UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

التخطيط القومي

Memo. No. 498

The Numerical Solution For The Roots of Polynomials (Part II)

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19th October 1964.

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The numerical solution for the roots of polynomials

1. Introduction

In [1] we have discussed the numerical solution for the real roots of equations. The bisecting method and the false position method as shown before are very simple, complete general and always convergent. General method of iteration and other methods with its convergence were explained.

This chapter deals with those methods which are applicable to finding the roots, real as well as complex, for the polynomials, such as the iteration, Lin-Bairstow and Dandelin-Graeffe methods. For computation, every method was followed with flow-charts.

The transition from numerical analysis to programming can generally be facilitated by a flow-chart. The flow-chart is a graphic representation of the procedures and shows how the alternatives fit together. When numerical analysis is complete and the transition from mathematical language to machine language begins, the flow-chart can be an excellent device for establishing continuity.

2. Determination of the limits for the roots of a polynomial

2.1 Limits for real roots by Maclaurin's theorem

The real roots of the equation

$$a_0x^n+a_1x^{n-1}+\dots+a_n = 0 \qquad (1)$$

where $a_0 > 0$, satisfy the inequality

$$x < 1 + \sqrt[m]{\frac{A}{a_0}} \qquad (2)$$

where m is the suffix of the first negative coefficient in the series $a_0, a_1, a_2, \dots, a_n$, and A is the largest of the moduli of the negative coefficients.

This method allows one to determine also a lower limit for the roots. For this, it is necessary to make the substitution $x=-y$ and to multiply the equation by $(-1)^n$ in order that the first coefficient remains positive; after this we can make use once again of formula (2).

If $|a_0|$ is considerably smaller than A, formula (2) gives a widely over estimated limit. In this case the polynomial may be broken down into the sum of several polynomials, the first coefficients of which are positive, and the upper limit for each of these may be determined. The greatest of these upper limits determines the upper limit of the roots of the initial polynomial. In a lucky breaking down of the polynomial, the limits are determined a good deal more accurately than by the first method. The decomposition is usually a good one if approximately the same values are obtained for all the upper limits.

Example (1)

The roots of the equation

$$2x^9+x^7-x^4+19x^3-24x^2+11=0 \qquad (3)$$

satisfy the inequality

$$x < 1 + \sqrt[5]{\frac{24}{2}} = 1+\sqrt[5]{12} \approx 2.7 .$$

Put $x = -y$ in (3) we get

$$2y^9+y^7+y^4+19y^3+24y^2-11 = 0$$

$$y < 1+ \sqrt[9]{\frac{11}{2}} \approx 2.3$$

from which $x > -2.3$. Thus the roots of the equation lie in the interval

$$-2.3 < x < 2.7 .$$

Example (2) : To determine an upper limit for the roots of the equation :

$$x^5+12x^4-8x^3+2x^2 - 5680x+112 = 0$$

According to formula (2) we get :

$b=1+\sqrt[2]{5680} \approx 76.5$. Thus $x < 76.5$.

Dividing the polynomial into two added components:

$$P_1(x) = 0.1x^5 - 8x^3$$

$$P_2(x) = 0.9x^5 +12x^4+2x^2 - 5680x+112.$$

We find upper limits for their roots:

$$b_1=1+\sqrt[2]{\frac{8}{0.1}} \approx 10 , \quad b_2=1+\sqrt[4]{\frac{5680}{0.9}} \approx 10.$$

whence $x < 10$.

Dividing the same polynomial into three added components :

$P_1(x) = 0.2x^5 - 8x^3$,

$P_2(x) = 0.8x^5 + 2x^2 - 1680x+112$,

$P_3(x) = 12x^4 - 4000\ x$,

we find

$b_1 = 1+ \sqrt{\frac{8}{0.2}} = 7.5$,

$b_2 = 1+ \sqrt[4]{\frac{1680}{0.8}} = 7.8$,

$b_3 = 1+ \sqrt[3]{\frac{4000}{12}} = 7.9$,

whence $x < 7.9$.

2.2 Limits for complex roots by Westerfield and Parodi

Consider the polynomial

$$x^n+a_1x^{n-1}+\ldots+a_n \quad (4)$$

with real and complex coefficients .

We shall denote by q_r the quantities[x)]

$$\sqrt[r]{|a_r|}\ ,\ r = 1,2,\ldots,n \quad (5)$$

arranged in order of decreasing magnitude

$$q_1 \geqslant q_2 \geqslant \ \ldots \geqslant q_n \quad (6)$$

It has been showed by Westerfield that all roots (real and complex) of the polynomial (4) satisfy the conditions:

$$|x| \leqslant q_1 + q_2 \quad (7)$$

x) The real positive value of the root is taken.

and

$$|x| \leq q_1+0.6180\ q_2+0.2213\ q_2+0.0883\ q_4 +$$
$$+0.0375\ q_5+0.0185\ q_6+0.0074\ q_7\ +0.0081\ q_8 \quad (8)$$

In the case of the coefficient a_1 of the polynomial (4) being much larger than the other coefficients, we can apply a simple and effective estimate found by M.Parodi:

Let $|a_1| > 2\sqrt{S}$

where

$$S = |a_2|+|a_3|+\dots+|a_n| \quad (9)$$

and

$$S > 1. \quad (10)$$

The polynomial (4) has one, and only one, root within the circle

$$|x+a_1| \leq \sqrt{S} \quad (11)$$

Example : Find the limits for the roots of the polynomial

$$x^4 - 48x^3+797x^2 - 5350x+12297 = 0$$

$\sqrt[1]{|-48|}=48$, $\sqrt[2]{|797|} \approx 28.2$,
$\sqrt[3]{|-5350|} \approx 17.5$, $\sqrt[4]{|12297|} \approx 10.5$

Thus $q_1=48$, $q_2=28.2$, $q_3=17.5$, $q_4=10.5$.

According to formula (7) we find :

$$|x| \leq 76.2 \ .$$

If we apply formula (8) we get the following value for the limits:

$$|x| \leq 48+17.4+3.9+0.9= 70.2$$

By Maclaurin's method,we find from (2) that $x < 49$; however this gave a limit only for real positive roots; while the value 70.2 is a limit for the moduli of all roots (real and complex).

3. Approximate Determination of the Roots of a polynomial by means of Vieta's Formula

The method provides the possibility of the easy determination of roots which are larger or smaller in magnitude than most of the other. Its advantage over other method lies in the fact that it requires a minimal quantity of calculation. However, the accuracy with which roots are determined is often very small : usually one succeeds indetermining only the order of magnitude of the largest and the smallest roots.

Vieta's formulae

These formulae connect the roots $x_1 , x_2 , \ldots , x_n$ of the polynomial

$$P(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

with its coefficients :

$$\left.\begin{aligned}
-\frac{a_1}{a_0} &= x_1 + x_2 + \ldots + x_n \\
\frac{a_2}{a_0} &= x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n \\
-\frac{a_3}{a_0} &= x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 + \ldots + x_{n-2} x_{n-1} x_n \\
&\vdots \\
(-1)^{n-1} \frac{a_{n-1}}{a_0} &= x_1 x_2 \ldots x_{n-2} x_{n-1} + x_1 x_2 \ldots x_{n-2} x_n + \ldots + \\
&\quad \ldots + x_2 x_3 \ldots x_{n-1} x_n \\
(-1)^n \frac{a_n}{a_0} &= x_1 x_2 \ldots\ldots x_n
\end{aligned}\right\} (A_1)$$

3.1 Calculation of the larger roots

Let

$$|x_1| \geqslant |x_2| \quad \ldots . \geqslant |x_n| . \qquad (A_2)$$

If $|x_1|$ is appreciably larger than the moduli of all the other roots, then it is possible to ignore the numbers x_2, x_3, ... , x_n

$$- \frac{a_1}{a_0} \approx x_1 \qquad (A_3)$$

Thus the largest roots approximately satisfies the equation

$$a_0 x + a_1 = 0 \qquad (A_4)$$

If the moduli of the first two roots are appreciably larger than the moduli of the remaining roots, we get from the first two of Vieta's formulae :

$$\left.\begin{aligned} - \frac{a_1}{a_0} &\approx x_1 + x_2 \\ \frac{a_2}{a_0} &\approx x_1 \quad x_2 \end{aligned}\right\} \qquad (A_5)$$

Thus the two larger roots of the given polynomial approximately satisfy the equation

$$a_0 x^2 + a_1 x + a_2 = 0 \qquad (A_6)$$

Analogously, if the moduli of three roots are appreciably larger than the moduli of the remaining ones, these roots are approximately determined by the equation :

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0 \qquad (A_7)$$

The truth of this statement follows from the relation :

$$\left.\begin{array}{l} -\dfrac{a_1}{a_0} \approx x_1 + x_2 + x_3 \\[2ex] \dfrac{a_2}{a_0} \approx x_1 x_2 + x_1 x_3 + x_2 x_3 \\[2ex] -\dfrac{a_3}{a_0} \approx x_1 x_2 x_3 \end{array}\right\} \qquad (A_8)$$

obtained from (A_1) and being Viet's formulae for equation (A_7).

