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TOPOLOGICAL KERNEL OF SETS AND APPLICATION ON FRACTALS

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ABSTRACT. In various mathematical sciences, sets and functions in topology have been extensively developed and exploited. Some novel separation axioms have been discovered through studying generalizations of closures duo to closed sets, ∧-sets, and V-sets. Self similar fractals have important role in some real life problems as physics and engineering. In this paper, the topology described by the family of $\delta\gamma$-∧ and $\delta\gamma$-V-sets in topological spaces is defined and studied in terms of $\wedge_{\delta\gamma}$-Sets and $\vee_{\delta\gamma}$-Sets. Also, the topological space $\mathbf{Top}_{(\square,\tau^{\wedge\delta\gamma})}$ is defined and studied. Additionally, several features of these sets are presented, as well as some associated new separation axioms. Finally, we improved theorems 4.3 [5] and 6.5 [6] for Caldas et al. We approximate self similar fractals through graph theory to topological spaces. Some topological properties such as separation axioms are studied. Finally, the kernel of topological approximations of fractals are calculated in terms of their connecting points.

KEYWORDS: Kernels, $\delta\gamma$-open sets, $\wedge_{\delta\gamma}$-sets and $\vee_{\delta\gamma}$-sets.

1. INTRODUCTION

Maki [25] extended the notions of Levine [24] and Dunham [10] on closure operators which based generalized closed sets in 1986 by introducing generalized ∧-sets a topology $(\square,\tau)$ ($\mathbf{Top}_{(\square,\tau)}$, for short) and defining its ∧-closure operator. In a continuation, Maki linked between the τ and $\tau^{\wedge}$ using generalized ∧-sets. Caldas and Dontchev [4], Ganster, Jafari, and Noiri [23], and Caldas, Jafari, and Noiri [7] recently acquired and explored three extensions of the concept of ∧-set. They continued Maki's efforts [25]. $\mathbf{Top}_{(\square,\tau)}$ used in some applications duo to graphs and related with some types of separation axioms, see [15, 16, 17, 18, 21, 22] which are used in physics [13, 14, 19, 20] and smart city [3].

The objective of this work is to continue study in similar lines, but using $\delta\gamma$-open sets this time. In given $\mathbf{Top}_{(\square,\tau)}$, we introduce $\wedge_{\delta\gamma}$-sets and $\vee_{\delta\gamma}$-sets and thereby gain novel topologies defined by these families of sets. Additionally, we discuss many essential characteristics of these novel topologies.

2. PRELIMINARIES

In a $\mathbf{Top}_{(\square,\tau)}$, the closure (resp. interior) of $\mathfrak{A}$ is denoted by $\mathfrak{C}(\mathfrak{A})$ and $\mathfrak{I}(\mathfrak{A})$. $\mathfrak{A}$ is regular open ($RO(\square,\tau)$, for short) [29] if $\mathfrak{A}=\mathfrak{I}(\mathfrak{C}(\mathfrak{A}))$ and their union forms δ-open ($\delta O(\square,\tau)$, for short) [30]. The complement of $RO(\square,\tau)$ (resp. $\delta O(\square,\tau)$) set is $RC(\square,\tau)$ (resp. $\delta C(\square,\tau)$). $\mathfrak{A}$ is b-open [2] (or γ-open [12], or sp-open [11]) ($\gamma O(\square,\tau)$, for short) if $\mathfrak{A}\subseteq\mathfrak{C}(\mathfrak{I}(\mathfrak{A}))\cup\mathfrak{I}(\mathfrak{C}(\mathfrak{A}))$. Its complement is γ-closed ($\gamma C(\square,\tau)$, for short).

The intersection of all $\delta C(\square,\tau)$ has $\mathfrak{A}$ is δ-closure [30] of $\mathfrak{A}$ ($\delta\mathfrak{C}(\mathfrak{A})$, for short). $\mathfrak{S}$ of a $\mathbf{Top}_{(\square,\tau)}$ is $\delta\gamma$-open [8] ($\delta\gamma O(\square,\tau)$, for short) if $\mathfrak{S}\subseteq\mathfrak{C}(\mathfrak{I}_{\delta}(\mathfrak{A}))\cup\mathfrak{I}(\mathfrak{C}_{\delta}(\mathfrak{S}))$ and its complement is $\delta\gamma$-closed [8] ($\delta\gamma C(X,\tau)$, for short). The union (resp. intersection) of all $\delta\gamma O(\square,\tau)$ contained in (resp. has) $\mathfrak{A}$ is the $\delta\gamma$-interior (resp. $\delta\gamma$-closure) of $\mathfrak{A}$ and is abbreviated with $\delta\gamma\mathfrak{I}(\mathfrak{A})$ (resp. $\delta\gamma\mathfrak{C}(\mathfrak{A})$).

3. $\wedge_{\delta\gamma}$-SETS AND $\vee_{\delta\gamma}$-SETS

We introduce the concepts of $\wedge_{\delta\gamma}$ and $\vee_{\delta\gamma}$-sets and examine some of their essential features in this section.

DEFINITION 3.1

In $\mathbf{Top}_{(\square,\tau)}$, $\mathfrak{D}$ is $\delta\gamma$-∧-set (abb. $\wedge_{\delta\gamma}(\mathfrak{D})$)(resp. $\delta\gamma$-V-set (abb. $\vee_{\delta\gamma}(\mathfrak{D})$)), if $\mathfrak{D}=\wedge_{\delta\gamma}(\mathfrak{D})$ (resp. $\mathfrak{D}=\vee_{\delta\gamma}(\mathfrak{D})$), where $\wedge_{\delta\gamma}(\mathfrak{D})=\cap\{\mathcal{G}:\mathcal{G}\supseteq\mathfrak{D},\mathcal{G}\in\delta\gamma O(\square,\tau)\}$ and $\vee_{\delta\gamma}(\mathfrak{D})=\cup\{\mathfrak{F}:\mathfrak{F}\subseteq\mathfrak{D},\mathfrak{F}^{c}\in\delta\gamma O(\square,\tau)\}$.

