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Interpretation of Duality of Linear Programming On Some Economic Problems.

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Interpretation of Linear Programming

On Economic Problems:-

Usually, linear programming problems are in the form of maximizing or minimizing an objective function under certain restrictions. Such maximum and minimum problems occur frequently in many branches of pure and applied mathematics. Also, such problems occur naturally when discussing economic problems, e.g. social planners attempt to maximize the welfare of the community; also, consumers wish to have the most use of their income to maximize their satisfaction. We shall consider some of the examples for the use of linear progrmaing in economic applications, and show how such problems can be tackled by linear programming.

A way of the interpretation of linear programming is by the use of duality principle, which will be explained later. In this paper we shall discuss first the duality principle, and then how to apply it in some linear programming problems.

Duality of Linear Programming Problems:

It is known in linear programming, that each maximum or minimum linear programming problems has a corresponding minimum or maximum problem, known as the dual of that problem.

e.g. If we have the linear programming problem:

- Maximize :
$$\sum_{j=1}^{n} A_j \cdot x_j \qquad (1.1)$$
$$(j = 1, 2, \ldots, n)$$

- Subject to :
$$\sum_{j=1}^{n} b_{ij} \cdot x_j \leqslant B_i \qquad (2.1)$$
$$(i = 1, 2, \ldots, m)$$

- Therefore, the dual to that problem is as follows:

- Minimize :
$$\sum_{i=1}^{m} B_i \cdot y_i \qquad (1.2)$$
$$(i = 1, 2, \ldots, m)$$

- Subject to :
$$\sum_{i=1}^{m} b_{ij} \cdot y_i \geqslant A_j \qquad (2.2)$$
$$(j = 1, 2, \ldots, n)$$

(e.g.) If we consider the example:

- Maximize : $2x_1 + 4x_2 + x_3 + x_4 = \max.$ ....(1)

- Subject to :

$$\left.\begin{array}{l} x_1 + 3x_2 + x_4 \leq 4 \\ 2x_1 + x_2 \leq 3 \\ x_2 + 4x_3 + x_4 \leq 3 \end{array}\right] \quad \text{.......}(2)$$

- Therefore, the dual of this problem will be;

- Minimize : $4y_1 + 3y_2 + 3y_3 = \min.$ ....(3)

- Subject to :

$$\left.\begin{array}{l} y_1 + 2y_2 \geq 2 \\ 3y_1 + y_2 + y_3 \geq 4 \\ 4y_3 \geq 1 \\ y_1 + y_3 \geq 1 \end{array}\right] \quad \text{......}(4)$$

- Meanning of Duality:

It is seen that when transforming from a standard linear programming problem to its dual that:

1) The bounds to the standard problem had been changed to be objectives in the dual problem.
2) The objectives in the standard problem had been changed into bounds for the dual problem.

This means that we look to the problem from an opposite point of view, as will be explained later when considering some linear programming problems.

- Some facts about duality:

(I) Assuming: $x_1 , x_2 , \dots , x_n$ to be a feasible solution of the standard maximum linear programming problem, relations (1.1), and $y_1 , y_2 , \dots , y_m$ to be a feasible solution of the dual problem, relations (1.2) , (2.2) , then :

$$\sum_{j=1}^{n} A_j \cdot x_j \leqslant \sum_{i,j} b_{ij} \cdot x_j \cdot y_i \leqslant \sum_{i=1}^{m} B_i \cdot y_i \quad \dots\dots(1)$$

- Proof:

since :

$$\sum_{j=1}^{n} b_{ij} \; x_j \leqslant B_i \quad \dots\dots\dots \text{[relation (2.1)]}$$

multiplying the j^{th} term of relation (2.1) by y_j and summing over j, therefore:

$$\sum_{i=1}^{m} B_j \cdot y_j \geqslant \sum_{i=1}^{m} y_i \cdot \sum_{j=1}^{n} b_{ij} \cdot x_j$$

$$\text{i.e.} \quad \sum_{i=1}^{m} B_i \cdot y_i \geqslant \sum_{i,j} b_{ij} \cdot x_j \cdot y_i \quad \dots\dots\dots(A)$$

Also, since:

$$\sum_{j=i}^{m} b_{ij} \cdot y_i \geqslant A_j \qquad \text{relation (2.2)} ,$$

multiplying the $i^{\underline{th}}$ term of relation (2.2) by x_j and summing over j, therefore:

$$\sum_{j=1}^{n} A_j \cdot x_j \leqslant \sum_{j=1}^{n} x_j \cdot \sum_{i=1}^{m} b_{ij} \cdot y_i$$

$$\text{i.e} \sum_{j=1}^{n} A_j \cdot x_j \leqslant \sum_{i,j} b_{ij} \cdot x_j \cdot y_i \quad \ldots\ldots \quad \text{(B)}$$