3.2 Calculation of the smaller roots

If we substitute into (A_3) a new argument $y = \frac{1}{x}$ and apply the results we have got for large roots, and then change back from y to the argument $x = \frac{1}{y}$, we get the following results.

If $|x_n|$ is appredicably smaller than the moduli of the other roots of the given polynomial, $|x_1|$ may be approximately determined by the equation

$$a_{n-1} x + a_n = 0 \qquad (A_8)$$

If the moduli of x_{n-1} and x_n are appreciably smaller than the moduli of the remaining roots, the three roots are approximately determined by the equation :

$$a_{n-3} x^3 + a_{n-2} x^2 + a_{n-1} x + a_n = 0 \qquad (A_9)$$

Analogous theorems hold also for any number of roots with larger or smaller moduli.

<u>Example</u> Determine the roots of the polynomials

$$P(x) = x^4 + 39 x^3 + 958 x^2 - 1080 x - 2000$$

we try to determine the largest root by means of the equation

$$x + 39 = 0$$

Then $x_1 = -39$. However a trial convinces us that $x_1 = -39$ is not even approximately a root.

We form the second equation :

$$x^2 + 39 x + 958 = 0$$

From which

$$x_1 = -19.5 + 24.04 i$$
$$x_2 = -19.5 - 24.04 i$$

The exact roots are $x_1 = -20 \pm 24.48 i$,

$$x_2 = -20 - 24.48 i$$

For determining the smallest root we take the equation

$$-1080 x - 2000 = 0,$$

from which $x_4 \approx -1.85$. A trial shows that the number found is not a root.

We take the equation $958 x^2 - 1080 x - 2000 = 0$

Then $x_4 = -0.99$, $x_3 = 2.12$ (exact values are $x_4 = -1$, $x_3 = 2$)

4. Iteration in the complex plane

A study closely analogous to that in [1] for the iteration methods may be applied to the solution of equations involving functions of a complex variable. For example, Newton's method may be applied readily if a suitable starting value is available.

4.1 Example

Using the starting value $x_o=i, i=\sqrt{-1}$, and applying Newton's formula to the equation :

$$f(x)=x^4+x^3+5x^2+4x+4=0 \tag{12}$$

we obtain

$$x_1=i-\frac{f(i)}{f'(i)}=i-\frac{3i}{1+6i}=0.486+0.919\,i \quad ,$$

$$x_2=0.486+0.919\,i-\frac{-0.292+0.174i}{1.780+6.005i}=-0.499+0.866i$$

as two approximations to the solution

$$x=\frac{-1+i\sqrt{3}}{2}$$

4.2 The square root of a real number

If we write $x=\sqrt{a}$, then $f(x)=x^2-a$ where $a \gtrless 0$. Newton's iteration method here assumes the form

$$x_{i+1}=x_i-(x_i^2-a)/2x_i \tag{13}$$

or, more simply

$$x_{i+1}=\frac{1}{2}\left(x_i+\frac{a}{x_i}\right) \tag{14}$$

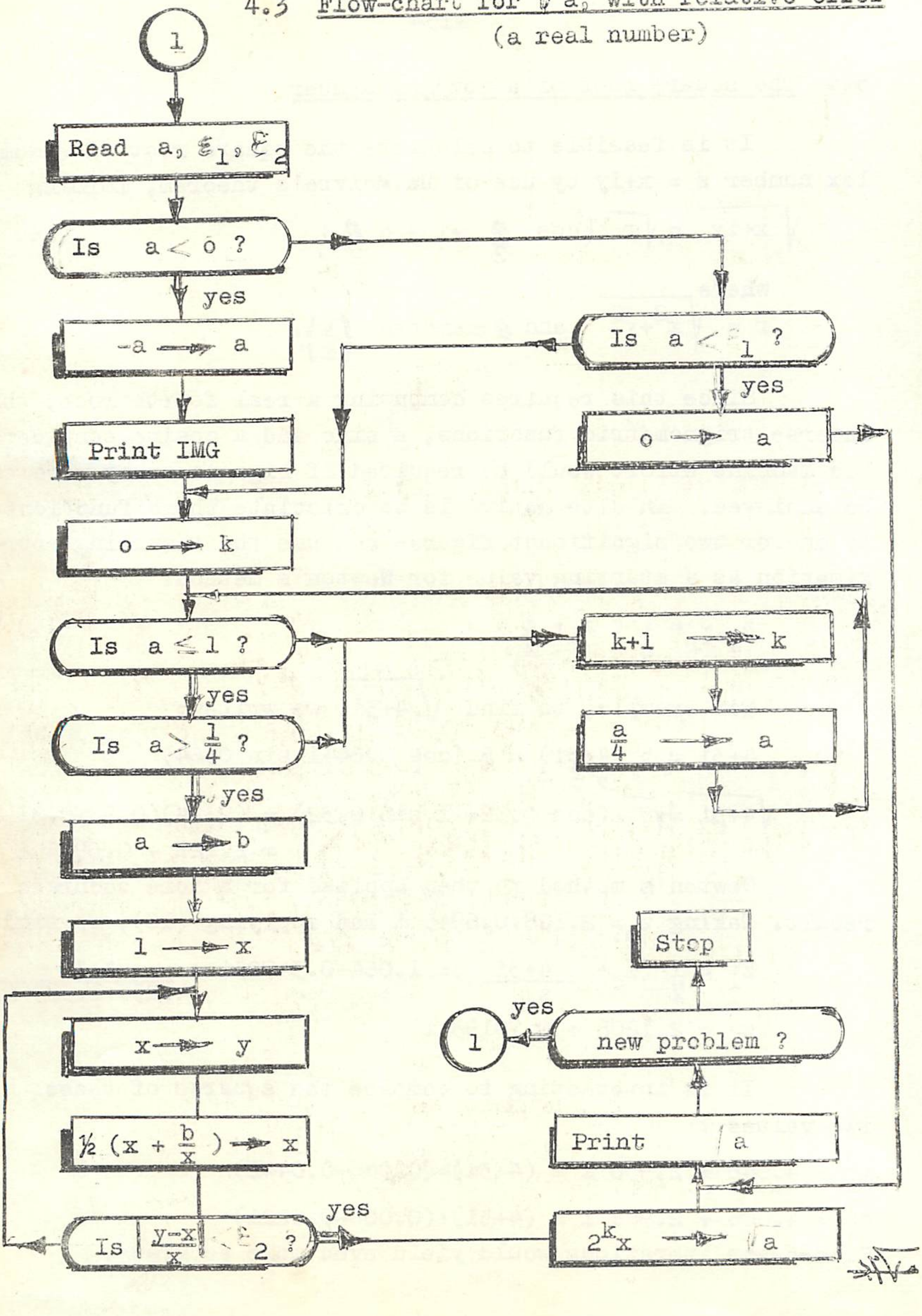
If recursion (14) is to be coded for a computer, it will be desirable to have the starting value x_o chosen to exceed the first iterate x_1. Since the code should be applicable to finding the square root of any positive, number a, large or small, and since it is convenient to start with a preasigned value, say $x_0 = 1$, we introduce a change of variables to meet these requirements. If the program is based on decimal arithmetic, we introduce a new quantity b which provides that $a = 10^{2k}b$, b an integer, and $\frac{1}{100} < b \leqslant 1$

We find $\sqrt{b}$ using (14) and convert to $\sqrt{a}$ through the relation $\sqrt{a} = 10^k \sqrt{b}$.