In [28], $\wedge_{\delta\gamma}(\mathfrak{D})$ or $\mathfrak{D}^{\wedge\delta\gamma}$ is $\delta\gamma$-kernel of a set $\mathfrak{D}$ ($\mathfrak{D}^{\wedge\delta\gamma}$, for short). The underlying holdings of sets $\wedge_{\delta\gamma}(\mathfrak{D})$ and $\vee_{\delta\gamma}(\mathfrak{D})$ can be resumed.

THEOREM 3.2

Let $\mathfrak{A}$, $\mathfrak{D}$ and $\{\mathfrak{D}_i : i \in I\}$ be subsets of a $\boldsymbol{Top}_{(\square,\tau)}$. Then, the followings are valid:

(1) $\mathfrak{D} \subseteq \wedge_{\delta\gamma}(\mathfrak{D})$;

(2) If $\mathfrak{A} \subseteq \mathfrak{D}$, then $\wedge_{\delta\gamma}(\mathfrak{A}) \subseteq \wedge_{\delta\gamma}(\mathfrak{D})$;

(3) If $\mathfrak{D} \in \delta\gamma O(\square,\tau)$, then $\mathfrak{D} = \wedge_{\delta\gamma}(\mathfrak{D})$;

(4) $\wedge_{\delta\gamma}(\wedge_{\delta\gamma}(\mathfrak{D})) = \wedge_{\delta\gamma}(\mathfrak{D})$;

(5) $\wedge_{\delta\gamma}(\cup\{\mathfrak{D}_i : i \in I\}) = \cup\{\wedge_{\delta\gamma}(\mathfrak{D}_i) : i \in I\}$;

(6) $\wedge_{\delta\gamma}(\cap\{\mathfrak{D}_i : i \in I\}) \subseteq \cap\{\wedge_{\delta\gamma}(\mathfrak{D}_i) : i \in I\}$;

(7) $\wedge_{\delta\gamma}(\mathfrak{D}^c) = (\vee_{\delta\gamma}(\mathfrak{D}))^c$.

Proof. (1) Follows duo to Definition 3.1.

(2) Since $\mathrm{I} \notin \wedge_{\delta\gamma}(\mathfrak{D})$, then $\exists\ \mathcal{G} \in \delta\gamma O(\square,\tau)$ s.t. $\mathcal{G} \supseteq \mathfrak{D}$ with $\mathrm{I} \notin \mathcal{G}$. Since $\mathfrak{D} \supseteq \mathfrak{A}$, then $\mathrm{I} \notin \wedge_{\delta\gamma}(\mathfrak{A})$ and so $\wedge_{\delta\gamma}(\mathfrak{A}) \subseteq \wedge_{\delta\gamma}(\mathfrak{D})$.

(3) Using Definition 3.1 and since $\mathfrak{D} \in \delta\gamma O(\square,\tau)$, we get $\wedge_{\delta\gamma}(\mathfrak{D}) \subseteq \mathfrak{D}$. By (1), $\wedge_{\delta\gamma} B) = B$.

(4) Follows from (3) and Definition 3.1.

(5) Let $\exists\ \mathrm{I}$ s.t. $\mathrm{I} \notin \wedge_{\delta\gamma}(\cup\{\mathfrak{D}_i : i \in I\})$. Then, $\exists\ \mathcal{G} \in \delta\gamma O(\square,\tau)$ s.t. $\cup\{\mathfrak{D}_i : i \in I\} \subseteq \mathcal{G}$ and $\mathrm{I} \notin \mathcal{G}$. Thus, $\forall\ i \in I$, we get $\mathrm{I} \notin \wedge_{\delta\gamma}(\mathfrak{D})$. This indicates that $\mathrm{I} \notin \cup\{\wedge_{\delta\gamma}(\mathfrak{D}_i) : i \in I\}$. On the other hand, suppose that $\exists\ \mathrm{I} \in \square$ s.t. $\mathrm{I} \notin \cup\{\wedge_{\delta\gamma}(\mathfrak{D}_i) : i \in I\}$. Then, by Definition 3.1, $\exists\ \mathcal{G}_i \in \delta\gamma O(\square,\tau)$, $\forall\ i \in I$ s.t $\mathrm{I} \notin \mathcal{G}_i$, $\mathfrak{D}_i \subseteq \mathcal{G}_i$. Let $\mathcal{G} = \cup\{\mathcal{G}_i : i \in I\}$. Then, we get $\mathrm{I} \notin \cup\{\mathcal{G}_i : i \in I\}$, $\cup\{\mathfrak{D}_i : i \in I\} \subseteq \mathcal{G}$ and $\mathcal{G} \in \delta\gamma O(\square,\tau)$. This indicates that $\mathrm{I} \notin \wedge_{\delta\gamma}(\cup\{\mathfrak{D}_i : i \in I\})$.

(6) Let $\mathrm{I} \notin \cap(\{\wedge_{\delta\gamma}(\mathfrak{D}_i) : i \in I\})$. Then, $\exists\ i_0 \in I$ s.t. $\mathrm{I} \notin \wedge_{\delta\gamma}(\mathfrak{D}_i)$ and $\exists\ \mathcal{G} \in \delta\gamma O(\square,\tau)$ s.t. $\mathrm{I} \notin \mathcal{G}$ and $\mathcal{G}_{i_0} \subseteq \mathcal{G}$ and so $\cap\{\mathfrak{D}_i : i \in I\} \subseteq \mathfrak{D}_{i_0} \subseteq \mathcal{G}$ and $\mathrm{I} \notin \mathcal{G}$. Therefore, $\mathrm{I} \notin \wedge_{\delta\gamma}(\cap\{\mathfrak{D}_i : i \in I\})$.

