approach. It takes from theory the logic of all structural and behavioral relations. It takes from mathematics the technique of expressing these relations in terms of a formal model. Finally, it takes from statistics the method of relating models to data and making any inference or significance tests.

Because of different problems relating to space and scope, we will review the shortcomings of the emperical approach in brief, and explain the econometric approach of studying economic fluctuations. For the sake of clarity, a small explanatory model will be added to show some of the mechanics involved in using the general model. The second Part of the paper introduces spectral analysis as another technique of studying fluctuations.

The introduction of spectral analysis is done in the easiest possible form. Due to the strange terms of spectral analysis which are foreign to economists, and due to the high level of the required mathematical background, this technique appears to be rather difficult.

However, economists who can follow mathematical analysis if every step is writtern down and explained without much detail will be able to benefit from this simplification. Therefore, an exposition to spectral analysis will be introduced first as a necessary background. We will try in this exposition to follow the mathematical form step by step. Nevertheless, we will avoid too rigorous proofs and theorems since the literature is already rich with them. Finally, a brief account of how such a technique can be used for studying fluctuations will be provided.

My deep gratitude is due to Prof. R. Campbell and Dr. R. Glenn Vice for being kind enough to have commented on this paper.

I. THE SHORTCOMINGS OF THE EMPERICAL APPROACH[1]

Koopman's criticism of Empiricism is directly related to the work of Burns and Mitchell[2]. We will review his criticism in brief and discuss the empiricist's answer to it.

The empirical approach of Burns and Mitchell lacks guidance from theoretical consideration, while "theoretical preconceptions" about the nature of observations cannot be dispensed with. This complete divorce from theory is clear in the following:

1 - In Chapters 9-12 they search for possible changes in cyclical behavior overtime. The average measure of cyclical behavior is computed from seven series chosen with no systematic discussion of the reasons of their selection.

2 - They do not study behavior in terms of the behavior of groups of economic agents whose modes of behavior, in the institutional and technological environment, are the ultimate determinants of the levels of economic variables and their fluctuations. They study behavior in a more mechanical sense of certain measurable joint effects of several actions and responses. This eliminates the benefits that might be received from economic theory. It also divorces the study of fluctuations from the explanation of the levels or trends around which the variables may fluctuate.

3 - There is no organizing principle to determine on what aspects of the observed variables attention should be concentrated, except for applying the formal definition of the cycle.

4 - The study is not related to the problem of prediction, its possibilities and limitations.

5 - Although Burns and Mitchell are aware of the problem of randomness, they do not discuss it in terms of definite distributional hypotheses. Their variance tests applied to durations, amplitudes, and time lags are not rigorous. The measure of these variables need not be independent over successive cycles.

Resorting to economic theory is also a practical necessity. Without resort to the theory in some systematic way, no conclusions relevant to the guidance of economic policy can be made.

Empiricists answer these criticisms on two bases:(3)

1 - The aggregate has an existence apart from its constituent parties. The behavior characteristics of its own are not deducible from the behavior of the particles. Therefore, seeking a basis for economic dynamics in the analysis of the economizing behavior of the individual may not be necessary or even particularly desirable.

2 - Social usefulness of economic policy is hardly a relevant criterion in the evaluation of economic research.

Our answer to the first point is simple. Aggregates do not "act" independently of their constituents. Even if their behavior is not directly related to their "Micro" units, a first step is necessary before dealing empirically with aggregates. There should be a theory which explains the economizing behavior of aggregate units. Then, this theory should be incorporated with any empirical work. Until such theory exists the empiricists should either use the existing theory or develop the necessary theory themselves.

The answer to the second point is also simple. Intelligent criteria are undoubtedly needed to evaluate economic policy. Economic research is the only source for such criteria. Therefore "social usefulness" is a relevant basis of evaluating research, but not an ultimate one.

II. THE GENERAL MODEL APPROACH

The econometric approach includes the simultaneous equation model approach, and the use of spectral analysis in studying economic fluctuations. The latter approach is discussed below, and this part takes up the former one.

In order to have a good understanding of the econometric approach it is necessary to understand its classification of variables and equations. Variables are classified into the following categories:

1. The Exogeneous or Independent Variables There are three principles according to which the exogeneous variables are determined.[4] The first is the departmental principle. It treats the variables which are wholly or partially outside the scope of economics, like population and time, as independent. The second is the causal principle. It regards as exogeneous the variables which influence the remaining exogeneous variables but are not influenced thereby. The third principle is time which classifies lagged (predetermined) variables as exogeneous.

2. The Endogeneous or Dependent Variables: They are those variables which are mutually dependent and determined within the system.

Equations can be classified as follows:(5)

1. Definitional Equations: They express definitions or identities which hold exactly in the system. As an example:

$Y = C + I$

Y : income
C : consumption
I : investment
W : total wage bill
w : wage rate
L : labor

$W = w L$

2. Stochastic Equations: They do not hold exactly. They are randomly determined. Three kinds of stochastic relations can be found:

   a. Behavioral : which expresses a behavioral relationship, as

   $C = a + b Y + u$

   a : constant
   b : marginal propersity to consume
   u : error term

   b. technological: which expresses a technical or institutional constrain as

   $O = a L^{\alpha} K^{\beta} u$

   α, β : parameters
   K : capital
   O : total output

   c. adjustment: which refers to the adjustment process taking place when there is a disequilibrium in a particular market, as

   $I - S = M^d - M^s + u$

   M^d: money demand
   M^s: money supply
   S : savings

The error term is included in the stochastic equations to count for the unknown variables and the errors of observations(6). To give the model a probablistic nature, we generally assume that the effects of the unexplained variables and errors happen at random.