If the computations indicated in (14) are done in the base 2, a natural choice of range for b is normally $\frac{1}{4} < b \leqslant 1$, Such that $a = 2^{2k} b$. The starting value $x_0 = 1$ again yields a decreasing sequence of iterates converging to $\sqrt{b}$ and hence $\sqrt{a} = 2^k \sqrt{b}$. The sequence of calculations is indicated in the following flow cahrt.

4.3 Flow-chart for $\sqrt{a}$, with relative error

(a real number)



4.4 The square root of a complex number

It is feasible to calculate the square root of a complex number $z = x+iy$ by use of De Moivre's theorem, forming

$$\sqrt{x+iy} = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \qquad (15)$$

where

$$r = \sqrt{x^2+y^2} \quad \text{and } \theta = \arctan\left(\frac{y}{x}\right).$$

Since this requires computing a real fourth root, three inverse trignometric functions, a sine and a cosine considerable machine effort would be required if high accuracy were to be achieved. An alternative is to calculate these functions to one or two significant figures and use the resulting approximation as a starting value for Newton's method:

$$z_{i+1} = \frac{1}{2}\left(z_i + \frac{z}{z_i}\right) \qquad (16)$$

For example, to find $\sqrt{4+3i}$, we write :

$$4+3i = 5\left(\frac{4}{5}+\frac{3}{5}i\right) = 5\,(\cos 0.64 + i \sin 0.64)$$

$$\sqrt{4+3i} = \sqrt{5}\,(\cos 0.32 + i \sin 0.32) = (2.24)(0.95+0.31\, i)$$
$$= 2.128 + 0.6945\, i$$

Newton's method is then applied for a more accurate result. Taking $z_o = 2.128+0.6945\, i$ and applying (16), we find

$$z_1 = \frac{1}{2}\left(z_o + \frac{4+3i}{z_o}\right) = 1.064+0.34725i + \frac{4+3i}{4.256+1.389\, i}$$

$$z_1 = 2.1208 + 0.70193\, i$$

It is interesting to compare the squares of these two values :

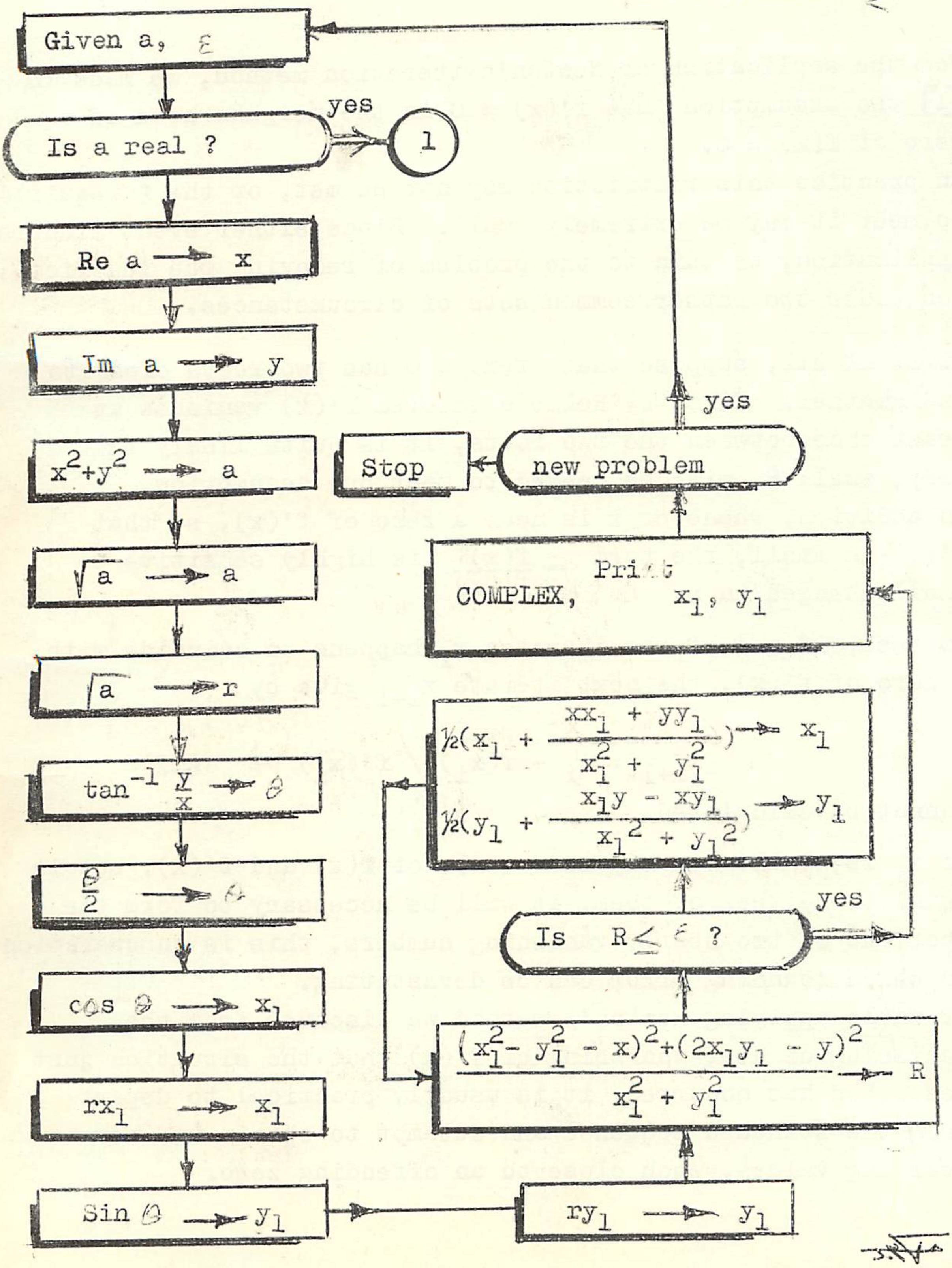
$$z_o^2 = 4.046 + 2.956\, i = (4+3i)+(0.046-0.044i)$$

$$z_1^2 = 4.006 + 2.978\, i = (4+3i)+(0.006-0.022i)$$

Subsequent iterations would yield even more accuracy.

4.5 Flow-chart for the square root of a real or complex number with relative error

$$a = \begin{cases} \text{a complex number } x+iy \\ \text{a real number } \geq 0 \end{cases}$$



4.6 A modification of Newtons method

For the application of Newton's iteration method, we made in [1] the assumption that $f'(x) \neq 0$ in the neighborhood of a zero of $f(x) = 0$.
In practice this restriction may not be met, or the ε required to meet it may be extremely small. Since either event hinders application, we turn to the problem of removing the restriction under two rather common sets of circumstances.

First of all, suppose that $f(x) = 0$ has two roots close to one another. Since by Rolle's theorem $f'(x)$ vanishes at least once between the two roots, it is quite likely that a very, small ε will be needed to meet our assumption.
In addition, whenever x is near a zero of $f'(x)$, so that $f'(x)$ is small, the term $-\frac{f(x)}{f'(x)}$ is highly sensitive to small changes in x.

In deed, if one of the iterates x_i happens to coincide with a zero of $f'(x)$, the next iterate x_{i+1} give by

$$x_{i+1} = x_i - f(x_i) / f'(x_i) \qquad (17)$$

cannot be calculated.

If x_i is simultaneously near zeros of $f(x)$ and $f'(x)$, but is equal to neither of them, it will be necessary to form the quotient of two nearly vanishing numbers, this is an operation in which rounding error can be devastating.
If while applying Newton's method we discover from the vanishing or near vanishing of $f'(x)$ that the situation just described has occurred, it is usually practical to depart from the standard sequence and attempt to obtain two new starting values, each close to an offending zero.

To obtain these values, we first apply Newton's method to the equation $f'(x) = 0$; that is, we perform the iteration

$$x_{i+1} = x_i - f'(x_i) / f''(x_i) \qquad (18)$$

using the last available iterate as a starting value. Suppose that the solution $x=c$ is obtained.