(7) Let $\mathfrak{D} \subseteq \square$. Then, $\square \backslash \vee_{\delta\gamma}(\mathfrak{D}) = \cap\{\square \backslash \mathfrak{F} : \square \backslash \mathfrak{D} \subseteq \square \backslash \mathfrak{F}$ and $\square \backslash \mathfrak{F} \in \delta\gamma O(\square,\tau)\} = \wedge_{\delta\gamma}(\square \backslash \mathfrak{D})$.

DEFINITION 3.3

$\mathfrak{D}$ of a $\boldsymbol{Top}_{(\square,\tau)}$ is $\wedge_{\delta\gamma}$-set if $\mathfrak{D} = \wedge_{\delta\gamma}(\mathfrak{D})$.

LEMMA 3.4

For $\mathfrak{D}$ and $\{\mathfrak{D}_i : i \in I\}$ of a $\boldsymbol{Top}_{(\square,\tau)}$, the followings hold:

(1) $\wedge_{\delta\gamma}(\mathfrak{D})$ is a $\wedge_{\delta\gamma}$-set.

(2) If $\mathfrak{D}$ is $\delta\gamma O(\square,\tau)$, then $\mathfrak{D}$ is a $\wedge_{\delta\gamma}$-set.

(3) If $\mathfrak{D}_i$ is a $\wedge_{\delta\gamma}$-set $\forall\ i \in I$, then $\cap\{\mathfrak{D}_i : i \in I\}$ is a $\wedge_{\delta\gamma}$-set.

(4) If $\mathfrak{D}_i$ is a $\wedge_{\delta\gamma}$-set, $\forall\ i \in I$, then $\cup\{\mathfrak{D}_i : i \in I\}$ is a $\wedge_{\delta\gamma}$-set.

Proof. Follows readily using Definition 3.1 and Theorem 3.2.

A $\boldsymbol{Top}_{(\square,\tau)}$ is Alexandröff [1] if each point has a minimum nbd or, equivalently, a unique minimal base.

THEOREM 3.5

For a $\boldsymbol{Top}_{(\square,\tau)}$, put $\tau^{\wedge\delta\gamma} = \{\mathfrak{A} : \mathfrak{A}$ is a $\wedge_{\delta\gamma}$-set of $\square\}$. Then, $(\square, \tau^{\wedge\delta\gamma})$ is Alexandröff.

Proof. Immediately consequence by Lemma 3.4.

REMARK 3.6

In general, $\wedge_{\delta\gamma}(\mathfrak{A} \cap \mathfrak{D}) \neq \wedge_{\delta\gamma}(\mathfrak{A}) \cap \wedge_{\delta\gamma}(\mathfrak{D})$ is held.

EXAMPLE 3.7

Let $\square = \{1,2,3\}$ and $\tau = \{\square, \phi, \{1\}\}$. If $\mathfrak{A} = \{2\}$ and $\mathfrak{D} = \{3\}$, then $\wedge_{\delta\gamma}(\mathfrak{A} \cap \mathfrak{D}) = \phi$, but $\wedge_{\delta\gamma}(\mathfrak{A}) \cap \wedge_{\delta\gamma}(\mathfrak{D}) = \{1\}$.

REMARK 3.8

Every $\delta\gamma$-open set is $\wedge_{\delta\gamma}$-set, while the reverse is not hold, in general.

EXAMPLE 3.9

Let $\square = \{1,2,3,4\}$ and let $\tau = \{\square, \phi, \{1\}, \{3\}, \{1,3\}, \{1,2\}, \{1,2,3\}, \{1,3,4\}\}$. Here, $\{4\}$ is $\wedge_{\delta\gamma}$-set, but $\{4\} \notin \delta\gamma(\square,\tau)$.

THEOREM 3.10

Let $\mathfrak{A}$, $\mathfrak{D}$ and $\{\mathfrak{D}_i : i \in I\}$ be in $\boldsymbol{Top}_{(\square,\tau)}$. Then, the followings are valid:

(1) $\vee_{\delta\gamma}(\mathfrak{D}) \subseteq \mathfrak{D}$;

(2) If $\mathfrak{A} \subseteq \mathfrak{D}$, then $\vee_{\delta\gamma}(\mathfrak{A}) \subseteq \vee_{\delta\gamma}(\mathfrak{D})$;

(3) $\vee_{\delta\gamma}(\vee_{\delta\gamma}(\mathfrak{D})) = \vee_{\delta\gamma}(\mathfrak{D})$;

(4) If $\mathfrak{D} \in \delta\gamma C(X,\tau)$, then $\mathfrak{D} = \vee_{\delta\gamma}(\mathfrak{D})$;

(5) $\vee_{\delta\gamma}(\cap\{\mathfrak{D}_i : i \in I\}) = \cap\{\vee_{\delta\gamma}(\mathfrak{D}_i) : i \in I\}$;

(6) $\vee_{\delta\gamma}(\cup\{\mathfrak{D}_i : i \in I\}) \supseteq \cup\{\vee_{\delta\gamma}(\mathfrak{D}_i) : i \in I\}$.

Proof. (1) It is clear by Definition 3.1 and Theorem 3.2.

(2) Let $\mathrm{I} \in \vee_{\delta\gamma}(\mathfrak{A})$. Then, $\exists\ \mathfrak{F} \subseteq \mathfrak{A}$ s.t. $\mathrm{I} \in \mathfrak{F}$ and $\mathfrak{F}^c \in \delta\gamma O(\square,\tau)$. Since $\mathfrak{A} \subseteq \mathfrak{D}$, then $\mathfrak{F} \subseteq \mathfrak{D}$. Therefore, $\mathrm{I} \in \vee_{\delta\gamma}(\mathfrak{D})$.