The econometric approach has three components(7). First, a specification of the process by which the independent variables are generated. Second, a specification of the process by which the observed distrubances generated. Third, a specification of the relationship connecting these to the observed independent variables. A given body of economic observations, cross section sample or time series, may be viewed as a sample from the population. Once we specify a parent population, the rules of statistical inference can be applied to develop a rational method of measuring a relationship of economic theory from a given sample.

Therefore, the quantification of economic theory is not a mechanical task. It is not a matter of measurement without theory. It is merely a specification of the probabilistic mechanisms that link economic observations to economic theory.

After constructing the structural equations of the model, we can solve the system for the endogeneous variables. This solution will be in terms of the exogeneous and predetermined variables, and the disturbances. This solution is called the reduced form. It can be used in forecasting the values of the dependent variables given the independent ones.

In order to examine the dynamic properties of the model, i.e., under what circumstances cycles, growth, etc., are found, we have to construct the "final or Timbergen equations".(8)

In each reduced form equation, the lagged variables, except for the one explained by the equation, are algebraically eliminated. Then, we get a difference equation in one endogeneous variable expressed as a function of a number of its lagged values, of parameters, of exogeneous variables, and of disturbances. If the exogeneous variables, the parameters, and the disturbances are held constant, the dependent variable will be a function of only its past values. This time path solution is called the solution of the final equation. It can be examined to see under what conditions it contains cycles, growth, decline,..., etc.

In order to explain the approach, a simple model follows as an example.

Let us assume we are interested in the time path of the price of a certain commodity, whose market equations are:

$$Q_t^d = \alpha_{11} P_t + \alpha_{12} P_{t-1} + \alpha_{13} Y_t + U_1 \qquad (1)$$

$$Q_t^s = \alpha_{21} P_t + \alpha_{22} P_{t-1} + \alpha_{23} Y_t + U_2 \qquad (2)$$

$$Q_t^d = Q_t^s \qquad (3)$$

where:

P : price.

Q : Q^d : quantity demanded, Q^s quantity supplied

Y : Income

U_i : disturbance variable

α_{ij}: parameters

t : time subscript.

Then, we have three equations in three dependent variables: Q_t^s, Q_t^d, and P_t, and one lagged variable, one exogeneous variable, and an error term.

The Reduced form:

By using (3), (1) and (2) are equal. Then:

$$\alpha_{11} P_t + \alpha_{12} P_{t-1} + \alpha_{13} Y_t + U_1 = \alpha_{21} P_t + \alpha_{22} P_{t-1} + \alpha_{23} Y_t + U_2 \qquad (4)$$

By solving (4) for P_t.:

$$P_t(\alpha_{11} - \alpha_{21}) = P_{t-1} (\alpha_{22} - \alpha_{12}) + Y_t (\alpha_{23} - \alpha_{13}) + (U_2 - U_1) \qquad (5)$$

$$P_t = \frac{\alpha_{22} - \alpha_{12}}{\alpha_{11} - \alpha_{21}} P_{t-1} + \frac{\alpha_{23} - \alpha_{13}}{\alpha_{11} - \alpha_{21}} Y_t + \frac{U_2 - U_1}{\alpha_{11} - \alpha_{21}} \qquad (6)$$

Let:

$$\frac{\alpha_{22} - \alpha_{12}}{\alpha_{11} - \alpha_{21}} = \alpha \qquad (7)$$

$$\frac{\alpha_{23}-\alpha_{13}}{\alpha_{11}-\alpha_{21}} = \beta \tag{8}$$

$$\frac{U_2 - U_1}{\alpha_{11}-\alpha_{21}} = V_t \tag{9}$$

By substituting (7), (8), (9) in (6):

$$P_t = \alpha P_{t-1} + \beta Y_t + V_t \tag{10}$$ *

By the same logic:

$$P_{t-1} = \alpha P_{t-2} + \beta Y_{t-1} + V_{t-1} \tag{11}$$

By substituting for P_{t-1} from (11) in (10):

$$P_t = \alpha[\alpha P_{t-2} + \beta Y_{t-1} + V_{t-1}] + \beta Y_t + V_t \tag{12}$$

$$= \alpha^2 P_{t-2} + (\beta Y_t + \alpha\beta Y_{t-1}) + (V_t + \alpha V_{t-1}) \tag{13}$$

Then, we can express P_{t-2} as in (10) and then substitute for it in (12), Repeating this operation s times will render:

$$P_t = \alpha^{s+1} P_{t-s-1} + \sum_{r=0}^{s} (\alpha^r\beta)\, Y_{t-r} \sum_{r=0}^{s} \alpha^r V_{t-r} \tag{14}$$

By letting s go to infinity, and assuming that $\lim_{r \to \infty} \alpha^r = 0$, then

$$P_t = \sum_{r=0}^{\infty} (\alpha^r\beta)\, Y_{t-r} + \sum_{r=0}^{\infty} \alpha^r V_{t-r} \tag{15}$$

In this case the system is said to be stable, and its final equation is (15). In other words, P_t is a function of some level of income and disturbances. If $\lim_{r \to \infty} \alpha^r \neq 0$, the final equation will show instability.

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* Needless to say, this is a difference equation of the first order.

If s = t-1, equation (14) will be:

$$P_t = \alpha^t P_0 + \sum_{r=0}^{t-1} (\alpha^t \beta) Y_{t-r} + \sum_{r=0}^{t-1} {}^r V_{t-r} \tag{16}$$

where P_0 is some initial value of P.