(If there is no such solution $x=c$ or if $f''(x)$ is small throughout a small neighlorhood of $x'=c$, further modification is abviously required.) We form the Taylor's series about $x=c$:

$$f(x) = f(c) + f'(c)\ (x-c) + \frac{1}{2}\ f''(c)\ (x-c)^2 + \ldots.$$

and use the fact that $f'(c) = 0$ to obtain

$$f(x) = f(c) + \frac{1}{2} f''(x)\ (x-c)^2 + R. \qquad (19)$$

If we assume that the remainder term R is small we may conclude that the zeros of $f(x)$ near $x=c$ are approximately equal to the roots of the quadratic equation

$$f(c) + \frac{1}{2} f''(c)\ (x-c)^2 = 0$$

namely

$$x = c \pm \sqrt{-2f(c)/f''(c)} \qquad (20)$$

Using these two numbers as starting values we can enter the iteration (17) with some hope of its converging.
Just as direct use of Newton's method may be difficult when the equation $f(x)= 0$ has two almost equal roots, so it is impossible with two equal roots in the sence that the expression $f(c)/f'(c)$ is indeterminante when $x=c$ is a multiple roote of $f(x)=0$.

It is simple to incorporate a test for equal roots into the modification provided to separate almost equal roots. First of all, the modification will almost certainly be entered when an iterate which is near a multiple root is reached. If x=c is a multiple root of f(x)=0, it is almost a root of f'(x)=0. If f"(c) ≠ 0, then x=c is a double root and will be found when the equation f'(x)=0 is solved. When f(c) is found we should test to see if it vanishes (or is less in magnitude than some preasigned value), and, if so, leave the modification with the information that x=c is a double root of f(x)=0. If f(c)≠ 0 we proceed through the original modification.

If x=c is a zero of f(x) of three or higher multiplicity, computation cannot proceed through the modified program, since division by f"(c) is required. Even though provision for this may be made by "pyramiding" use of the modification, manual intervention may be preferred unless problems frequently involving zeros of high multiplicity are to be expected.

Further Remarks

When an electronic computer is employed to perform the calculation it is possible to employ methods in which very large numbers of arithmetic operations are performed in the course of computation. Thus it is necessary to employ methods in which small, individual errors will not have a serious, cumulative effect on the results.

It should be mentioned in conclusion that we have not considered here the possible effects of uncertainly in the evaluation of f(x) or correspondingly, g(x) on the uncertainly in the solutions of f(x) = 0 or x = g(x).

J.H.Wilkinson has constructed an example in which the effect is so extreme as to be almost humorous.
Whereas

$$f(x) = (x-1)(x-2) \dots (x-20) = x^{20} - 210\, X^{19} + \dots + 20!$$

has zeros 1, 2, ..., 20 , the equation

$f(x) - 2^{-23} X^{19} = 0$ has among its solutions numbers near 20.846 and $13.99 \pm 2.5\, i$. Since there is no easy way to anticipate when this effect will render a given computation unusable there seems to be no substitute for prudence in the use of root - finding techniques.

5. Lin-Bairstow Method for Complex Roots

5.1 The basic idea of the method

Ageneral method for determining the complex roots of a polynomical equation

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x+a_n = 0 \quad (21)$$

involves finding a quadratic factor of the polynomial by an iterative procedure. If $P_n(x)$ is divided by a trial factor $x^2 + rx + s$, one obtains as a quotient a polynomial $Q(x)$ of degree n-2 and a remainder $Rx+S$. One may therefore write

$$\sum_{k=0}^{n} a_k x^{n-k} = (x^2+rx+s) \sum_{k=0}^{n-2} b_k x^{n-k-2} + Rx + S \quad (22)$$

from which it follows that

$$\begin{aligned}
a_0 &= b_0 \\
a_1 &= b_1 + rb_0 \\
a_2 &= b_2 + rb_1 + sb_0 \\
&\dots\dots\dots\dots\dots\dots \\
a_k &= b_k + rb_{k-1} + sb_{k-2} \\
&\dots\dots\dots\dots\dots\dots \\
a_{n-1} &= R + rb_{n-2} + sb_{n-3} \\
a_n &= S + sb_{n-2}
\end{aligned} \quad (23)$$

If one sets

$$\begin{aligned}
b_{-1} &= b_{-2} = 0 \\
b_{n-1} &= R \\
b_n &= S - rR
\end{aligned} \quad (24)$$

one may write Eqs. (23) in the form

$$b_k = a_k - rb_{k-1} - sb_{k-2} \quad , \quad k = 0,1,2, \ldots, n \qquad (25)$$

The coefficients b_k of the polynomial $Q(x)$ and the coefficents R and S of the remainder are functions of r and s. We now attempt by an iterative process to solve the simultaneous equations

$$R(r,s) = 0 \qquad \text{and} \qquad S(r,s) = 0 \qquad (26)$$

which, if satisfied by r^* and s^*, make $x^2 + r^* x + s^*$ a factor of the polynomial. To find r^* and s^* we asume that we have an r and S near these values so that

$$\begin{aligned} r^* &= r + \Delta r \\ s^* &= s + \Delta s \end{aligned} \qquad (27)$$

where Δr and Δs are small. Using Taylor's expansion for functions of two variables and neglecting terms of higher power than the first in these increments, we have

$$\begin{aligned} R(r,s) + \Delta r \frac{\partial R}{\partial r} + \Delta s \frac{\partial R}{\partial s} &\cong R(r^*, s^*) = 0 \\ S(r,s) + \Delta r \frac{\partial S}{\partial r} + \Delta s \frac{\partial S}{\partial s} &\cong S(r^*, s^*) = 0 \end{aligned} \qquad (28)$$

We next find the four partial derivatives in Eqs. (28) and solve these equations for Δr and Δs. Use of Eqs. (27) then yields an r^* and s^*, which, of course because of the approximate nature of Eqs. (28) are now only an improved estimate of the roots of Eqs. (26).

To find the partial derivatives in Eqs. (28) we differentiate Eqs. (25); thus

$$\left.\begin{aligned}\frac{\partial b_k}{\partial r} &= -b_{k-1} - r\frac{\partial b_{k-1}}{\partial r} - s\frac{\partial b_{k-2}}{\partial r}\\ \frac{\partial b_k}{\partial s} &= -b_{k-2} - r\frac{\partial b_{k-1}}{\partial s} - s\frac{\partial b_{k-2}}{\partial s}\end{aligned}\right\}\quad (29)$$

Since from Eqs. (23) $b_o = a_o$, it is not a function of r or s; and therefore from the above equations :

$$\begin{aligned}\frac{\partial b_o}{\partial r} &= 0 & \frac{\partial b_1}{\partial s} &= 0\\ \frac{\partial b_1}{\partial r} &= -b_o & \frac{\partial b_2}{\partial s} &= -b_o - r\frac{\partial b_1}{\partial s} = -b_o\\ \frac{\partial b_2}{\partial r} &= -b_1 - r\frac{\partial b_1}{\partial r} & \frac{\partial b_3}{\partial s} &= -b_1 - r\frac{\partial b_2}{\partial s} - s\frac{\partial b_1}{\partial s}\\ &= -b_1 + rb_o & &= -b_1 + rb_o\end{aligned}\quad (30)$$

Thus for k=0, 1 and 2

$$\frac{\partial b_k}{\partial r} = \frac{\partial b_{k+1}}{\partial s} = -b_{k-1} + r b_{k-2} = -c_{k-1} \quad (31)$$

and by mathematical induction this equation holds for all k. For suppose that Eq. (31) holds for all k up to n-1; then by Eq. (29)

$$\begin{aligned}\frac{\partial b_m}{\partial r} &= -b_{m-1} - r\frac{\partial b_{m-1}}{\partial r} - s\frac{\partial b_{m-2}}{\partial r}\\ &= -b_{m-1} - r\frac{\partial b_m}{\partial s} - s\frac{\partial b_{m-1}}{\partial s} - s\frac{\partial b_{m-1}}{\partial s}\\ &= \frac{\partial b_{m+1}}{\partial s}\end{aligned}$$

and thus it holds for m.

Making use of Eq. (31) one may write in place of the equations in (29) the singl recurrence relation

$$c_k = b_k - rc_{k-1} - sc_{k-2} \tag{32}$$

In particular, from Eqs. (30) and (31) $c_{-1} = 0$ and $c_o = b_o$. Thus the c's are obtained from the b's in exactly the same way the b's were obtained from the a's.

