UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

التخطيط القومي

Memo. No. 163

OUTPUT-PRODUCTIVITY AND VALUE ADDED PRODUCTIVITY

by

Prof. Bent Hansen

14th March, 1962.

Reprint: July 1966.

CAIRO
3. MOHAMED MAZHAR - ZAMALEK

Output-Productivity and Value Added-Productivity.

by Bent Hansen.

i) The Problem.

The physical volume of production of a sector has to be measured by means of a production index. Depending upon how the concept of production is defined, a production index may be an output-index or a real value added-index.[1] An output-index is based upon the physical outputs leaving the sector during a unit of time (possibly including the increase of the sector's stocks of finished goods), certain fixed prices being used as weights. A value added-index is based upon the physical outputs leaving the sector (plus increase of stocks of finished goods) with deduction for such inputs as are used by the sector during the unit of time considered, once more with certain fixed prices as weights.

Corresponding to these two kinds of production indexes, we have two kinds of productivity measures. We can talk about output-productivity and value added-productivity, respectively, depending upon whether productivity is measured in terms of output or real value added. For both kinds of productivity measures, the concepts of average and marginal productivity apply.

In practical index-making, we find both kinds of production indexes side by side. The tendency seems to be to give preference to value added-indexes often so-called net-production indexes are nothing but output-indexes.[2] When G.N.P. in constant prices is used to estimate productivity, we are obviously confronted with a true value added-index. In practical measurements of productivity we shall therefore also find the two corresponding kinds of measures. Here too, I think, there is a tendency

1) Sometimes the terms "gross" and "net production" indexes are used.

2) Richard Stone, Quantity and Price Indexes in National Account. OEEC, Paris, 1956.

to give preference to the value added measures. Since productivity measures are used very often for solving practical problems, for instance as a guidance in wages policies, it is obviously of importance to know the properties of and the relationships between output-and value added-measures of productivity. We need here only mention the fact that while, according to traditional production theory, the value of the marginal output-productivity of labour should in competitive equilibrium be equal to the money wage rate, guidance for wage changes is sometimes sought in measurements of the average value added-productivity of labour. To what extent will such methods of procedure lead us astray? The intention of this note is to treat this problem from a purely theoretical point of view.

ii) Production Theory and Value Added.

Consider a sector producing one commodity only. The physical quantity produced is called Q. We choose the unit of this commodity in such a way, that its price, P, is 1. The number of inputs (factors) used is n, q_i denoting physical quantities used-up, and P_i prices of the inputs. The traditional production function (a being an auxiliary shift variable) is then

(1) $$Q = (q_1, \ldots, q_n, a).$$

We have now, recalling that the output price is 1,

(2) $$Q = \sum_{i=1}^{n} p_i q_i,$$

which simply says that total value of output is equal to total costs (in a wide sense). In a competitive equilibrium, we shall, however, also have

(3) $$p_i = PQ'_i = Q'_i.$$

Q'_i denotes the partial derivative of the production function with respect to input i. (3) says that the value of the marginal product is equal to the price of the input. From (2) and (3) it follows that in competitive equilibrium

$$(4) \qquad Q = \sum_{i=1}^{n} Q'_i q_i.$$

This equation says that total output is equal to the sum of the products of the marginal products and the quantities of the input factors.[1] The equation can also be interpreted in value terms (recall that the output price is 1) to say that the total value of output is equal to the sum of the values of marginal products times the quantities of the input factors, which in its turn (see eq. (3)) is equal to total costs.

Value added, V, is defined as

$$(5) \qquad V=Q - \sum_{i=j}^{n} p_i q_i,$$

where the input factors j to n are those "intermediary" inputs which are used up in the sector and which should be deducted from the output value to obtain value added, while the input factors 1 to j-1 are the "originary" input factors. Exactly how the line of demarkation between these two classes of inputs shall be drawn, we need not bother about in this connection.

We have then four concepts of productivity:

Average output productivity	$=Q/q_i$	, i=1, ...,n,
Marginal output productivity	$=Q'_i$	, i=1, ...,n,
Average value-added productivity	$=V/q_i$	, i=1, ...j-1, and
Marginal value-added productivity	$=V'_i$	, i=1, ...j-1,

1) This is not Euler's Theorem about homogeneous functions, but simply a result of the assumption about competitive equilibrium, see P.A. Samuelson, Foundations of Economic Analysis, Cambridge, Mass., 1947, p.81 f.f. We could also have assumed homogeneity of degree one in all inputs and dropped the assumption of competitive equilibrium. To be able to continue the analysis, we would then, however, have to add a corresponding assumption concerning the inputs to be deducted to obtain value added.