If the exogeneous variable Y is indefinitely sustained at $\bar{Y}$, then P will approach its equilibrium value $\bar{P}$ which is equal to:

$$\bar{P} = \sum_{r=0} (\alpha^r \beta) \bar{Y} \tag{17}$$

$$= \sum_{r=0}^{t-1} (\alpha^r \beta) \bar{Y} + \sum_{r=t}^{\infty} (\alpha^r \beta) \bar{Y} \tag{18}$$

$$= \sum_{r=0}^{t-1} (\alpha^r \beta) \bar{Y} + \alpha^t \sum_{r=0}^{\infty} (\alpha^r \beta) \bar{Y} \tag{19}$$

By comparing with 17,

$$\bar{P} = \sum_{r=0}^{t-1} (\alpha^r \beta) \bar{Y} + \alpha^t \bar{P} \tag{20}$$

By subtracting (20) from (16).

$$P^{\bullet} = \alpha^t P_0^{\bullet} + \sum_{r=0}^{t-1} (\alpha^r \beta) Y^{\bullet} + \sum_{r=0}^{t-1} \alpha^r V_{t-r} \tag{21}$$

where $P^{\bullet} = P_t - \bar{P}$

and $Y^{\bullet} = Y_t - \bar{Y}$

and $P_0^{\bullet} = P_0 - \bar{P}$.

In other words the enodgeneous and exogeneous variables are expressed as derivatives from their equilibrium values.

Equation (21) expresses the time path of P in terms of three components:

a. its initial position, P_0^*

b. the time path of Y

c. the time path of the disturbances.

The inherent dynamic properties of the model refer to the characteristics of the time path of P following an initial deviation from equilibrium without further changes in the time pathes of Y or U. By concentrating on the matrix of α's we can estimate the characteristic roots of the model, which determine the specification of the cycles if they exist[9].

III. SPECTRAL ANALYSIS[10]

Studying cycles in economics is merely a description of economic time series after certain modifications. Time series can either be described in the time domain or in the frequency domain. The time domain is more familiar to economists. The frequency domain is more familiar to communications engineers and its terminology is more related to their language habits.

Expressing time series in the time domain takes two steps. to construct an autocovariance function between present and lagged values of the series. Second to construct the autocorrelation function, by simple normalization.

Let x(t) be a time series, with zero mean observed at points of time t = ... -2, -1, 0, 1, 2, ... and assume that x(t) is a continuous function of time. Taking τ as any time lag, the autocovariance of x(t) is equal to:

$$(t,\tau) = E\{[x(t) - Ex(t)]\,[(t+\tau) - Ex(t+\tau)]\} \qquad (1)$$

Since $Ex(t) = 0$, $Ex(t+\tau) = 0$

$$(t,\tau) = E[x(t)x(t+\tau)] \qquad ,\tau = 0, \pm 1, \pm 2, \ldots \qquad (2)$$

Moreover, if we have a limited number of observations equal to 2T which starts from -T, -T+1,..., 0,... up to +T, and if the number of lags is equal to M, τ will be either negative or positive. By using

(2) in a summation form, and assuming that $x(t)$ is stationary*
(variance of τ = variance of $-\tau$), then the autocovariance function will be:

$$C(\tau) = \frac{1}{2T} \sum_{t=-T+M}^{T-M} \left\{ x(t)x(t+\tau) + x(t)x(t+\tau) \right) \qquad (3)$$

Notice that $C(0)$ is equal to the variance of $x(t)$, and $C(\tau)$ is independent of t. (3) is also called the mean lagged product. By dividing the autocovariance $C(\tau)$ by the variable $C(0)$, we get the autocorrelation for lag τ:

$$R(\tau) = C(\tau)/C(0) \qquad (4)$$

Plotting $R(\tau)$ against τ gives the correlogram of $x(t)$. $C(\tau)$ may directly without normalization be plotted against τ as a description of $x(t)$.

Before reviewing the description of time series in the frequency domain, it would be of interest to state some definitions (11). The frequency is a measure of rate of repetition over time. Theoretically it can be the number of cycles per second. In economics, it is the fraction of cycle completed in one time period. The amplitude is the maximum value of the oscillation. The phase is the fraction of a cycle within a certain time at a given frequency.

Spectral analysis starts with a basic assumption. The value of a time series at each time can be expressed as a function of a particular sinusoidal wave. In other words:

$$x(t) = A \sin \lambda t \qquad (5)$$

where A is the amplitude and λ is the frequency.

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* A process $x(t)$ is stationary if $P\left[x(t+\tau)\right] = P\left[x(t)\right]$

In estimating the spectrum, it is of special interest to concentrate on Gaussian normal time series. A series is of this kind if for any T observations and any choice of t and points $t_1 < t_2 < \ldots \ldots < T$, the series has a T-variate normal distribution. If the means in that distribution are all zero, it will be possible to specify the distribution by its covariance function or $E(x(t_i)x(t_k))$ where $i,k = 1,\ldots,T$, if the process x(t) is stationary, its variance will be a function of the time lag $\tau = t_i - t_k$ only.