From Eqs. (24) and (31)

$$\begin{aligned} R &= b_{n-1} \\ \frac{\partial R}{\partial r} &= \frac{\partial b_{n-1}}{\partial r} = -c_{n-2} \\ \frac{\partial R}{\partial s} &= \frac{\partial b_{n-1}}{\partial s} = -c_{n-3} \\ S &= b_n + rb_{n-1} \end{aligned} \qquad \begin{aligned} \frac{\partial S}{\partial r} &= \frac{\partial b_n}{\partial r} + b_{n-1} + r\frac{\partial b_{n-1}}{\partial r} = -c_{n-1} - rc_{n-2} + b_{n-1} \\ \frac{\partial S}{\partial s} &= \frac{\partial b_n}{\partial s} + r\frac{\partial b_{n-1}}{\partial s} = -c_{n-2} - rc_{n-3} \end{aligned} \tag{33}$$

and therefore the equations (28) for Δr and Δs may be written

$$\begin{aligned} c_{n-2}\,\Delta r + c_{n-3}\,\Delta s &= b_{n-1} \\ (c_{n-1} - b_{n-1})\,\Delta r + c_{n-2}\,\Delta s &= b_n \end{aligned} \tag{34}$$

Note that c_n is not needed in these equations and that in place of c_{n-1} we may compute directly

$$\bar{c}_{n-1} = c_{n-1} - b_{n-1} = -rc_{n-2} - sc_{n-3} \tag{35}$$

the coefficient needed in Eqs. (34)

To summarize, one computes the b_k and c_k from the given coefficients a_k using recurrence relations (25), (32) and (35) and assuming that

$$\left.\begin{array}{l} a_{-1} = b_{-1} = c_{-1} = 0 \\ a_o = b_o = c_o \end{array}\right\} \qquad (36)$$

It is convenient for this purpose to arrange the coefficients in the following array :

$$\left.\begin{array}{lll} a_o & b_o & c_o \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \cdots & \cdots & \cdots \\ a_{n-2} & b_{n-2} & c_{n-2} \\ a_{n-1} & b_{n-1} & \bar{c}_{n-1} \end{array}\right\} \qquad (37)$$

$$\begin{array}{ll} a_n & b_n \end{array}$$

Having these coefficients, one may write down the two simultaneous equations in (34) for Δr and Δs. These increments added on to r and s, respectively, give by Eqs. (27) improved estimates of coefficients r^* and s^* in a quadratic factor $x^2 + r^* x + s^*$ of the given polynomial. When such a quadratic factor is found with sufficient accuracy, two roots of the given equation (21) are determined by setting $x^2 + r^* x + s^* = o$

Such a method permits one to compute a pair of complex roots of a polynomial having real coefficients by operating only with real numbers.

5.2 Summary of Lin-Bairstow method

$$P_n(x) = \sum_{k=0}^{n} a_k x^{n-k}$$

Step 1 : $b_k = a_k - rb_{k-1} - sb_{k-2}$,

$k = 1, 2, \ldots, n$;

$a_{-1} = b_{-1} = 0$,

$a_0 = b_0$.

Step 2 : $c_k = b_k - rc_{k-1} - sc_{k-2}$,

$k = 1, 2, \ldots, n-2$;