(3) It follows from (1) and (2) that $\vee_{\delta\gamma}(\vee_{\delta\gamma}(\mathfrak{A})) \subseteq \vee_{\delta\gamma}\mathfrak{A})$. If $\mathrm{I} \notin \vee_{\delta\gamma}(\vee_{\delta\gamma}(\mathfrak{A}))$, then $\mathrm{I} \notin \mathfrak{F}$, $\forall\ \mathfrak{F} \subseteq \vee_{\delta\gamma}(\mathfrak{A})$. Since $\vee_{\delta\gamma}(\mathfrak{A}) \subseteq \mathfrak{A}$, then $\mathrm{I} \notin \mathfrak{F}$, $\forall\ \mathfrak{F} \subseteq \mathfrak{A}$ and $\mathfrak{F}^c \in \delta\gamma O(\square,\tau)$. This indicates that $\mathrm{I} \notin \vee_{\delta\gamma}(\mathfrak{A})$.

(4) By Definition 3.1 and $\mathfrak{D} \in \delta\gamma C(\square,\tau)$, we get $\mathfrak{D} \subseteq \vee_{\delta\gamma}(\mathfrak{D})$. By (1) we get $\mathfrak{D} = \vee_{\delta\gamma}(\mathfrak{D})$.

(5) Let $\exists\ \mathrm{I}$ s.t. $\mathrm{I} \in \vee_{\delta\gamma}(\cap\{\mathfrak{D}_i : i \in I\})$. Then, $\exists\ \mathfrak{F} \subseteq \cap\{\mathfrak{D}_i : i \in I\}$ s.t $\mathrm{I} \in \mathfrak{F}$ and $\mathfrak{F}^c \in \delta\gamma O(\square,\tau)$. So, $\mathfrak{F} \subseteq \mathfrak{D}_i$, $\forall\ i \in I$. Then, $\mathrm{I} \in \vee_{\delta\gamma}(\mathfrak{D}_i)$. Therefore, $\mathrm{I} \in \cap\vee_{\delta\gamma}\{\mathfrak{D}_i : i \in I\}$. On the other hand, suppose that $\exists\ \mathrm{I} \in \square$ s.t $\mathrm{I} \in \cap\vee_{\delta\gamma}\{\mathfrak{D}_i : i \in I\}$. Then, $\mathrm{I} \in \vee_{\delta\gamma}\{\mathfrak{D}_i$, $\forall\ i \in I$. Then, $\exists\ \mathfrak{F} \subseteq \mathfrak{D}_i$, $\forall\ i \in I$. Then, $\mathfrak{F} \subseteq \cap\{\mathfrak{D}_i : i \in I\}$ and $\mathfrak{F}^c \in \delta\gamma O(\square,\tau)$. Therefore, $\mathrm{I} \in \vee_{\delta\gamma}(\cap\{\mathfrak{D}_i : i \in I\})$.

(6) Let $\text{I} \in \cup \vee_{\delta\gamma} \{\mathfrak{D}_i : i \in I\}$, $\exists\ i_0 \in I$ s.t. $\text{I} \in \vee_{\delta\gamma} (\mathfrak{D}_{i_0})$. So, $\exists\ \mathfrak{F} \subseteq \mathfrak{D}_{i_0}$ for $i_0 \in I$ and $\mathfrak{F}^c \in \delta\gamma O(\square,\tau)$. Thus, $\text{I} \in \cup \{\mathfrak{D}_{i_0} : i_0 \in I\}$ and so $\text{I} \in \cup \{\mathfrak{D}_i : i \in I\}$. Therefore, $\text{I} \in \vee_{\delta\gamma} (\cup \{\mathfrak{D}_i : i \in I\})$.

There is more equivalent definitions of $\wedge_{\delta\gamma}$ and $\vee_{\delta\gamma}$-sets.

DEFINITION 3.11

For $\mathfrak{D}$ of a $\boldsymbol{Top}_{(\square,\tau)}$, $\wedge_{\delta\gamma}$ and $\vee_{\delta\gamma}$-sets are defined by (1) $\mathfrak{D}$ is $\wedge_{\delta\gamma}$-set if $B = \wedge_{\delta\gamma} (\mathfrak{D})$ and $\wedge_{\delta\gamma} (\square) = \{\mathfrak{D} : \mathfrak{D} \subseteq \square$ and $B = \wedge_{\delta\gamma} (\mathfrak{D})\}$.

(2) $\mathfrak{D}$ is $\vee_{\delta\gamma}$-set if $\mathfrak{D} = \vee_{\delta\gamma} (\mathfrak{D})$ and $\vee_{\delta\gamma} (\square) = \{\mathfrak{D} : \mathfrak{D} \subseteq \square$ and $\mathfrak{D} = \vee_{\delta\gamma} (\mathfrak{D})\}$.

PROPOSITION 3.12

In a $\boldsymbol{Top}_{(\square,\tau)}$, the followings are satisfied: (1) $\delta\gamma O(\square) \subseteq \wedge_{\delta\gamma} (\square)$; (2) $\delta\gamma O(\square) \subseteq \vee_{\delta\gamma} (\square)$.

Proof. Directly from Definitions 3.1 and 3.11.

4. THE ASSOCIATED $\mathbf{Top}_{(\square,\tau^{\wedge\delta\gamma})}$

DEFINITION 4.1

A $\boldsymbol{Top}_{(\square,\tau)}$ is (δ,γ)-T_1 if $\forall\ \text{I} \neq \text{ŧ}$ in $\square$, $\exists\ \text{I} \in \mathcal{G} \in \delta\gamma O(\square,\tau)$, but $y \notin \mathcal{G}$ and $\text{ŧ} \in \mathfrak{H} \in \delta\gamma O(\square,\tau)$, but $\text{I} \notin \mathfrak{H}$, or equivalently, $\boldsymbol{Top}_{(\square,\tau)}$ is (δ,γ)- T_1 iff $\{\text{I}\} \in \delta\gamma\mathcal{C}(\square,\tau)$, $\forall\ \text{I} \in \square$.