To give an example, let a_i and b_i be two independent, zero-mean normal variates with common variance σ_i^2 for each i. Therefore:

$$E\, a_i a_k = E\, b_i b_k = \sigma_i^2 \quad i = k,\ i = 1,\ldots,n$$

$$0 \quad i \neq k$$

$$E\, a_i b_k = 0 \qquad i,k = i,\ldots,n$$

and let λ_i be n discrete frequencies expressed in radians such that $0 \leq {}_1 < {}_2 < \ldots < \lambda_n \leq \pi$. Then, x(t) is a zero mean, stationary Gaussian process if:

$$x(t) = \sum_{i=1}^{n} (a_i \cos \lambda_i t + b_i \sin \lambda_i t) \qquad (6)$$

In this case, σ_i^2 is a function of frequency and is called the spectrum of x(t). To show this relation, we will derive the variance σ_i^2 as follows:

$$x(t) = \sum_{i=1}^{n} (a_i \cos \lambda_i t + b_i \sin \lambda_i t)$$

$$x(t+\tau) = \sum_{k=1}^{n} \left[a_k \cos \lambda_k (t+\tau) + b_s \sin \lambda_k (t+\tau) \right]$$

$$\gamma(t) = E[x(t)\ x(t+\tau)]$$

$$= E\Big\{ \sum_{i=1}^{n} \sum_{k=1}^{n} a_i a_k \cos \lambda_i\, t \cos \lambda_k\, (t+\tau)$$

$$+ a_i b_k \cos \lambda_i t \sin \lambda_k (t+\tau)$$

$$+ a_k b_i \sin \lambda_i t \cos \lambda_k (t+\tau)$$

$$+ b_i b_k \sin \lambda_i t \sin \lambda_k (t+\tau) \Big\}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \sigma_i^2 \cos \lambda_i t \cos \lambda_k (t+\tau) + 0$$

$$+ 0 + \sigma_i^2 \sin \lambda_i t \sin \lambda_k (t+\tau)$$

$$= \sum_{i=1}^{n} \sigma_i^2 \Big[(\cos \lambda_i t \cos \lambda_k\, (t+\tau) + \sin \lambda_i t \sin \lambda_k (t+\tau))\ .$$

But $\sin(\Theta + \emptyset) = \sin \Theta \cos \emptyset + \cos \Theta \sin \emptyset$

and $\cos(\Theta + \emptyset) = \cos \Theta \cos \emptyset - \sin \Theta \sin \emptyset$

then:

$$\gamma(t) = \sum_{i=1}^{n} \sigma_i^2 \ \cos \lambda_i t\ (\cos \lambda_k t \cos \lambda_k t - \sin \lambda_k t \sin \lambda_k \tau)$$

$$+ \sin \lambda_i t\ (\sin \lambda_k t \cos \lambda_k \ \ + \cos \lambda_k t \sin \lambda_k \tau)$$

$$\gamma(t) = \sum_{i=1}^{n} \sigma_i^2 \cos \lambda_i \tau. \tag{7}$$

By integrating both sides over $\pm\pi$ with respect to , and dividing through by π, we get

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \gamma(t)\, d\tau = \frac{1}{\pi} \int_{-\pi}^{\pi} \Big(\sum_i \sigma_i^2 \cos \lambda_i \tau\Big)\, d\tau .$$

By multiplying through by $\cos\lambda_k\tau$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\gamma(t)\cos\lambda_k\tau d\tau = \frac{1}{\pi}\int_{-\pi}^{\pi}(\sum_i \sigma_i^2 \cos\lambda_i\tau\cos\lambda_k\tau)d\tau$$

$$= \frac{1}{\pi}\sum_i \sigma_i^2 \int_{-\pi}^{\pi}\cos\lambda_i\tau\cos\lambda_k\ d\tau$$

$$= \frac{1}{\pi}\sum_i \sigma_i^2 \begin{cases}\pi;\lambda_i = \lambda_k \\ 0;\lambda_i \neq \lambda_k\end{cases}$$

$$= \sigma_i^2$$

$$\therefore\ \sigma_i^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\gamma(t)\cos\lambda_i\tau d\tau \qquad (8)$$

The spectrum of x(t), σ_i^2, is expressed in (8) as a function of the autocovariance function γ(t), which is in turn a function of the time lag τ. In general, not restricted with a finite parameter scheme, any covariance stationary random process x(t) can be expressed as:

$$x(t) = \int_0^\infty \cos\lambda td\ U(\lambda) + \int_0^\infty \sin\lambda td\ V(\lambda) \qquad (9)$$

where d U (λ) and d V (λ) are random variables and

$$E[d\ U(\lambda)\ d\ U(\lambda')] = E[d\ V(\lambda)\ d\ V(\lambda')] = 0 \qquad (10)$$

$$\text{all } \lambda \neq \lambda'$$

$$E\ d\ U(\lambda)\ d\ V(\lambda') = 0 \qquad \text{all } \lambda \text{ and } \lambda' \qquad (11)$$

$$E[d\ U(\lambda)]^2 = E[d\ V(\lambda)]^2 = d\ F(\lambda) \qquad (12)$$

The function d F (λ) is called the power spectrum of x(t). The reader will notice the similiarity between a_1, b_i and U(λ), V(λ).

Now (9) shows that the time series is decomposed into a superposition of sine and cosine waves of different frequencies and random amplitudes d U (λ) and d V (λ). The source equation expresses real time series in terms of real numbers of sines, cosines and random amplitudes. On the other hand the sine and cosine functions can be expressed respectively as infinite series:(12)*

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* $$\sin x = x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - + \ldots \quad (13)$$

$$\text{cosin } x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots \quad (14)$$

But, we know that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Then:

$$e^{jx} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \ldots \quad (15)$$

If $j = \sqrt{-1}$, $j^2 = -1$, $j^3 = 1-$, $j^4 = 1$ and so forth,

Then:

$$e^{jx} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \ldots$$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots) + (jx - j\frac{x^3}{3!} + j\frac{x^5}{5!} - \ldots)$$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots) + j(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots)$$

===

This gives us the following equation:

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \tag{19}$$

Euler's equation together with expressions (18) and (19) can be used to express the time series x(t) in a <u>complex form</u>. This enables us to combine the phase, the particular frequency, and the amplitude into one complex exprension of the power spectrum d F (λ).