Therefore, from (A) and (B), it can be seen that:

$$\sum_{j=1}^{n} A_j \cdot x_j \leqslant \sum_{i,j} b_{ij} \cdot x_j \cdot y_i \leqslant \sum_{i=1}^{m} B_i \cdot y_i$$

(II) - If there exist feasible solutions $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ for the maximum problem above and its dual, such that:

$$\sum_{j=1}^{n} A_j \cdot x_j = \sum_{i=1}^{m} B_i \cdot y_i$$

then these feasible solutions are optimal to the respective problems.

— Proof:

Let: $x'_1, x'_2, \ldots, x'_n$ be any other feasible solution of the maximum problem, therefore from (1) we have:

$$\sum_{j=1}^{n} A_j \cdot x'_j \leqslant \sum_{i=i}^{m} B_i \cdot y_i$$

and since it is assumed that $\sum_{j=i}^{n} A_j \cdot x_j = \sum_{i=1}^{n} B_i \cdot y_i$

therefore $\sum_{j=1}^{n} A_j \cdot x'_j \leqslant \sum_{j=1}^{m} A_j \cdot x_j$

Showing that x'_js are the optimum solution. By the same method, the optimality of y'_js can be proved.

(III) If a standard maximum or minimum linear programming problem and its dual are both feasible, then they both have optimal solutions and both have the same value. If either is not feasible, then neither has an optimal solution.

This is considered as a fundamental duality theorem, the proof is not simple and it will not be considered here.

Now after stating the duality principles, we are going to discuss some applications of duality to some of the economic problems.

(I.a) The Diet Problem:-

Such problems appear when trying to select a diet for a group of persons, an army say, satisfying certain nutritional reqirements while by the same time regarding the most economical conditions.

Confronted with different foods, F_1 , F_2 , ... , F_n, a dietitian is to select a diet by choosing the amount of each of these different foods to be consumed annually. The diet chosen must contain for example some nutritional elements such as protiens, calories, minerals, etc. Assuming that there are varieties of these nutrients, N_1 , N_2 , ... , N_m and assume that each person is to consume A_1 units of N_1, A_2 units of N_2,...., and A_m units of N_m per year. To meet these requirements, the dietition must determine the amount of each nutrient contained in each food.

Assuming the amount of the i^{th} nutrient in the j^{th} food to be a_{ij}, where j=1,2, ..., n; and i=1,2, ..., m.; we can write the matrix of coefficients of the problem as follows:

i \ j	F_1	F_2	$\cdots$	F_j	$\cdots$	F_n
N_1	a_{11}	a_{12}	$\cdots$	$\cdots$	$\cdots$	a_{1n}
N_2	a_{21}	a_{22}	$\cdots$	$\cdots$	$\cdots$	a_{2n}
$\vdots$	$\vdots$			$\vdots$		$\vdots$
N_i	$\cdots$	$\cdots$	$\cdots$	a_{ij}	$\cdots$	$\vdots$
N_m	a_{n1}	a_{n2}	$\cdots$	$\cdots$	$\cdots$	a_{mn}

Table (1)

The problem matrix of coefficients.

or

Nutrition matrix

If the dietitian chooses a certain diet composed of the amounts x_1 of F_1, x_2 of F_2, ..., and x_j of F_j, therefore, the amount of nutrients in x_j units of F_j containing a_{ij} units of that nutrient is equal to $(x_j \cdot a_{ij})$. Considering the condition that each diet must contain at least A_i units of the nutrients N_i, this can be stated mathematically as follows:

$$x_1 \cdot a_{11} + x_2 \cdot a_{12} + x_3 \cdot a_{13} + \cdots + x_1 \cdot a_{1j} + \cdots + \cdots + x_n \cdot a_{1n} \geqslant A_1 \quad (\text{for nutrient } N_1)$$

and

$$x_1 \cdot a_{21} + x_2 \cdot a_{22} + x_3 \cdot a_{23} + \cdots + x_j \cdot a_{2j} + \cdots + \cdots + x_n \cdot a_{2n} \geqslant A_2 \quad (\text{for nutrient } N_2)$$

$$\vdots \qquad\qquad \vdots$$

$$x_1 \cdot a_{i1} + x_2 \cdot a_{i2} + x_3 \cdot a_{i3} + \dots + x_j \cdot a_{ij} + \dots +$$
$$+ \dots + x_n \cdot a_{in} \geqslant A_i \quad \text{(for nutrient } N_i)$$

$$\vdots \qquad\qquad \vdots$$

$$x_1 \cdot a_{m1} + x_2 \cdot a_{m2} + x_3 \cdot a_{m3} + \dots + x_j \cdot a_{mj} + \dots +$$
$$+ \dots + x_n \cdot a_{mn} \geqslant A_m \quad \text{(for nutrient } N_m)$$

or this set of relations can be stated as follows:

$$\sum_{j=1}^{n} x_j \cdot a_{ij} \geqslant A_i \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

for nutrients : $i = 1, 2, \dots, m$.