THEOREM 4.2

A $\boldsymbol{Top}_{(\square,\tau)}$ is (δ,γ)-T_1 iff $\{\text{I}\}$ is $\wedge_{\delta\gamma}$-set, $\forall\ \text{I} \in \square$.

Proof. Necessity. Consider $\text{ŧ} \in \wedge_{\delta\gamma} (\{\text{I}\})$ for some ŧ different from I. Then, $\text{ŧ} \in \cap \{\mathcal{G}_{\text{I}} : \text{I} \in \mathcal{G}_x$ and $\mathcal{G}_{\text{I}} \in \delta\gamma O(\square,\tau)\}$ and so $\text{ŧ} \in \mathcal{G}_{\text{I}}$, $\forall\ \text{I} \in \mathcal{G}_{\text{I}} \in \delta\gamma O(\square,\tau)$, which contradict with the assumption. Sufficiency. Consider $\{\text{I}\}$ is a $\wedge_{\delta\gamma}$-set, $\forall\ \text{I} \in \square$. Let $\text{I} \neq \text{ŧ}$. Then, $\text{ŧ} \notin \wedge_{\delta\gamma} (\{\text{I}\})$ and $\exists$ a $\mathcal{G}_{\text{I}} \in \delta\gamma O(\square,\tau)$ s.t. $\text{I} \in \mathcal{G}_{\text{I}}$ and $\text{ŧ} \notin \mathcal{G}_{\text{I}}$. Similarly, $\text{I} \notin \wedge_{\delta\gamma} (\{\text{ŧ}\})$ and $\exists$ a $\mathcal{G}_{\text{ŧ}} \in \delta\gamma O(\square,\tau)$ s.t. $\text{ŧ} \in \mathcal{G}_{\text{ŧ}}$ and $\text{I} \notin \mathcal{G}_{\text{ŧ}}$.

$\mathfrak{A}$ in $\mathbf{Top}_{(\square,\tau)}$ is generalized closed (briefly, g-closed) [24] if $\mathfrak{C}(\mathfrak{A}) \subseteq \mathcal{U}$ whenever $\mathfrak{A} \subseteq \mathcal{U}$ and $\mathcal{U} \in \tau$. A $\mathbf{Top}_{(\square,\tau)}$ is a $T_{\frac{1}{2}}$ if $\forall$ g-closed of $\square$ is closed. Dunham [9] pointed out that a $\mathbf{Top}_{(\square,\tau)}$ is $T_{\frac{1}{2}}$ iff $\{x\} \in O(\square,\tau)$ or $\{x\} \in \mathcal{C}(\square,\tau)$.

THEOREM 4.3

The followings hold: (1) $\boldsymbol{Top}_{(\square,\tau)}$ is (δ,γ)-T_1 iff $(\square,\tau^{\wedge\delta\gamma})$ is discrete.

(2) $\mathbf{Top}_{(\square,\tau^{\wedge\delta\gamma})}$ is a $T_{\frac{1}{2}}$.

Proof. (1) Necessity. Consider $(\square,\tau)$ is (δ,γ)-T_1. By Theorem 4.2, $\{\text{I}\}$ is a $\wedge_{\delta\gamma}$-set and $\{\text{I}\} \in \tau^{\wedge\delta\gamma}$, for $\text{I} \in \square$. By Lemma 3.4, $\mathfrak{A} \in \tau^{\wedge\delta\gamma}$, for $\mathfrak{A} \subseteq \square$. Hence, $(\square,\tau^{\wedge\delta\gamma})$ is discrete. Sufficiency. $\{\text{I}\} \in \tau^{\wedge\delta\gamma}$ and so $\{\text{I}\}$ is a $\wedge_{\delta\gamma}$-set, for $\text{I} \in \square$. By Theorem 4.2, $(\square,\tau)$ is (δ,γ)-T_1.

(2) Let $\text{I} \in \square$. Then, $\{\text{I}\}$ is $\delta\gamma$-clopen in $(\square,\tau)$. If $\{\text{I}\} \in \delta\gamma(\square,\tau)$, then by Lemma 3.4, $\{\text{I}\}$ is a $\wedge_{\delta\gamma}$-set and $\{\text{I}\} \in \tau^{\wedge\delta\gamma}$. If $\{\text{I}\} \in \delta\gamma\mathcal{C}(\square,\tau)$, then $\square\backslash\{\text{I}\} \in \delta\gamma O(\square,\tau)$, and $\square\backslash\{\text{I}\} \in \tau^{\wedge\delta\gamma}$. Therefore, $\{\text{I}\}$ is either open or closed in $(\square,\tau^{\wedge\delta\gamma})$.

DEFINITION 4.4

A function $\boldsymbol{f} : (\square,\tau) \to (\lozenge,\sigma)$ is

(1) Strongly γ-cont. [27] if $\forall\ \text{I} \in \square$ and γ-open set $\mathbf{f}(\text{I}) \in \mathcal{G} \in \gamma O(\lozenge,\tau)$, $\exists\ \text{I} \in \mathfrak{H} \in \gamma O(\square,\tau)$ s.t $\mathbf{f}(\mathcal{G}) \subseteq \mathfrak{H}$.

(2) (δ,γ)-irresolute ((δ,γ)-**Irr**, for short) if $\forall\ \text{I} \in \square$ and $\mathbf{f}(\text{I}) \in \mathcal{G} \in \delta\gamma O(\lozenge,\tau)$, $\exists\ \text{I} \in \mathfrak{H} \in \delta\gamma O(\square,\tau)$ s.t $\mathbf{f}(\mathcal{G}) \subseteq \mathfrak{H}$.