$c_{-1} = 0$

$c_0 = a_0$

$-rc_{n-2} - sc_{n-3} = \bar{c}_{n-1}$

Step 3 : $D = c_{n-2}^2 - \bar{c}_{n-1} c_{n-3}$,

$\Delta r = (b_{n-1} c_{n-2} - b_n c_{n-3})/D$

$\Delta s = (c_{n-2} b_n - \bar{c}_{n-1} b_{n-1})/D$

Step 4 : $r^* = r + \Delta r$

$s^* = s + \Delta s$

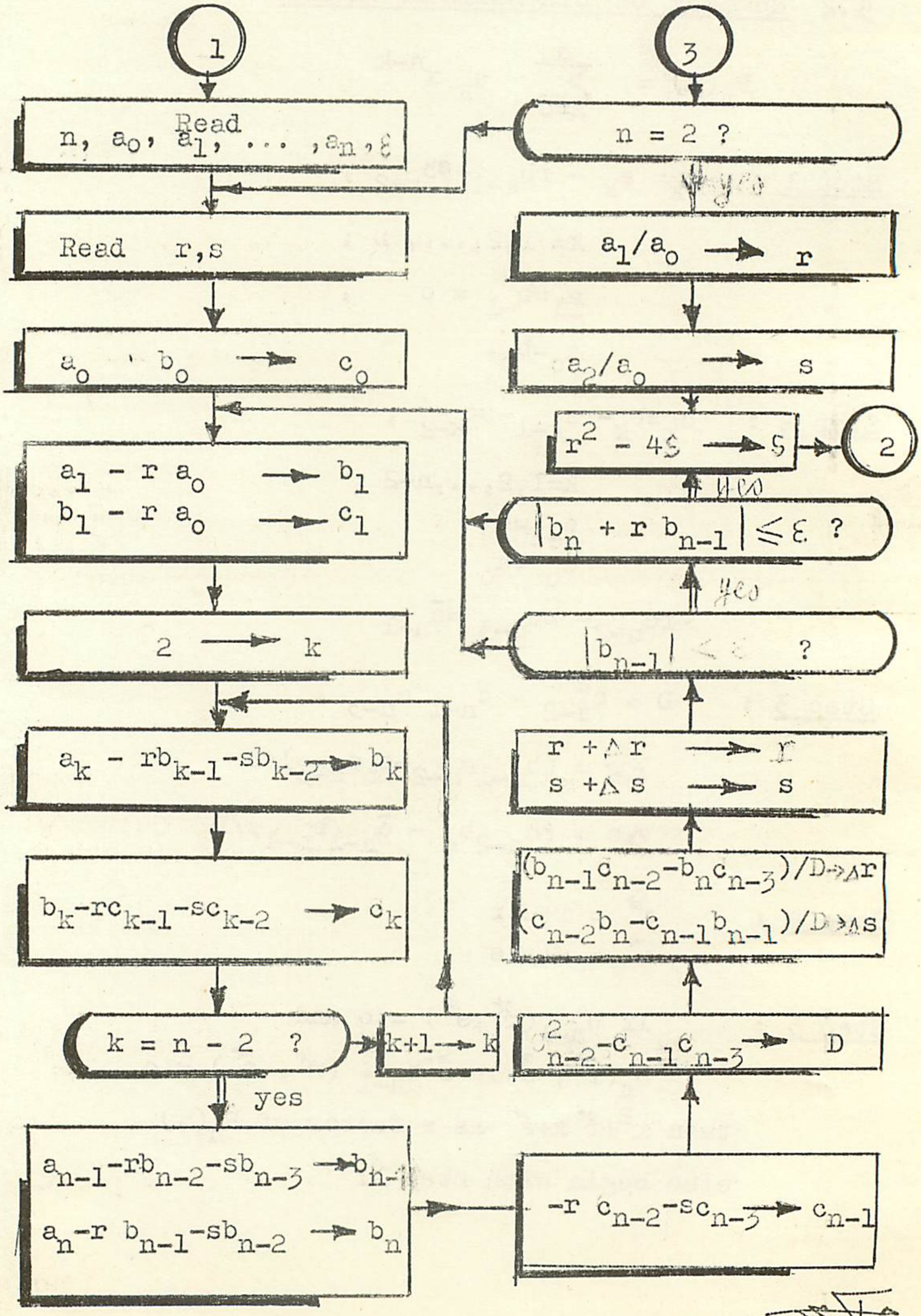
Step 5 : If $b_{n-1}(r^*, s^*) = 0$ and

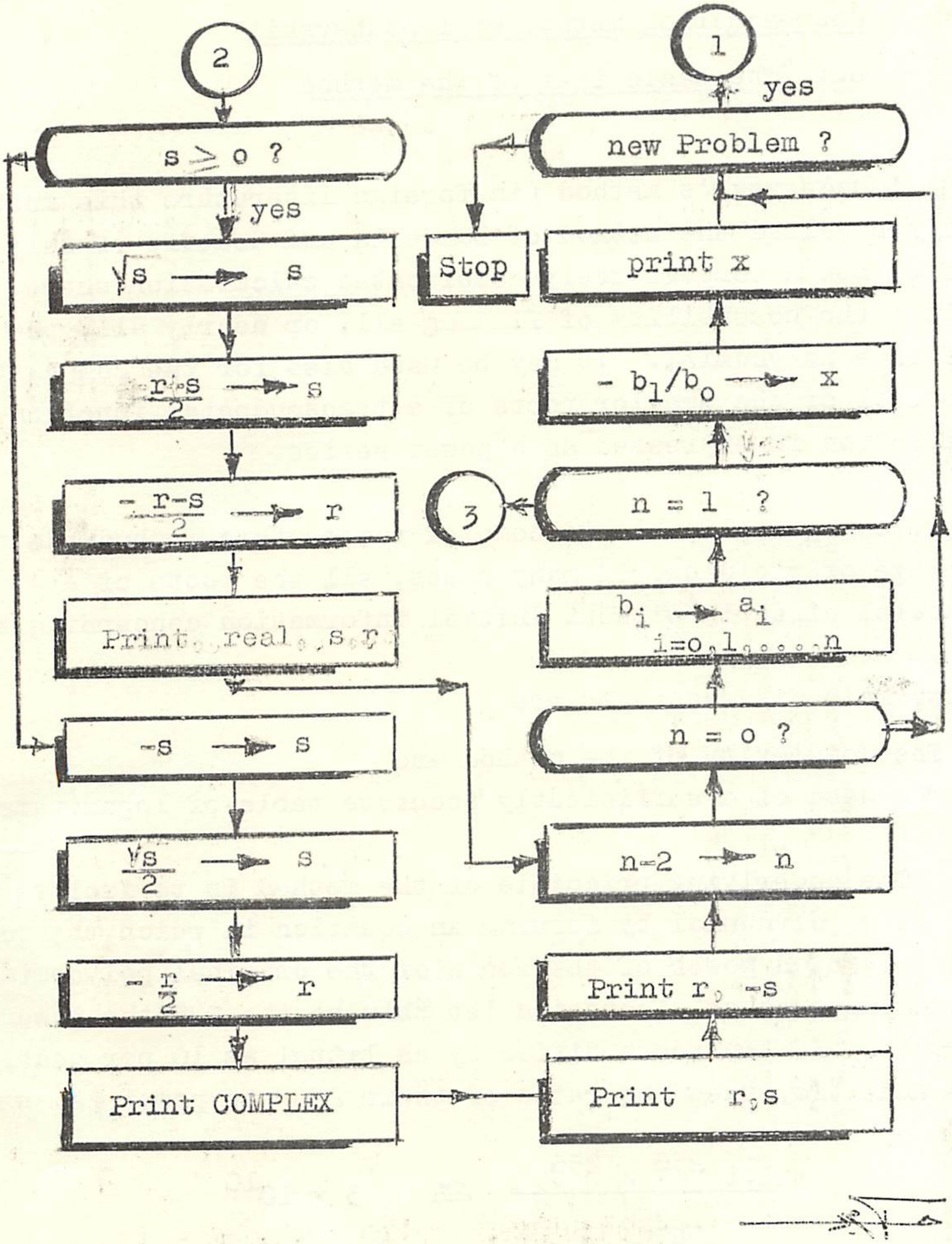
$b_n(r^*, s^*) + r^* b_{n-1}(r^*, s^*) = 0$

then $x^2 + r^* x + s^*$ is a factor of $P_n(x)$

else begin with step 1.

5.3 Flow Chart for Lin-Bairstow method




2
s ≥ o ?
yes
√s → s
r+s/2 → s
-r-s/2 → r
Print real s, r
-s → s
√s/2 → s
-r/2 → r
Print COMPLEX
1
yes
new Problem ?
Stop
print x
- b1/bo → x
n = 1 ?
3
bi → ai
i=o,1,...,n
n = o ?
n-2 → n
Print r, -s
Print r,s

6. Root-squaring method of Lobachevskii

6.1 The basic idea of the method

Lobachevskii's method (in foreign literature this method is often called the method of Dandelin and Graeffe or only Graeffe) has a comparatively complicated calculation sheme, but provides the possibility of finding all, or nearly all, the roots of a polynomial. It may be used also for the calculation, of several of the smaller roots of a transendental function, if the function is expressed as a power series.

Besides being applicable to complex roots, this method has the advantage of yielding, in many cases, all the roots of a polynomial directly without initial information concerning the roots.

Its two main disadvantages are

(1) The complexity of the method and

(2) The need of a sufficiently accurate table of logarithms.

The underlying principle of the method is to isolate the roots of a polynomial by forming an equation in which the roots are a very high power of the roots of the original polynomial. For the purposes of discussion let this be the 256 the power of the roots. If two roots differ by as littel as 10 per cent, say a and 1.1a, then the ratio of their 256 the power is

$$\frac{1.1^{256} a^{256}}{a^{256}} \cong 3 \times 10^{10}$$

Since the new equation has roots differing greatly in magnitudes it can oten be solved rather simply for the magnitude of the new roots. By the use of logarithm tables the magnitude of the roots of the original equation can then be determined.

The amplitude (or angle) of the root is determined in turn, by making use of the original equation, e.g., direct substitution will fix the sign of the real roots.

Suppose we wish to form an equation whose roots are the negative of the square of the roots of

$$f(x) = \sum_{k=o}^{n} a_k x^{n-k} = o \qquad (38)$$

This can be done, as is proved below, by forming the even function of x

$$F(-x^2)=f(x)f(-x)= \sum_{k=o}^{n} a_k x^{n-k} \sum_{k=o}^{n} (-1)^{n-k} a_k x^{n-k} \qquad (39)$$

Put $y = -x^2$ we get

$$F(y) = \sum_{k=o}^{n} b_k y^{n-k} \qquad (40)$$

Let the zeros of f(x) be $x= P_1, P_2, \ldots$ and P_n ; then the zeros of

$F(-x^2) = f(x)f(-x)$

are $x = \pm P_1, \pm P_2, \ldots$ and $\pm P_n$

and since $y = -x^2$

the roots of Eq.(40) are $y = -P_1^2, -P_2^2, \ldots$ and $-P_n^2$.

Thus the required equation is :

$$\sum_{k=o}^{n} b_k x^{n-k} = o \qquad (41)$$

where :

$$b_k=a_k^2 +2 \sum_{s=1}^{\min(n-j,j)} (-1)^s \, a_{j-s} \, a_{j+s}, \qquad (42)$$

$k=o,1,2,\ldots,n$

Suppose for example that the given equation is :

$$a_0x^6+a_1x^5+a_2x^4+a_3x^3+a_4x^2+a_5x+a_6 = o \qquad (43)$$

and the required equation is :

$$b_0x^6+b_1x^5+b_2x^4+b_3x^3+b_4x^2+b_5x+b_6 = o \qquad (44)$$

According to (42) we get :

$$\begin{aligned} b_0 &= a_0^2 \\ b_1 &= a_1^2 - 2\,a_0a_2 \\ b_2 &= a_2^2 - 2\,a_1a_3 - 2\,a_0a_4 \\ b_3 &= a_3^2 - 2a_2a_4 + 2a_1a_5 - 2\,a_0a_6 \\ b_4 &= a_4^2 - 2\,a_3a_5 + 2\,a_2a_6 \\ b_5 &= a_5^2 - 2\,a_4a_6 \\ b_6 &= a_6^2 \end{aligned} \qquad (45)$$

This procedure can be performed schematically as follows: one writes down the detached coefficients $a_o, a_1 \ldots, a_6$ as shown in (46) and under them their squares and the products indicated. The coefficients $b_o, b_1, \ldots, b_6$ are now determined by adding the quantities beneath the corresponding $a_o, a_1, \ldots., a_6$

a_o	a_1	a_2	a_3	a_4	a_5	a_6	
a_o^2	a_1^2	a_2^2	a_3^2	a_4^2	a_5^2	a_6^2	
	$-2\,a_oa_2$	$-2\,a_1a_3$	$-2\,a_2a_4$	$-2\,a_3a_5$	$-2\,a_4a_6$		(4
		$+2\,a_oa_4$	$+2\,a_1a_5$	$+2\,a_2a_6$			
			$-2\,a_oa_6$				
b_o	b_1	b_2	b_3	b_4	b_5	b_6	

6.2 Roots all Real and Unequal in Magnitude

The root-squaring method is most easily applied to an equation having only real roots and these all of different magnitudes. Suppose that

$$P(x) = a_o x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0 \qquad (47)$$

has n real roots $P_1, P_2, \dots, P_n$ and that

$$|P_1| > |P_2| > |P_3| > \dots > |P_n| \qquad (48)$$

After the first root-squaring process, the root corresponding to P_k is $-P_k^2$; after the second, the corresponding root is $-P_k^4$; after the third, it is $-P_k^8$; and after the m the; it is $q_k = -P_k^{2^m}$ (thus for m = 8 it is $-P_k^{256}$).