The diet satisfying conditions (1) is termed as a feasible diet. Such feasible diets are all satisfying the required conditions, however they have not the same cost. Then from the set of feasible diets, the dietitian is to choose the most economic one, i.e. that diet satisfying the conditions with the least costs. Now considering the economic part of the problem, assuming that c_j is the cost of one unit of F_j, therefore the cost of the diet will be given by:

$$c_1 \cdot x_1 + c_2 x_2 + c_3 \cdot x_3 + \dots + c_j \cdot x_j + \dots +$$
$$+ \dots + c_n x_n = \text{cost of the diet.}$$

or :

$$\sum_{j=1}^{n} c_j \cdot x_j = \text{cost of the diet.}$$

and for the economy of the problem, the cost of the diet chosen must be the minimum.

$$\text{i.e.} \quad \sum_{j=1}^{n} c_j \cdot x_j = \text{minimum.} \qquad \ldots\ldots\ldots\ldots(2)$$

Now, our problem can be described as :

- Minimize : $\sum_{j=1}^{n} c_j \cdot x_j$

$(j = 1,2,3, \ldots , n) \ldots\ldots(I.I)$

- such that:

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \geqslant A_i$$

$(i = 1,2,3, \ldots , m) \ldots\ldots(I,II)$

The diet satisfying both I and II is known as the "optimal diet". It is clear for such a problem, that a feasible diet exists if each nutrient N_i occurs in at least one of the foods F_j, i.e. by using a sufficient amounts of that food we can satisfy the nutrients requirements. Now the problem can be divided into two parts, first find the feasible diets, and then choose from them the optimal diet.

(I.b) Interpretation of duality for the Diet problem :

As was explained before, the dual to equations (I.I) and (I.II), will be:

- Maximize: $$\sum_{i=1}^{m} A_j \cdot y_i \quad \text{.....................} \quad (I.III)$$

- such that $$\sum_{i=1}^{m} a_{ij} \cdot y_j \leqslant c_j \quad \text{.................} \quad (I.IV)$$

$$(j = 1,2,3, \ldots , n).$$

Now, Considering the left hand side of relation (I.IV), since c_j is expressing a cost, i.e. it is expressed in money value, then, the right hand side, which is ($a_{ij} \cdot y_i$) will have the same units i.e. it will be expressed in money value too. But since a_{ij} is representing the amount of nutrient "i" in the food "j" then y_i represents a cost for that nutrient.

Now we are going to discuss what the dual problem here economically means. The original problem, was the problem of the dietitian who is trying to find a diet containing certain amounts of the different nutrients, relations (I.II), and by the same time will have the minimum cost, relation (I.I). Now for a diet salesman, assume he did find a way to provide the diet with the required nutrients the latter needs, e.g. if he is to provide the dietitian with some vitamin pills, iron capsules, ... etc. , to substitute for some kinds of the vegitables or meat.

Then the dietitian whose aim is to minimize the cost,will willingly substitute the pills for the other foods provided that he will save money. Now suppose that the salesman sets the prices of a unit of N_i at some value y_i regarding that:

$$\sum_{i=1}^{m} a_{ij}.\ y_i \leq c_j$$ for all j where a_{ij} in this

case is the amount of the nutrient N_i to substitute for that existing in the food F_j. This means that the total value of the nutrients existing in a unit of F_j will be no greater than the unit cost of F_j, which is c_j , for all j. Now, the dietitian, confronted with this new offer, he is going to buy pills instead of the previous foods since it will be alwys more economical, no matter what he was to choose.By the same time, the pills salesman would like to charge the dietitian as much as possible subject to (1.1V) , therefore since the adequate diet calls for "A_i" units of nutrient N_i , the salesman would like to set his prices "y_i" such that ($A_i \cdot y_i$) would be a maximum.

This may be a discription to what the dual of the diet problem may mean. In such a discription we can be some what less concrete by saying that the nutrient prices "y_i" are these which enable the pill salesman to realise maximum return and by the same time compete with the grocer. This may give an idea of the competitive prices which is characteristic of the interpretation of the duality theorem in such a case.