This transformation is done as follows:

$$x(t) = \int_0^{\infty} \cos \lambda t d\, U(\lambda) + \int_0^{\infty} \sin \lambda t d\, V(\lambda) \quad \text{(equation 9)}$$

$$= \int_0^{\infty} \left(\frac{e^{j\lambda t} + e^{-j\lambda t}}{2}\right) d\, U(\lambda) + \int_0^{\infty} \left(\frac{e^{j\lambda t} - e^{-j\lambda t}}{2j}\right) d\, V(\lambda).$$

===

By comparing this expression with (13) and (14) we get:

$$e^{jx} = \cos x + j \sin x \tag{16}$$

similarly

$$e^{-jx} = \cos x - j \sin x \tag{17}$$

Both (16) and (17) are different expressions of "Euler's Equation".

Their sus is

$$e^{jx} + e^{-jx} = 2 \cos x$$

$$\therefore \quad \cos x = \frac{e^{jx} + e^{-jx}}{2} \tag{18}$$

The difference between (16) and (17) is:

$$e^{jx} - e^{-jx} = 2 j \sin x$$

By changing the integration limits we get:

$$x(t) = \int_{-\infty}^{\infty} \frac{1}{2} e^{j\lambda t} \, d\,U(\lambda) - \int_{-\infty}^{\infty} \frac{1}{2} e^{j\lambda t} \, j \, d\,V(\lambda)$$

$$= \int_{-\infty}^{\infty} e^{j\lambda t} \frac{1}{2} (d\,U(\lambda) - j\,d\,V(\lambda))$$

$$= \int_{-\infty}^{\infty} e^{j\lambda t} \, d\,Z(\lambda) \qquad (20)$$

Where $d\,Z(\lambda) = \frac{1}{2}(d\,U(\lambda) - j\,d\,V(\lambda))$.

Also $d\,Z(\lambda)$ is a complex random variable.

To express the power spectrum in terms of the complex number $e^{j\lambda t}$, it is necessary to express the covariance function in terms of the latter. We already know that the covariance function is

$$\gamma(\tau) = E[x(t)\, x(t+\tau)].$$

By using (9)

$$\gamma(\tau) = E\left(\int_0^{\infty} \cos\lambda t \, d\,U(\lambda) + \int_0^{\infty} \sin\lambda t \, d\,V(\lambda)\right).$$

$$\left(\int_0^{\infty} \cos\lambda'(t+\tau) \, d\,U(\lambda') + \int_0^{\infty} \sin\lambda'(t+\tau) d\,V(\lambda')\right)$$

By multiplying through and applying property (11):

$$\gamma(\tau) = \int_0^{\infty} \cos\lambda t \cos (t+\tau)[d\,U(\lambda)]^2 + \int_0^{\infty} \sin\lambda t \sin\lambda(t+\tau)$$

$$[d\,V(\lambda)^2]$$

From equation (12), we get:

$$\gamma(\tau) = \int_0^{\infty} [\cos\lambda t \cos\lambda(t+\tau) + \sin\lambda t \sin\lambda(t+\tau)] \, d\,F(\lambda).$$

By applying the formulas of sine and cosine of the sum of two angles used to derive (7) above, we get:

$$\gamma(\tau) = \int_0^{\infty} \cos \lambda\tau \, d F(\lambda) \qquad (21)$$

From (18)

$$\gamma(\tau) = \int_0^{\infty} \frac{e^{j\lambda t} + e^{-j\lambda\tau}}{2} \, d F(\lambda)$$

$$\therefore \quad \gamma(\tau) = \int_{-\infty}^{\infty} \frac{1}{2} e^{j\lambda\tau} \, d F(\lambda) \qquad (22)$$

By the same token:

$$d F(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-j\lambda\tau} \gamma(\tau) \qquad (23)$$

Finally, we are able to deduce from x(t) an expression of its power spectrum which is a function of the frequency λ, the lag τ and the amplitudes d $\dot{U}$ (λ) and d V (λ). In the next section, we will discuss how to use this function in studying cycles in general and long swings in particular.

IV. CYCLES AND POWER SPECTRA

So far we have assumed that the time series x(t) is probabilistic in nature, i.e., it is a stochastic process. The observed value of x at a particular point of time t is a sample chosen in some way from a universe containing all possible values of x at time t. Then for convenience we studied the properties of certain functions of x(t) assuming certain distribution, normal or Gaussian, and a stationary condition. If normality is not fulfilled, the spectrum and the autocovariance functions will still describe x(t) although in a less exact way.

The observed sequence of x will be used to infer the characteristics of the suposedly random function x(t). In other words, we infer from the random function the characteristics of the sample realization, which is an opposite to what pure statistics usually does.

Through this procedure of using the power spectra, we can determine the expected value of the amplitude associated with each frequency of oscillation. By using a specific weighting structure, we can separate important cyclical components from insignificant ones. The power spectrum itself expresses the variance associated with the corresponding frequency. Therefore, the power spectrum specifies the contribution of each frequency to the total variance. Thus, spectral analysis is merely a variance analysis in terms of frequency.