THEOREM 4.5

(1) If $f : (\square,\tau) \to (\lozenge,\sigma)$ is (δ,γ)-$\boldsymbol{Irr}$, then $\boldsymbol{f} : (\square,\tau^{\wedge\delta\gamma}) \to (\lozenge,\sigma^{\wedge\delta\gamma})$ is cont..

(2) The identity function $\mathbf{Id}_{\square} : (\square,\tau^{\wedge\delta\gamma}) \to (\square,\tau)$ is strongly γ-cont..

Proof. (i) Let $\mathcal{G}$ be any $\wedge_{\delta\gamma}$-set in $\lozenge$. Then, $\mathcal{G} = \wedge_{\delta\gamma} (\mathcal{G}) = \cap\ \{\mathfrak{W} : \mathcal{G} \subseteq \mathfrak{W}$ and $\mathfrak{W} \in \delta\gamma O(\lozenge,\sigma)\}$. Since $\mathbf{f}$ is (δ,γ)-**Irr**, then $\mathbf{f}^{-1}(\mathfrak{W}) \in \delta\gamma O(\square,\tau)$, $\forall\ \mathfrak{W}$. Hence, $\mathbf{f}^{-1}(\mathcal{G}) \supseteq \cap \{\mathbf{f}^{-1}(\mathfrak{W}) : \mathbf{f}^{-1}(\mathcal{G}) \subseteq \mathbf{f}^{-1}(\mathfrak{W})$ and $\mathfrak{W} \in \delta\gamma O(\lozenge,\sigma) \supseteq \cap \{\mathfrak{H} : \mathbf{f}^{-1}(\mathcal{G}) \subseteq \mathfrak{H}$ and $\mathfrak{H} \in \delta\gamma O(\square,\tau) = \wedge_{\delta\gamma} (\mathbf{f}^{-1}(\mathcal{G}))$. On the other hand, by assumption, $\mathbf{f}^{-1}(\mathcal{G}) \subseteq \wedge_{\delta\gamma} (\mathbf{f}^{-1}(\mathcal{G}))$. Hence, we get $\mathbf{f}^{-1}(\mathcal{G}) = \wedge_{\delta\gamma} (\mathbf{f}^{-1}(\mathcal{G}))$. Therefore, $\mathbf{f}^{-1}(\mathcal{G}) \in \tau^{\wedge\delta\gamma}$ and $\mathbf{f} : (\square,\tau^{\wedge\delta\gamma}) \to (\lozenge,\sigma^{\wedge\delta\gamma})$ is cont..

(ii) Let $\mathcal{G}$ be γ-open in $(\square,\tau)$. Since $\mathcal{G}$ is $\delta\gamma$-open, by Theorem 3.2, $(\mathbf{Id}_{\square})^{-1}(\mathcal{G}) = \mathcal{G} \in \tau^{\wedge\delta\gamma}$ and hence $\mathbf{Id}_{\square}$ is strongly γ-cont.

5. SOBER $(\boldsymbol{\delta},\boldsymbol{\gamma})$-$\boldsymbol{\mathcal{R}_0}$-SPACES

DEFINITION 5.1 [7]

In $\boldsymbol{Top}_{(\square,\tau)}$, the γ-kernel os a set $\mathfrak{A}$ ($\gamma\boldsymbol{Ker}(\mathfrak{A})$, for short) is $\gamma\boldsymbol{Ker}(\mathfrak{A}) = \mathfrak{A}^{\wedge\gamma} = \cap \{\mathcal{G} \in \gamma O(\square,\tau) : \mathcal{G} \supseteq \mathfrak{A}\}$.

DEFINITION 5.2

Let $\mathfrak{A}$ be a subset of a $\boldsymbol{Top}_{(\square,\tau)}$. The $\mathfrak{A}^{\wedge\delta\gamma} = \cap \{\mathcal{G} \in \delta\gamma OX,\tau) : \mathcal{G} \supseteq \mathfrak{A}\}$.

LEMMA 5.3

In a $\boldsymbol{Top}_{(\square,\tau)}$ $\mathfrak{A}^{\wedge\delta\gamma} = \{\text{I} \in \square : \delta\gamma\mathfrak{C}(\{\text{I}\}) \cap \mathfrak{A} \neq \phi\}$, for $\text{I} \in \square$.

Proof. Let $\text{I} \in \mathfrak{A}^{\wedge\delta\gamma}$ and $\delta\gamma\mathfrak{C}(\{\text{I}\}) \cap \mathfrak{A} = \phi$. Then, $\text{I} \notin \square\backslash\delta\gamma\mathfrak{C}(\{\text{I}\}) \in \delta\gamma O(\square,\tau)$ which has $\mathfrak{A}$, for $\text{I} \in \mathfrak{A}^{\wedge\delta\gamma}$. Consequently, $\delta\gamma\mathfrak{C}(\{\text{I}\}) \cap \mathfrak{A} \neq \phi$. Next, let $\delta\gamma\mathfrak{C}(\{\text{I}\}) \cap \mathfrak{A} \neq \phi$ and $\text{I} \notin \mathfrak{A}^{\wedge\delta\gamma}$. Then, $\exists\ \mathcal{D} \in \delta\gamma O(\square,\tau)$ has $\mathfrak{A}$ and $\text{I} \notin \mathcal{D}$. Let $\text{ŧ} \in \delta\gamma\mathfrak{C}(\{\text{I}\}) \cap \mathfrak{A}$. Therefore, $\mathcal{D}$ is a (δ,γ)-nbd of ŧ which has not I. By a contradiction, $\text{I} \in \mathfrak{A}^{\wedge\delta\gamma}$.

DEFINITION 5.4

A $\boldsymbol{Top}_{(\square,\tau)}$ is sober (δ,γ)-$\mathcal{R}_0$ if $\cap_{\text{I}\in\square}\ \delta\gamma\mathfrak{C}(\{\text{I}\}) = \phi$.