(II.a) . The Transportation Problem.

Let a certain commodity can be produced in any of "m" plants P_1 , P_2 ,..., P_i,..., P_m each supplying an amount of the commodity A_1 , A_2 , ... , A_i , ...A_m. respectively. Assume there are "n" markets M_1 , M_2 , M_j , M_n to have amounts of that product B_1 , B_2 , ... , B_j , ... , B_n respectively. Let c_{ij} be the cost of shipping per unit of the product from plant "i" to the market "j".

Now the problem can be stated as to find the minimum total cost of shipping from the different plants such that:

a) The markets' demand must be satisfied.

b) The plants' supply must not be exeeded.

Assuming the amount of the product to be supplied from plant "i" to the market "j" to be x_{ij} , then a table as shown can be constructed representing the amounts to be shipped from each plant to each of the markets as follows:

i \ j	M_1	M_2	...	M_i	......	M_n
A_1	x_{11}	x_{12}	---	x_{ij}	-----	x_{1n}
A_2	x_{21}	x_{22}	---	x_{2j}	-----	x_{2n}
⋮	⋮	⋮		⋮		⋮
A_i	...	...	...	x_{ij}	......	...
⋮		⋮		⋮		⋮
A_m	x_{m1}	x_{m2}	...	...	......	x_{mn}

Table(II.I): Matrix of supply of the plants to the different markets.

- The problem can be represented mathematically as follows:

- Minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$ .... (II.1)

- Such that

$$\sum_{j=1}^{n} x_{ij} \leqslant A_i \quad \text{............. (II.2)}$$

for $i = 1, 2, \ldots, m.$

- and $\sum_{i=1}^{m} x_{ij} \geqslant B_j$ .... (II.3)

for $j = 1, 2, \ldots, n.$

- Where $x_{ij} \geqslant 0$ .... (II.4)

(for all i's and j's.)

A solution to this problem is feasible if we can find a shipping schedule such that the numbers "c_{ij}'s" are postive and by the same time satisfying conditions (2) and (3).

It is clear that a feasible solution exists if the total supply from all the plants is greater than or equal to the total demand of all the markets.

$$\text{i.e. if} \quad \sum_{i=1}^{m} A_i \geqslant \sum_{j=1}^{n} B_j$$

(II.b) The Dual to the Transportation Problem:-

In the transportation problem shown above, we see that it is discribed by an objective function to be minimized, relation (II.1), subject to two types of relationships: one is upper bounded, relations (II.2), and the other is lower bounded, relation (II.3). In such a problem with the two types of restrictions, the dual will be composed of two types of variables which we will assume as "y's" and "z's" relative to the dual to the original problem, relations (II.1), (II.2), (II.3), will as follows:

- Maximize $\sum_{j=i}^{n} B_j \cdot Z_j - \sum_{i=1}^{m} A_i \, y_i$ ....(11.5)

- Subject to $Z_j - y_i \leqslant C_{ij}$ ....(11.6)

(for all i and j).

(j= 1,2, ... , n) , and (i= 1,2, ... , m)

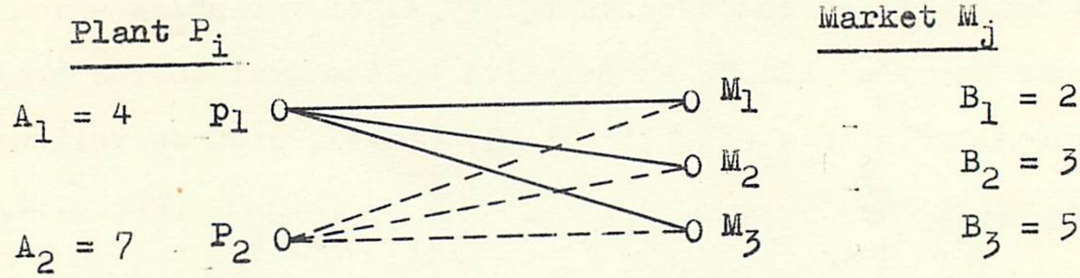
(e.g.) Consider the problem of two plants P_1 and P_2 producing steel for example, which is needed for three markets M_1, M_2 and M_3. The maximum amount produced yearly in plant $P_1 = 4$ units and that in plant $P_2 = 7$ units. The annual demand for steel in markets M_1, M_2, M_3 must not be less than 2, 3, 5 units respectively. What is the optimum transportation schedule that minimizes the cost of transportation from the different plants to the different markets assuming that the cost of transportation from plants to markets are as shown in table (II.2).

i \ j	M_1	M_2	M_3
P_1	1	2	3
P_2	2	4	6

Table (II.2)

Cost matrix c_{ij} .