Since spectral analysis is based on an explicitly formulated stochastic model, tests of the statistical significance of the components of individual spectra are possible. This means that we can test if the contribution of cycles of a particular duration is significantly different from zero.

Economic time series are of a particular type. They are non-stationary; their expected values and covariances are functions of time. Therefore, it is necessary to adjust them before estimating their spectra. However, eliminating the trend may eliminate with it relatively more of the functions of time which are of some interest, (which are called signal). It might eliminate relatively less of those undesirable time functions (called noise). This is technically referred to as lowering the signal-noise ratio. First differences usually cause this deficiency. Adelman eliminated the trend from the series she studied by fitting a least squares linear trend to the logarithms of the original data and used the deviations from the fitted trend to estimate the spectra.

In estimating the spectra, we will be interested in studying a cycle of a special frequency. This makes it necessary to use a weighting system that gives more power around the frequency of concern. A physical example is to "look at the spectrum through a certain window". A weighting system of this kind is thus called a spectral window. The window helps to estimate the average power in a frequency band centered around the frequency in question. However, there is an opposite relation between the band width and the variance of the estimated power spectrum. It is desirable to make the band width minimum so that we keep our estimate closest to the desired frequency. But this will reduce the accuracy of our estimate. Adelman used a window suggested by Parzen which had the smallest variance and the largest bandwidth of all windows suggested so far.

In view of the incomplete elimination of the trend and of the limited number of observations in economic time series, the estimated spectra will show relatively more power at very low frequencies. These difficulties cause the power at high frequencies to leak to low frequencies. This is called power leakage. Figure (1) shows a spectra estimated by Nerlove for the Fed. Res. index for industrial production plotted against different frequencies. The figure shows a clear leakage of power.

The leaked power should be filtered back to lower frequencies. There are different mathematical filters designed for this purpose. Adelman uses one designed by Parzen which filters out most of the power at a frequency lower than 1/18 of a cycle per year. Figure (2) shows a comparison between a filtered and unfiltered spectrum estimated by Adelman.

By choosing the proper frequency, and looking at the spectra through the proper windows, we can isolate the cycle of interest. Nevertheless, filtering the data has to be in order.

V. LONG SWINGS THROUGH THE WINDOW & AN APPLICATION OF SPECTRAL ANALYSIS

Do long cycles exist? This simple question can be answered using spectral analysis[13]. After using certain smoothing procedures, long waves of 10-20 years were observed. The question is whether these long movements are a class of economic phenomena independent, but interacting with shorter cyclical fluctuations. A great deal of doubt would be justifiable if the smoothing processes used can add cyclical characteristics to the data.

Random shocks can usually produce long swings. Adelman proved that by working with the Klien-Goldberger[14] model. The traditional smoothing techniques used by Kusnets and Burns[15] can produce spurious cyclical fluctuations into the basic series. In spectral terms, their smoothing procedures can be interpreted as a filtering process. Mathematically, this can be proved unnecessary in the context of using difference equations[16]. Nevertheless, the methods used by Abramovitz are too complicated to construct a priori formulation for them.

As we have seen in spectral analysis, it is not necessary to eliminate shorter cycles from the series before studying longer cycles. On the other hand, it is possible to determine the cycles of all durations simultaneously. This is a greater improvement over the traditional techniques.

Figures 3-11 show the empirical results of Adelman's spectral analysis for the U.S. time series of output, investment consumption, employment, capital stock, productivity of labor, productivity of capital, and the wholesale price index. She used many series pertaining to each economic variable (17). Only one representative spectrum is shown for each variable because of the qualitative similarity between the spectra of each variable. The spectrum of construction showed a large amount of distortion after the filtering process, and hence it was omitted.

The filtered spectra of figures 3-10 show no existence of long cycles since 1890. The entire variance can be attributed solely to leakage from power at frequencies lower than 1/18. The power remaining in the long swing domain of 15 years can be attributed to the removal of the entire trend from the data. Therefore, Adelman attributed the long cycles observed in the U.S. economy since 1890 to two factors:

a. The introduction of spurious long cycles by the smoothing process.

b. The necessity for averaging over a statistically small number of random shocks.

This is consistent with the observation of the direct association of long cycles with a random distribution of frequencies. It is also consistent with the long swing's association with random shocks (18). Strong exogeneous shocks have characterized almost all the beginnings of long cycles, which is still consistent with these results (19).

Figure 11, however, shows that any long cycle which may exist in population must be longer than 15 years.

Adelman's conclusion is that long cycles are a myth. However, we may note that here research is subject to the following limitations:

1 . The assumption of normality, and how "imperfect" the spectra are would misrepresent the process if normality does not exist.

2 . How efficient the weighting system, or the window used, is.

3 . How efficient the filter used in relation to variance and band width is.

4 . The importance of the population factor, if it has long swings of longer than 15 years, and its effects on other variables.

5 . The importance of the construction factor, if its spectra can be reasonably estimated.

In the light of these restrictions, we see that Adelman's conclusion can still be subject to further mathematical and emperical work.

FOOTNOTES

1 . The Main reference to this critic is T. J. Koopman's "Measurement Without Theory". Review of Economics and Statistics, Vol. 29, August 1947.

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5 . C. F. Christ, "Aggregate Econometric Models," American Economic Review, June 1956.

6 . L. B. Klien, An Introduction to Econometrics, Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 91-92.

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R. B. Blackman and J. W. Tukey, The Measurement of Power Spectra, New York and Dover, 1958.

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11. R. B. Blackman and J. W. Tukey, pp. 167-178.

Adelman, p. 448.