V'_i being the partial derivative of value added with respect to input number i. Following traditional procedures, the value added concepts are only defined for the "originary" inputs; we could in principle also talk about the value added productivities of the "intermediary" goods, but we don't need this concept.

iii) A Simple Case: Labour Only.

Assume first for the sake of simplicity, that there is only one "originary" input factor, say number 1, which may be labour. We have then

(6) $V = p_1 q_1 = Q'_1 q_1,$

which immediately gives us the average value added-productivity of labour,

(7) $V/q_1 = Q'_1,$

and the marginal value added-productivity of labour,

(8) $V'_1 = Q'_1 + Q''_{11} q_1.$

In words: The average value added-productivity of labour is equal to its marginal output-productivity (eq. (7)). And the marginal value added-productivity of labour is equal to its "marginal revenue" in terms of output (eq. (8)).

If we keep labour constant and assume a change in the shift-variable a (to indicate a change in technical knowledge), we find

(9) $\frac{dV}{V} = \frac{Q''_{1a} da}{Q'_1}$

i.e., the relative increase in total value added is equal to the relative increase in marginal output-productivity.

In this simple case, the average value added-productivity gives information about the marginal output-productivity. The concept of marginal value added-productivity seems less interesting in itself. It is important to notice that the product of the marginal value-added-productivity of labour and the quantity of labour is not equal to value added (with $Q''_{11} < 0$ this product is actually smaller than value added). From the total differential (keeping $q_2, \dots, q_n$ constant or including them in a)

(10) $dV = Q'_1 dq_1 + Q''_{11} q_1 dq_1 + Q''_{1a} q_1 da,$

and (6), we finally get

$$\frac{dV}{V} = \underbrace{\frac{dq_1}{q_1}}_{\text{relative increase in input of labour}} + \underbrace{\frac{Q''_{11} dq_1 + Q''_{1a} da}{Q'1}}_{\text{relative increase in marginal output productivity of labour}}$$

The relative increase in marginal output-productivity of labour can obviously be found either from (7), or from (11), namely through deducting the relative increase in input of labour from the relative increase of output. This will of course yield the same result as (7).

iv) <u>A Less Simple Case: Labour and Capital</u>.

Consider then a case which corresponds quite well to standard aggregative thinking. We assume that there are two "originary" input factors, Labour and Capital, and one "intermediary" factor, Raw Materials. The symbols explain themselves, Value added is now

(12) $V = Q - p_R q_R = p_L q_L + p_C q_C.$

The average value added-productivities become (see eq.(3))

(13) $V/q_L = Q/q_L - p_R q_R/q_L = Q'_L + Q'_C q_C/q_L$, and

(14) $V/q_C = Q/q_C - p_R q_R/q_C = Q'C + Q'_L q_L/q_C$

We assume, that both output price and raw-material price are constant and consider first eq. (13). The treatment of (14) is completely analogous to the treatment of (13).

Equation (13) gives us the relationship between average value added-productivity, average output-productivity, and the marginal output-productivity of labour and capital. As will be seen, we have the obvious relations

$$V/q_L < Q/q_L > Q'_L \quad \text{and} \quad V/q_L > Q'_L.$$

But this is all we can say without further assumptions, and unfortunately there seems to be no obvious prior knowledge to be applied here. If, for instance, we assume, that the marginal output-productivities are always proportional, i.e. $Q'_C = kQ'_L$, then the relative change from one point of time to another in average value added-productivity of labour will be helpful in estimating the relative change of marginal output-productivity of labour (and capital as well) provided that the input-proportions are also unchanged. Letting the superscripts 1 and 0 denote time 1 and 0, respectively, and assuming that $Q'_C = kQ'_L$, we get from (11),

$$(15) \qquad \frac{(Q'_L)^1}{(Q'_L)^0} = \frac{(V/q_L)^1}{(V/q_L)^0} \cdot \frac{1+k\frac{q_C^0}{q_L^0}}{1+k\frac{q_C^1}{q_L^1}}$$

Since we assume that Q'_L and Q'_C are not directly observable, we don't know the numerical value of k. Therefore (15) is only helpful directly if $q_C^0/q_L^0 = q_C^1/q_L^1$, in which case the relative increase in the marginal output-productivity of labour will be exactly equal to the relative increase in the average value added-productivity of labour. In case q_C and q_L do not change in the same proportion, we should, it is true, be able to say that

$$(16)\qquad \frac{(Q'_L)^1}{(Q'_L)^0} \gtrless \frac{(V/q_L)^1}{(V/q_L)^0} \quad \text{as} \quad \frac{q_C{}^1}{q_L{}^1} \lessgtr \frac{q_C{}^0}{q_L{}^0},$$

which would at least be better than nothing. But, when the input-proportions change our assumption of proportionality between the marginal output-productivities becomes untenable (competitive equilibrium requires decreasing marginal output-productivities), and unfortunately the effects of changed input-proportions work in a way to make (16) break down. (In the first inequality ">" holds if "<" holds in the second inequality; but this means that labour increases relatively more than capital and this in its turn tends to decrease the left side of the first inequality).