Each root - squering process increases greatly the inequalities of (48), When these inequalities are so great that, to the number of significent figures decided upon,

$$1 + \frac{q_{i+1}}{q_i}^{(n-1)} = 1 \qquad , \qquad i = 1, 2, \dots, n-1 \qquad (49)$$

the root-squaring process should be terminated.

Suppose that the final equation after the m root squarings required, is

$$Q_{(x)} = b_o x^n + b_1 x^{n-1} + \dots + b_{n-1} x + b_n = 0 \qquad (50)$$

and that its roots $q_1, q_2, \dots, q_n$ satisfy Eqs.(49)

By Vieta's theorem

$$\frac{b_1}{b_o} = -(q_1+q_2 + \ldots + q_n) = -q_1 \left(1 + \frac{q_2}{q_1} + \ldots + \frac{q_n}{q_1}\right)$$

therefore by Eqs. (49) , and to the approximation there indicated,

$$\frac{b_1}{b_o} = -q_1 .$$

Likewise, to the same order of approximation ,

$$\frac{b_2}{b_o} = q_1 (q_2 + q_3 + \ldots + q_n) + q_2 (q_3 + \ldots + q_n) + \ldots + q_{n-1} (q_n)$$

$$= q_1 q_2 \left[\left(1 + \frac{q_3}{q_2} + \ldots + \frac{q_n}{q_2}\right) + \left(\frac{q_3}{q_1} + \ldots + \frac{q_n}{q_1}\right) + \ldots + \frac{q_{n-1}}{q_1} \cdot \frac{q_n}{q_2} \right] \cong q_1 q_2$$

Continuing in this way, we have

$$\left.\begin{aligned} b_1 &\cong -q_1 b_o \\ b_2 &\cong q_1 q_2 b_o \\ b_3 &\cong -q_1 q_2 q_3 b_o \\ b_4 &\cong q_1 q_2 q_3 q_4 b_o \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (51)$$

$$\left.\begin{array}{l} b_k = (-1)^k \; q_1 q_2 \; \cdots \; q_k \, b_o \\ \cdots\cdots\cdots\cdots\cdots \\ b_n = (-1)^n \, q_1 \, q_2 \; \cdots \; q_n \, b_o \end{array}\right\}$$

From these equations, we have

$$\begin{array}{l} b_o \, q_1 + b_1 = 0 \\ b_1 \, q_2 + b_2 = 0 \\ \cdots\cdots\cdots\cdots \\ b_{k-1} q_k + b_k = 0 \\ \cdots\cdots\cdots\cdots \\ b_{n-1} q_n + b_n = 0 \end{array} \tag{52}$$

Since, by Eqs. (52)

$$|q_k| = \frac{|b_k|}{|b_{k-1}|} = |p_k|^{2^m}$$

$$\log |p_k| = 2^{-m} \left(\log |b_k| - \log |b_{k-1}| \right) \tag{53}$$

from which one can determine $|p_k|$. Substitution in the original equation (47) will determine whether the roots are positive or negative.

Termination of the Root-squaring Process.

As stated in the above discussion, the root squaring should be terminated when q_i is negligible compared to q_{i-1} and when, therefore, the coefficients of the equation are given in terms of the root by Eqs. (51). One needs, however to be able to judge from the root-squaring process itself when these conditions are met.

Suppose that an additional root squaring is performed on Q (x) (see Eq. (50)) to obtain.

$$\overline{Q}(x) = \overline{b}_o x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_{n-1} x + \overline{b}_n = 0 \qquad (54)$$

with the roots $\overline{q}_1$, $\overline{q}_2$, ... , $\overline{q}_n$, where by the nature of the process

$$\overline{q}_k = - q_k^2 \qquad (55)$$

Now the roots will be separted even further then before. Therefore the approximations used in obtaining Eqs. (51) are certainly valid, and hence

$$\overline{b}_k = (-1)^k \overline{q}_1 \overline{q}_2 \ldots \overline{q}_k \overline{b}_o \qquad (56)$$

Directly from the root-squaring process $\overline{b}_o = b_o^2$.

Substituting this in Eq. (56) together with the expressions for $\overline{q}_k$ from Eq. (55) on obtains the approximate relation

$$\overline{b}_k = q_1^2 \; q_2^2 \; \ldots \; q_k^2 \; b_o^2 = b_k^2 \qquad (57)$$

This will be true if the corss-product terms in the root-squaring process are negligible.

Thus the vanishing of all the cross-product terms in the root-squaring process can be used to indicate that the roots are sufficiently separated for the degree of accuracy required that the process should be terminated.

Consider the equation $x^3 - 2x^2 - 5x + 6 = 0$ (58)