This problem can be represented by the diagramatic representation shown in fig (II.1)



- This problem can be represented mathematically as follows:

- Minimize:

$$\sum_{i,j} C_{ij} \cdot x_{ij} \qquad \text{..............} (1)$$

- Subject to:

$$\sum_{j=1}^{n} x_{ij} \leqslant A_i$$

(for all i = 1,2, ... , m).

i.e.

$$\left.\begin{aligned} \sum_{j=1}^{3} x_{1j} &\leqslant 4 \\ \sum_{j=1}^{3} x_{2j} &\leqslant 7 \end{aligned}\right] \qquad \text{..................} (2)$$

and: $$\sum_{i=1}^{m} x_{ij} \geqslant B_j$$

(for all $j = 1, 2, \ldots, n$).

i.e.
$$\left.\begin{array}{l} \sum_{i=1}^{2} x_{i1} \geqslant 2 \\ \sum_{i=1}^{2} x_{i2} \geqslant 3 \\ \sum_{i=1}^{2} x_{i3} \geqslant 5 \end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

or, these relations can be expanded as follows:

- Mimimize :
$$x_{11} + 2x_{12} + 3x_{13} + $$
$$+ 2x_{21} + 4x_{22} + 6x_{23} \quad \ldots\ldots\ldots\ldots(4)$$

- Subject to
$$\left.\begin{array}{l} x_{11} + x_{12} + x_{13} \leqslant 4 \\ x_{21} + x_{22} + x_{23} \leqslant 7 \end{array}\right] \ldots\ldots(5)$$

- and
$$\left.\begin{array}{l} x_{11} + x_{21} + x_{31} \geqslant 2 \\ x_{12} + x_{22} + x_{32} \geqslant 3 \\ x_{13} + x_{23} + x_{33} \geqslant 5 \end{array}\right] \ldots\ldots(6)$$

where x_{ij} is the amount to be shipped from plant "i" to the market "j." Now the dual to this problem can be stated as follows:

— Maximize :

$$\sum_{i=1}^{n} B_i \cdot z_i - \sum_{i=1}^{m} A_i \, y_i$$

i.e. Mimimize:

$$2 Z_1 + 3 Z_2 + 5 Z_3 - 4 y_1 - 7 y_2 \quad \ldots\ldots (7)$$

— Such that : $Z_j - y_i \leqslant c_{ij}$

(for all i and j)

i.e.

$$\left.\begin{array}{lcl} Z_1 - y_1 & \leqslant & 1 \\ Z_1 - y_2 & \leqslant & 2 \\ Z_2 - y_1 & \leqslant & 2 \\ Z_2 - y_2 & \leqslant & 4 \\ Z_3 - y_1 & \leqslant & 3 \\ Z_3 - y_2 & \leqslant & 6 \end{array}\right] \quad \ldots\ldots\ldots\ldots (8)$$

— Interpretation of duality to the transportation problem:

To show the meaning of the dual to the transportation problem, considering relation (II.6) since the left hand-side c_{ij} is in cost units, then the right hand-side $Z_j - y_i$ must be in cost units too. The original problem was the problem of the planner whether he may be the manufacturer or consumer who would like to fullfill the demand with the least transportation cost. Now to explain what the dual represents, assume a middle person between the manufacturer and the consumer, who would like to buy the product from the manufacturer and then sell them for him to the consumer. Suppose this middle person would buy each unit of the product from the manufacturer by "y_i" and would sell it for him to the consumer for "z_j" units per unit of the product noticing that: $z_j - y_i \leqslant C_{ij}$ for all i and j . We see that the manufacturer will agree for the benifit of increasing his profits. Now regarding that middle person, he would like to gain as much profit as he can by maximizing the amount :

$$\sum_{j=1}^{n} (B_j - Z_j) - \sum_{i=1}^{m} A_i \cdot y_i$$ which is the difference

between the value of his sales to the market "j" less the cost he paid to plant "i," subject only to condition (II.6) .

This may give an idea to what the dual to the transportation problem may mean. However, it is clear that the two concepts of the original problem and its dual are leading to the same aim, and the optimum solution to any one of them if exists, will be the same as the optimum to the other.

Duality of linear programming may be useful and of great help in the solution of some linear programming problems, where it may be difficult to solve the original standard problem. In this paper some economic problems were invistigated by linear programming, and the meaning of duality of these problems were explained. However, by same method, we can interprete the principles of linear programming and duality to other economic and other general problems.