Similar remarks apply to (14).

Assume on the other hand, that $Q'_C q_C/q_L = k_1$ and $Q'_L q_L/q_C = k_2$ (k_1 and k_2 being two positive constants). Then it would follow that i.e.,

$$\frac{\Delta V/q_L}{V/q_L} < \frac{\Delta Q'_L}{Q'_L} \quad \text{and} \quad \frac{\Delta V/q_C}{V/q_C} < \frac{\Delta Q'_C}{Q'_L},$$

the relative changes in the average value added-productivity of labour (capital) would always be smaller than the relative changes in the marginal output-productivities of labour (capital). The two assumptions on which this last result was based may be sound in case of changes in the quantities of the input factors, but they preclude changes in the production function, i.e. in technical knowledge, and for that reason they cannot be accepted.

So far, we are left in mid-air. The average value added-productivities tell us nothing directly about the marginal output-productivities except in special cases. Let us turn then to the marginal value added-productivities and consider the total differential

$$(17)\quad dV=(Q'_L+Q''_{LL}q_L+Q''_{CL}q_C)dq_L+(Q'_C+Q''_{LC}q_L+Q''_{CC}q_C)dq_C+$$
$$+(Q''_{La}q_L+Q''_{Ca}q_C)da.$$

Putting dq_C and da = 0, we find the marginal value added-productivity of labour

(18) $$V'_L = Q'_L + Q''_{LL}q_L + Q''_{CL}q_C.$$

It is seen that we can only assume equality between the marginal value added-productivity and the marginal output-productivity of labour if the second derivatives Q''_{LL} and Q''_{CL}, or at the least the sum $Q''_{LL}q_L + Q''_{CL}q_C$, are equal to zero, and although they may be zero there is not a period reason why they should be. It will be understood also that we can say nothing a priori about the orders of magnitude of V'_L and Q'_L because $Q''_{LL} < 0$ and $Q''_{CL} > 0$.

A similar expression holds for the marginal value added-productivity of capital

(19) $$V'_C = Q'_C + Q''_{CC}q_C + Q''_{LC}q_L.$$

The comments to (18) apply here, mutatis mutandis.

From the total differential (17), and (12) and (3) we can finally form the following expression (compare with eq. (11)):

(20) $$\frac{dV}{V} \quad \underbrace{\frac{Q'_L dq_L + Q'_C dq_C}{Q'_L q_L + Q'_C q_C}}_{\substack{\text{relative, weighted} \\ \text{average in "originary"} \\ \text{input quantities}}} +$$

$$+ \underbrace{\frac{Q''_{LL}q_L + Q''_{CL}q_C}{Q'_L q_L + Q'_C q_C} dq_L + \frac{Q''_{LC}q_L + Q''_{CC}q_C}{Q'_L q_L + Q'_C q_C} dq_C + \frac{Q''_{La}q_L + Q''_{Ca}q_C}{Q'_L q_L + Q'_C q_C} da}_{\substack{\text{relative, weighted average increase in} \\ \text{the marginal output-productivities of} \\ \text{the "originary" inputs}}}$$

What we have done here is to split the relative increase in value added (at constant prices) into two parts: one giving us the relative weighted average change in the quantities of the "originary" inputs, the weights being the marginal input-productivities, the other giving us the relative weighted average change in the marginal output-productivities of the "originary" inputs, the weights being the quantities of the "originary" inputs.

The second part is further devided into three components, showing us the relative weighted average increase in the marginal output-productivities of the "originary" inputs due to changes, dq_L and dq_C, in the quantities of the "originary" inputs themselves and to changes, da, in technical knowledge[1], respectively, the weights everywhere being the quantities of "originary" inputs.

It will then be seen, that even if we adjust the relative increase in value added for the changes in input quantities (i.e. deduct the first part of the right hand side of (20) from dV/V), (20) will at most give us the weighted average change in the marginal output-productivities unless additional assumptions are made. But to be able to compute the weighted average change in the input quantities, we have to know the weights, i.e., we have to know the proportion between the marginal output-productivities, unless the inputs happen always to be in constant proportion. But this means that the situation is exactly the same as with eq. (13) above.

v) The General Case.