12. H. A. Simmons, Plane and Spherical Trigonometry, J. Wiley and Sons, Inc., 1945, pp. 344-345.

13. Irma Adelman, op. cit.

14. L. Klien, A. S. Goldberger, An Econometric Model of the United States, 1929-1952, Amesterdam, 1955.

I. Adelman, "Long Cycles, A Simulation Experiment", F. Balderston, A. Haggatt, ed., Proceedings of a Conference on Simulation, Cincinnati, 1964.

15. See the reference page.

16. P. J. Taubman, "On the Existence of Long Cycles", Discussion Paper No. 2, Economic Research Service Unit, Univ. of Pennsylvania.

17. See the appendix of Adelman's article in the AER.

18. See Adelman, Proceedings of a Conference on Sumulation.

19. M. Abramovitz..

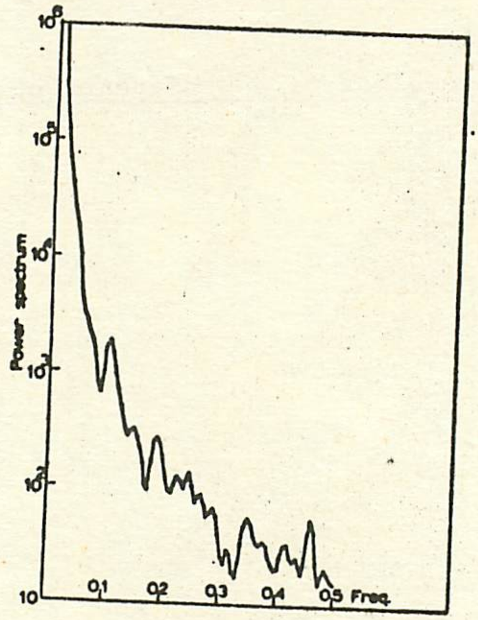

Figure 1

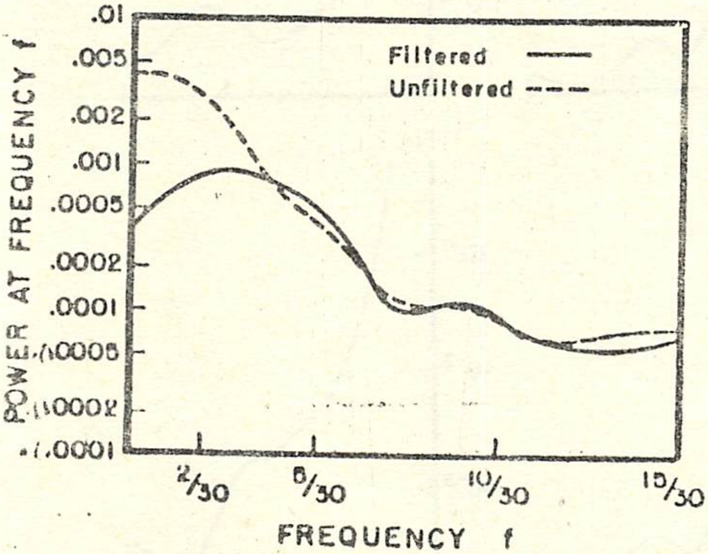

Figure 2

Power Spectrum of Natural Logs of Deviations From Trend of GNP

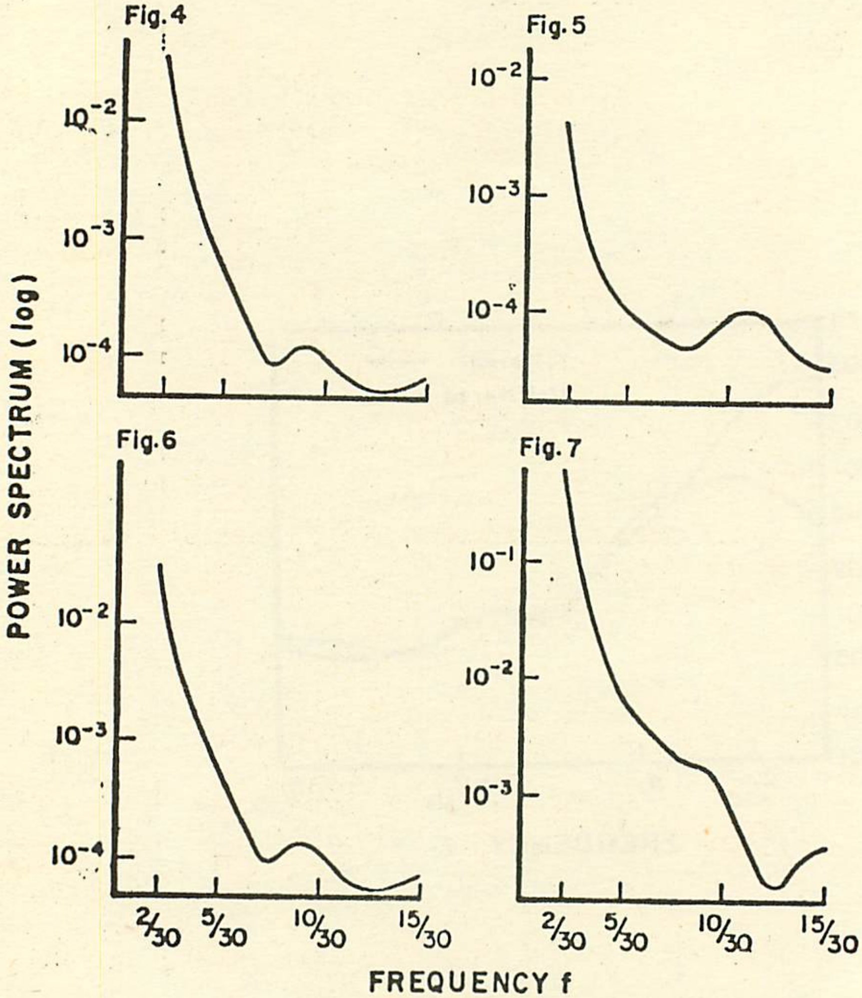

Figure 3. Power Spectrum of Natural Log of Deviations From Trend of Output Series.