THEOREM 5.5

A $\boldsymbol{Top}_{(\square,\tau)}$ is sober (δ,γ)-$\mathcal{R}_0$ iff $\{\text{I}\}^{\wedge\delta\gamma} \neq \square\ \forall\ \text{I} \in \square$.

Proof. Let $\mathbf{Top}_{(\square,\tau)}$ be sober (δ,γ)-$\mathcal{R}_0$. Assume that $\exists\ \mathfrak{k}$ in $\square$ s.t. $\{\mathfrak{k}\}^{\wedge\delta\gamma}=\square$. Then, for $\mathfrak{l}\in\square$, $\mathfrak{l}\in\mathcal{V}$, $\forall\ \mathcal{V}\in\delta\gamma O(\square,\tau)$ has $\mathfrak{k}$ and so $\mathfrak{k}\in\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$ for any $\mathfrak{l}\in\square$. This indicates that $\mathfrak{k}\in\bigcap_{\mathfrak{l}\in\square}\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$ which give a contradiction. Conversely, assume that $\{\mathfrak{l}\}^{\wedge\delta\gamma}\neq\square$, $\forall\ \mathfrak{l}\in\square$. If $\exists\ \mathfrak{k}\in\square$ s.t. $\mathfrak{k}\in\bigcap_{\mathfrak{l}\in\square}\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$, then $\forall\ \delta\gamma$-open set has $\mathfrak{k}$ must contain every point of $\square$. So, $\square$ is the unique $\delta\gamma$-open set has $\mathfrak{k}$. Hence, $\{\mathfrak{k}\}^{\wedge\delta\gamma}=\square$ which is a contradiction.

DEFINITION 5.6

$\boldsymbol{f}:(\square,\tau)\to(\diamond,\sigma)$ *is* $\delta\gamma$*-closed if* $\boldsymbol{f}(\mathfrak{F})\in\delta\gamma\mathcal{C}(\diamond,\tau)$, $\forall$ $\mathfrak{F}\in\delta\gamma\mathcal{C}(\square,\tau)$.

THEOREM 5.7

If $\boldsymbol{f}$ *is an injective* $\delta\gamma$*-closed function and* $\square$ *is sober* (δ,γ)*-*$\mathcal{R}_0$*, then* $\diamond$ *is sober* (δ,γ)*-*$\mathcal{R}_0$.

Proof. Since $\square$ is sober (δ,γ)-$\mathcal{R}_0$, then $\bigcap_{\mathfrak{l}\in\square}\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})=\phi$. Since $\mathbf{f}$ is a $\delta\gamma$-closed injection, we get $\phi=\mathbf{f}(\bigcap_{\mathfrak{l}\in\square}\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}))=\bigcap_{\mathfrak{l}\in\square}\mathbf{f}(\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}))\supseteq\bigcap_{\mathfrak{l}\in\square}\delta\gamma\mathfrak{C}(\{\mathbf{f}(\mathfrak{l})\})\supseteq\bigcap_{\mathfrak{k}\in\diamond}\delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$.

THEOREM 5.8

Let $\mathbf{Top}_{(\square,\tau)}$ *be sober* (δ,γ)*-*$\mathcal{R}_0$*. Then, for any* $\mathbf{Top}_{(\diamond,\sigma)}$*,* $\square\times\diamond$ *is sober* (δ,γ)*-*$\mathcal{R}_0$.

Proof. It is enough to prove that $\bigcap_{(\mathfrak{l},\mathfrak{k})\in\square\times\diamond}\delta\gamma\mathfrak{C}(\{(\mathfrak{l},\mathfrak{k})\})=\phi$. We get $\bigcap_{(\mathfrak{l},\mathfrak{k})\in\square\times\diamond}\delta\gamma\mathfrak{C}(\{(\mathfrak{l},\mathfrak{k})\})\subseteq\bigcap_{(\mathfrak{l},\mathfrak{k})\in\square\times\diamond}(\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})\times\delta\gamma\mathfrak{C}(\{\mathfrak{k}\}))=\bigcap_{\mathfrak{l}\in\square}\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})\times\bigcap_{\mathfrak{k}\in\diamond}\delta\gamma\mathfrak{C}(\{\mathfrak{k}\})\subseteq\phi\times\diamond=\phi$.

6. $(\boldsymbol{\delta},\boldsymbol{\gamma})$- $\boldsymbol{\mathcal{R}_0}$-SPACES

DEFINITION 6.1

A $\mathbf{Top}_{(\square,\tau)}$ *is a* (δ,γ)*-*$\mathcal{R}_0$ *(resp.* γ*-*$\mathcal{R}_0$*) if* $\forall\ \mathcal{G}\in\delta\gamma O(\square,\tau)$ *(resp.* $\gamma O(\square,\tau)$*),* $\mathcal{G}\supseteq\delta\gamma\mathfrak{C}(\{x\})$ *(resp.* $\gamma\mathfrak{C}(\{x\})$*),* $\forall\ \mathfrak{l}\in\mathcal{G}$.

DEFINITION 6.2

A $\mathbf{Top}_{(\square,\tau)}$ *is (i)* (δ,γ)*-*T_0 *if* $\mathfrak{l}\neq\mathfrak{k}$ *in* $\square$*,* $\exists\ \mathcal{U},\mathcal{U}\in\delta\gamma O(\square,\tau)$ *s.t. either* $\mathfrak{l}\in\mathcal{U}$*,* $\mathfrak{k}\notin\mathcal{U}$ *or* $\mathfrak{k}\in\mathcal{V}$*,* $\mathfrak{l}\notin\mathcal{V}$.