We could now develop the same analysis on a more general level, for instance through including land among the originary inputs, or on a completely general level considering j-1 unspecified originary inputs. We abstain from this, however, because it will bring us nothing new in principle.

1) Changes in the quantity of raw-material inputs will act upon V exactly as a change in a.da can therefore be interpreted to include changes in raw-materials inputs.

vi) The Practical Problem.

We have seen that a study of real value added-productivity will not in general help us to come across the fundamental marginal output-productivities. They should in principle be studied directly upon the output. In many countries, however, only value added indexes of production are available, or, existing output series may be too short to permit statistical inference about the marginal output-productivities. The practical man, i.e. the policy-maker who has to form an opinion about the marginal output-productivities whether he likes it or not, may be in the situation where he has to judge about marginal output-productivities on the basis of value added. What's to do then?

A challenging answer to this question would be the following: If it is really true that society is always in competitive equilibrium and we have $p_i = Q'_i$, why not simply study the development of p_i?! This would of course be true if we really knew that society is always in competitive equilibrium, but this is something about which we have to assure ourselves through showing that (except for stochastical deviations) $p_i = Q'_i$ and this in its turn requires observation of Q'_i. And for this purpose we cannot-even if it really is true that $p_i = Q'_i$- substitute V/q_i or V'_i for Q'_i. The practical problem may just be that the policy-maker wants to control whether his economy has been in competitive equilibrium or not.

Having convinced himself that society uptil now has actually been in the desired competitive equilibrium, the policy-maker may then be in the position where he believes that he is able to forecast a certain increase in value-added and a certain increase in the originary inputs, and he may ask himself: At constant output prices and raw material prices how much should money wages and the price of capital services, (land services etc.) be changed to keep the society in competitive equilibrium in future too. Here again, it may be objected that the policy-maker cannot make a sensible forecast of value-added without going back to the output, but don't let us bother about this.

The only thing to do seems then to try to judge about the marginal output-productivities from (13) or (2o). Provided that the input proportions do not change too much, the hypothesis of proportional changes (due to changes in technical knowledge) in the marginal output-productivities <u>may</u> be sound at the same time as the knowledge of the constant proportion between the marginal output-productivities becomes unnecessary. In a full employment society, the proportion between capital input and labour input cannot change much from year to year. For society as a whole, labour input may increase 1-2 pct. per year; capital may be by 3-5 pct. This will change the ratio capital/labour very little between two successive years and unless the k in eq. (15) is very large in relation to the ratio capital/labour, we can safely disregard the correction factor on the right hand side of (15). It is obvious that both the assumption of constant input proportion and constant marginal-productivity proportion will more easily be fulfilled the shorter the period considered is, and the larger the sector considered is in relation to society as a whole. In a society with non-full employment, conditions may be different. If a certain year total employment decreases by 5 pct. at the same time as capital increases by 5 pct. (which may happen just after a boom year), the input proportions change more, but even here it is obvious that unless k is very large, the correction factor in eq. (15) will still be unimportant. So, we still return to the question of k. What may be the order of magnitude of k?

Let us consider an example which may show the orders of magnitude. Assume that we have a society in competitive equilibrium with a value added = 60 billions (units of currency), number of workers equal to 0.003 billion and a stock of capital=

300 billion (units of currency).[1] Assume further that we know that about 2/3 of value added is labour income and 1/3 capital income. We have then P_L = 13 333 while p_C = 0.066. In competitive equilibrium where input prices equal marginal output-productivities, k would then be equal to p_C/p_L = 0.066/13 333 = 0.000005 while q_C/q_L = 300/0.003 = 100 000 and kq_C/q_L accordingly equal to 0.5. If k is constant (approximately), the correction factor may (in the extreme case of unemployment above) become at most 1.5/1.4 = 1.1. At most an error of about 10 pct should then be done in assuming that the marginal output-productivity of labour changes in the same proportion as the average value added-productivity of labour. If the latter increases by 4 pct., the former should then be known to increase by between 3.6 and 4.4 pct., which is quite a good precision. For obvious reasons these figures should be taken as nothing but an example.

Finally, if society is really seriously out of competitive equilibrium, all the reasonings we have done will break down. No conclusions at all can then be drawn from value added-productivities to output-productivities. It may, of course, still simply be postulated that e.g. average value added-productivity is equal to marginal output productivity, but no a priori reasoning can justify such postulates. The only thing to do is then to study output directly.

1) The figures correspond roughly to Swedish conditions.