Figure 4. Power Spectrum of Natural Log of Deviations From Trend of Productivity of Labor.

Figure 5. Power Spectrum of Natural Log of Deviations From Trend of Productivity of Capital.

Figure 6. Power Spectrum of Natural Log of Deviations From Trend of Investment.

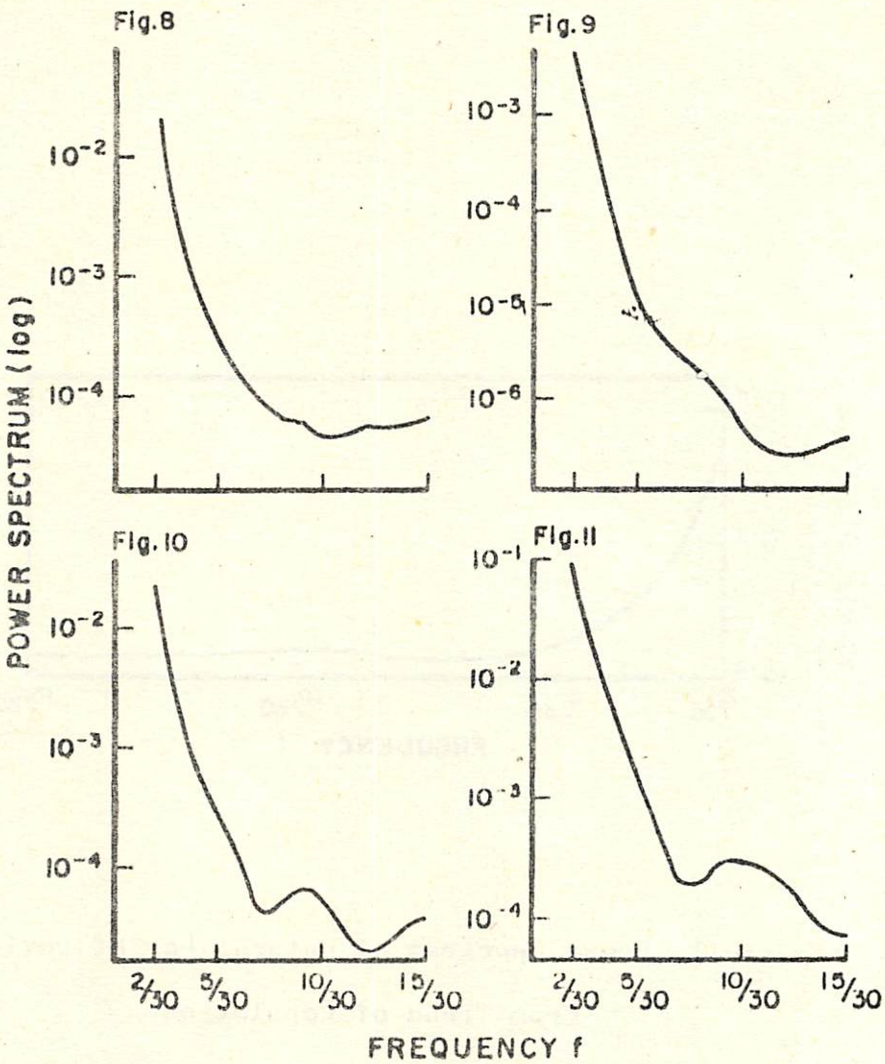

Figure 7. Power Spectrum of Natural Log of Deviations From Trend of Consumption.

Figure 8. Power Spectrum of Natural Log of Deviations From Trend of Capital Stock.

Figure 9. Power Spectrum of Natural Log of Deviations From Trend of Employment.

Figure 10. Power Spectrum of Natural Log of Deviations From Trend of Wholesale Price Index.

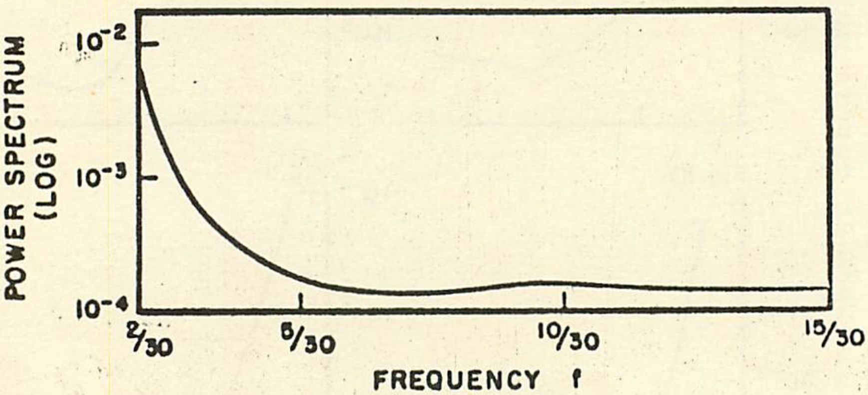

Figure 11. Power Spectrum of Natural Log of Deviations From Trend of Population

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