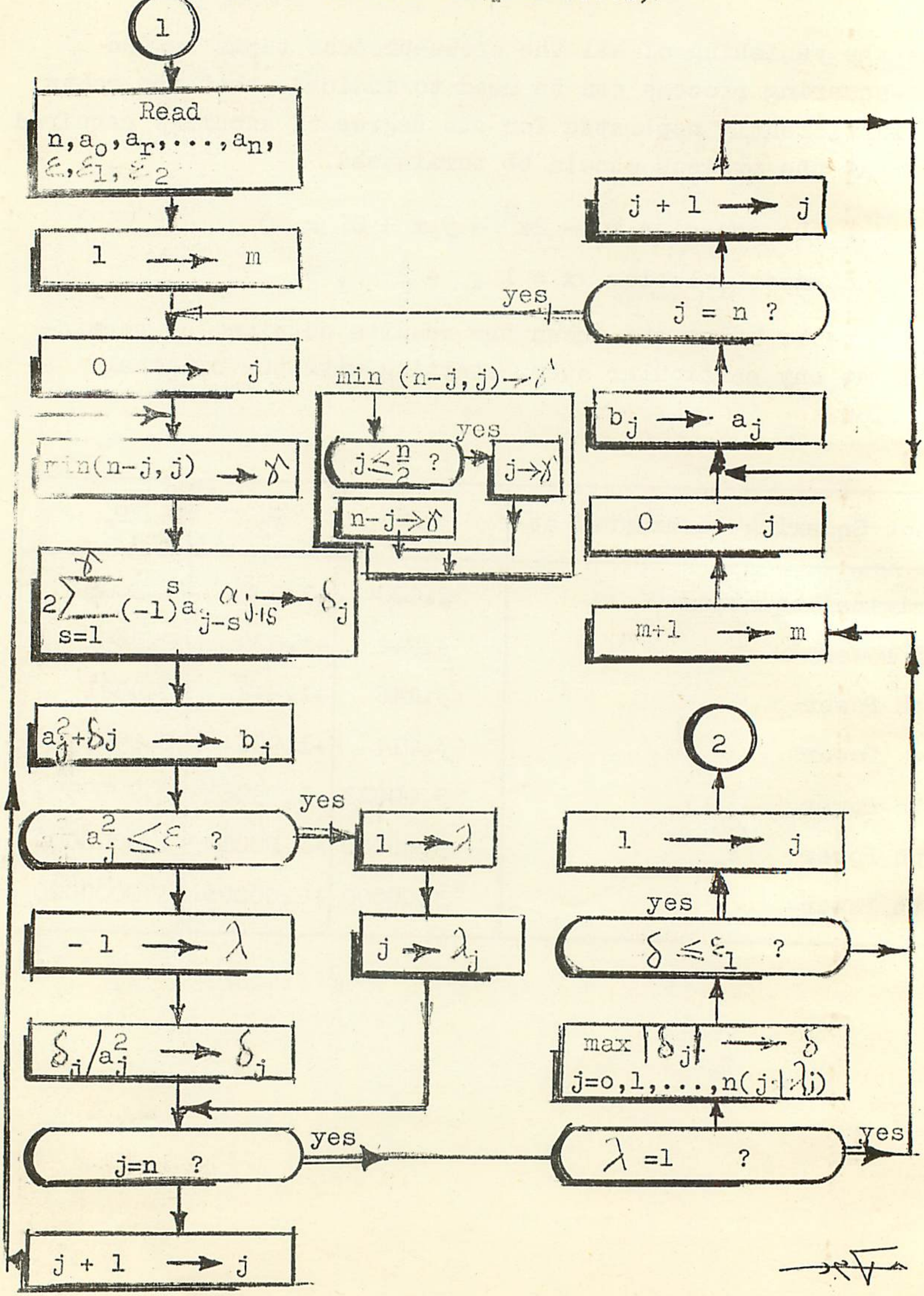
with the exact solution $x = 1, -2, 3$

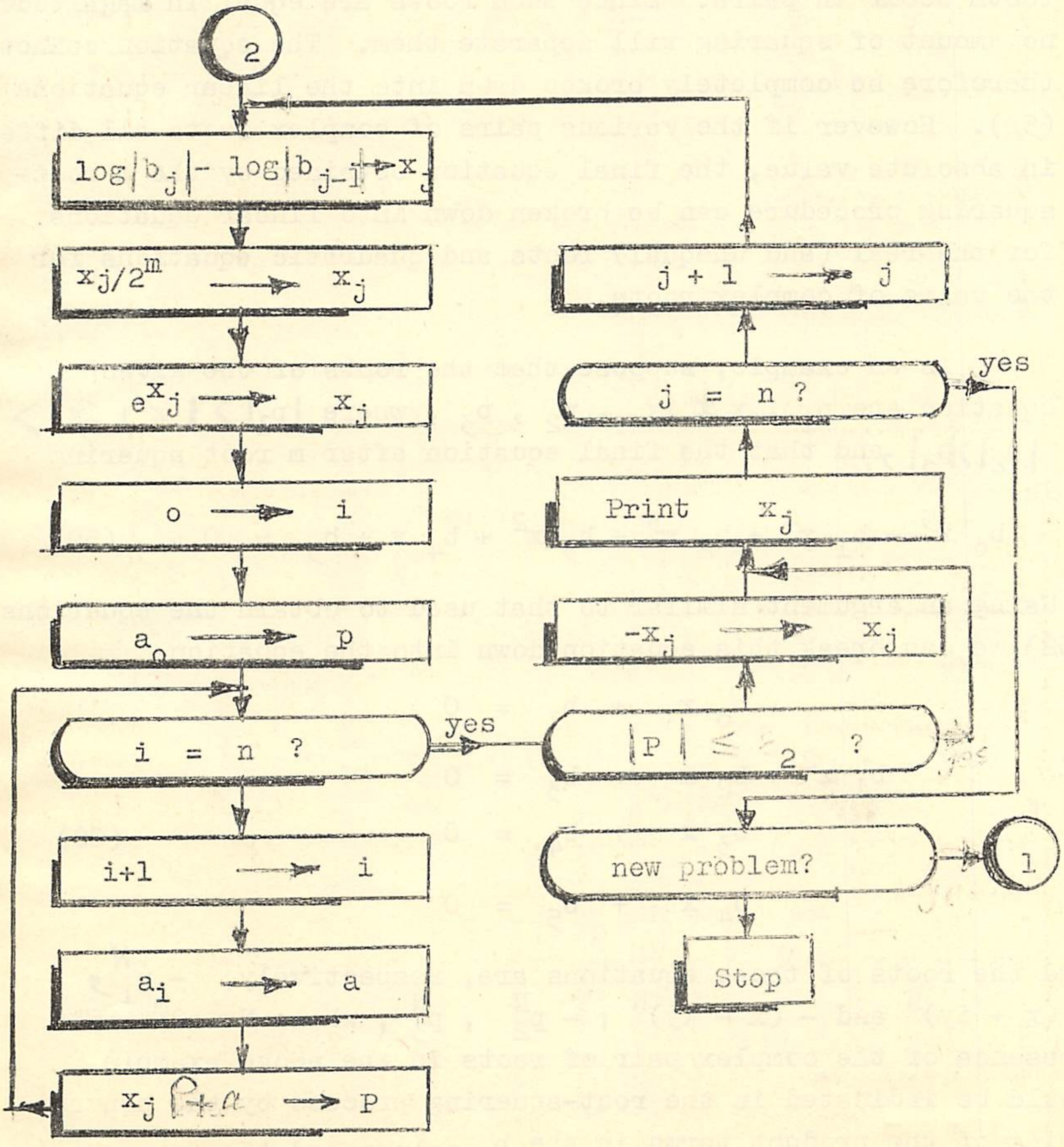
In the table below are shown the results obtained by terminating at any particular step, starting with the original equation.

Root Squaring terminated at	P_1	P_2	P_3
Original equation	2.000	-2.500	1.2000
2d Power	3.742	-1.870	0.8571
4th Power	3.146	-1.942	0.9821
8th Power	3.014	-1.991	0.9996
16th Power	3.00028	-1.99981	0.999995
32th Power	3.00000	-2.00000	1.0000000
64th Power	3.00000	-2.00000	1.0000000

Flow-chart for Graeffe method

(case of real unequal roots)




2
log|b_j| - log|b_{j-1}| → x_j
x_j/2^m → x_j
e^{x_j} → x_j
o → i
a_o → p
i = n ?
yes
i+1 → i
a_i → a
x_j p+a → p
|p| ≤ ε_2 ?
-x_j → x_j
Print x_j
j = n ?
yes
j + 1 → j
new problem?
1
Stop

6.4 Complex Roots and Roots Equal in Magnitude

For a polynomial with real coefficients the complex roots occur in pairs. Since such roots are equal in magnitude no amount of squaring will separate them. The equation cannot, therefore be completely broken down into the linear equations (52). However if the various pairs of complex roots all differ in absolute value, the final equation obtained by the m root-squaring procedure can be broken down into linear equations for the real (and unequal) roots and quadratic equations for the pairs of complex roots.

As an example, suppose that the roots of the given equation are p_1 , $x \pm iy$, p_2 , p_3 , where $|p_1| > |x \pm iy| > |p_2| > |p_3|$, and that the final equation after m root squaring is

$$b_o x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5 = 0 \qquad (59)$$

Using an argument similar to that used to obtain the equations in 52) we can break this equation down into the equation

$$\begin{aligned} b_o x + b_1 &= 0 \\ b_1 x^2 + b_2 x + b_3 &= 0 \\ b_3 x + b_4 &= 0 \\ b_4 x + b_5 &= 0 \end{aligned} \qquad (60)$$

and the roots of those equations are, respectively, $-p_1^N$, $-(x + iy)^N$ and $-(x - iy)^N$; $-p_2^N$, p_3^N , where $N = 2^m$. The presence of the complex pair of roots in the above example would be indicated in the root-squaring process by the nonvanishing of the product terms in the b_2 column and the frequent changes in sign of b_2 , although the latter may not always occure.

If two or more complex pairs are equal in magnitude, quartic equations must be solved; and therefore the method is of little or no use.

The presence of two real roots of equal magnitude will like wise give rise to quadratic equations, such as found in Eqs. (60) for complex roots, the presence of such roots is marked by the nonvanishing of cross-product terms.
These cross-product terms, in this case, approach a value equal to half the squared term. Again the method is of little or no use if three or more roots are equal in magnitude.

In general, the root-squaring method of Dandelin and Graeffe is rather unsatisfactory because of the complexity of rules for its proper application, the uncertainty of obtaing an answer if too many roots are equal in magnitude, and the tendency for numerical errors to occur in the root-squaring process. These errors are not corrected by the subsequent calculations. For real roots, it is therefore best to use the method of false position or the Birge-Vieta method. However, for finding the complex roots of polynomials of the eighth order and higher, the root-squaring method has few sucessful competitors.
(This statement is based on the assumption that all real roots have previously been determined and the order of the polynomial reduced accordingly).

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