(ii)(δ,γ)-T_1 $\mathfrak{l}\neq\mathfrak{k}$ in $\square$, $\exists\ \mathcal{U},\mathcal{U}\in\delta\gamma O(\square,\tau)$ s.t. $\mathfrak{l}\in\mathcal{U}$, $\mathfrak{k}\notin\mathcal{U}$ and $\mathfrak{k}\in\mathcal{V}$, $\mathfrak{l}\notin\mathcal{V}$.

(iii) (δ,γ)-T_2 $\mathfrak{l}\neq\mathfrak{k}$ in $\square$, $\exists\ \mathcal{U},\mathcal{U}\in\delta\gamma O(\square,\tau)$ s.t. $\mathfrak{l}\in\mathcal{U}$ and $\mathfrak{k}\in\mathcal{V}$ and $\mathcal{U}\cap\mathcal{V}=\phi$

REMARK 6.3

Every (δ,γ)*-*T_2 *space is* (δ,γ)*-*T_1*. Also, every* (δ,γ)*-*T_1 *space is* (δ,γ)*-*T_0.

LEMMA 6.4 [30]

For $\mathfrak{A}$ *of a* $\mathbf{Top}_{(\square,\tau)}$*. The followings hold: (i) If* $\mathfrak{A}\in O(\square,\tau)$*, then* $\delta\mathfrak{C}(\mathfrak{A})=\mathfrak{C}(\mathfrak{A})$*. (ii) If* $\mathfrak{A}\in\mathcal{C}(\square,\tau)$*, then* $\delta\mathfrak{I}(\mathfrak{A})=\mathfrak{I}(\mathfrak{A})$.

LEMMA 6.5

The followings hold, for $\mathfrak{A}$ *of a* $\mathbf{Top}_{(\square,\tau)}$*. (i) If* $\mathfrak{A}\in\gamma O(\square,\tau)$*, then* $\mathfrak{A}\in\delta\gamma O(\square,\tau)$.

(ii) $\mathfrak{A}\in\delta\gamma O(\square,\tau)$ iff $\mathfrak{A}\in\gamma O(\square,\tau_s)$, where τ_s is a semi-regularization on $\square$.

(iii) $\mathfrak{A}\in\delta\gamma\mathcal{C}(\square,\tau)$ iff $\mathfrak{A}\in\gamma\mathcal{C}(\square,\tau_s)$

Proof. (i) This is obvious, since $\mathfrak{C}(\mathfrak{A})\subseteq\delta\mathfrak{C}(\mathfrak{A})$ for $\mathfrak{A}\subseteq\square$.

(ii) Since $\delta\mathfrak{C}(\mathfrak{A})\in\mathcal{C}(\square,\tau)$, then by Lemma 6.4 and that $\mathfrak{I}(\delta\mathfrak{C}(\mathfrak{A}))=\delta\mathfrak{I}(\delta\mathfrak{C}(\mathfrak{A}))=\mathfrak{I}_{\tau_s}(\mathfrak{C}_{\tau_s}(\mathfrak{A}))$, where $\mathfrak{I}_{\tau_s}(\mathfrak{A})$(resp. $\mathfrak{C}_{\tau_s}(\mathfrak{A})$) is the interior (resp. closure) of $\mathfrak{A}$ w.r.to τ_s.

(iii) Obvious by (ii).

LEMMA 6.6 [26]

In a $\mathbf{Top}_{(\square,\tau)}$*,* $\{\mathfrak{l}\}$ *is either preopen or preclosed, for* $\mathfrak{l}\in\square$.

PROPOSITION 6.7

In a $\mathbf{Top}_{(\square,\tau)}$*,* $\{\mathfrak{l}\}$ *is either* γ*-open or* γ*-closed, for* $\mathfrak{l}\in\square$.

Proof. By Lemma 6.6, $\{\mathfrak{l}\}$ is preopen or preclosed. Since each preopen (resp. preclosed) is γ-open [12] (resp. γ-closed [12]), then $\{\mathfrak{l}\}$ is either γ-open or γ-closed, for $\mathfrak{l}\in\square$.

THEOREM 6.8

A $\mathbf{Top}_{(\square,\tau)}$ *is* (δ,γ)*-*$\mathcal{R}_0$ *iff* $(\square,\tau_s)$ *is* γ*-*$\mathcal{R}_0$.

Proof. By Lemma 6.5, we get: (i) $\mathfrak{A}\in\delta\gamma O(X,\tau)$ iff $A\delta\gamma O(X,\tau_s)$;

(ii) $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})=\bigcup\ \{\mathfrak{F}:\mathfrak{l}\in\mathfrak{F}\in\delta\gamma\mathcal{C}(\square,\tau)\}=\{\mathfrak{F}:\mathfrak{l}\in\mathfrak{F}\in\gamma\mathcal{C}(\square,\tau_s)\}=\tau_s-\gamma\mathfrak{C}(\{x\})$, for $\mathfrak{l}\in\square$. The proof follows immediately from (i) and (ii).

DEFINITION 6.9

A $\mathbf{Top}_{(\square,\tau)}$ *is* (δ,γ)*-symmetric if* $\mathfrak{l}\in\delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$ *implies* $\mathfrak{k}\in\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$*, for* $\mathfrak{l},\mathfrak{k}\in\square$.

THEOREM 6.10

The followings are equivalent: (i) $\mathbf{Top}_{(\square,\tau)}$ *is* (δ,γ)*-symmetric;*

(ii) $\{\mathfrak{l}\}\in\delta\gamma\mathcal{C}(\square,\tau)$;

(iii) $\mathbf{Top}_{(\square,\tau)}$ is (δ,γ)-T_1.

Proof. (i)⇒ (ii): Let $\mathfrak{l},\mathfrak{k}\in\square$ s.t. $\mathfrak{l}\neq\mathfrak{k}$. By Proposition 6.7, $\{\mathfrak{k}\}$ is γ-open or γ-closed in $(\square,\tau)$. So, (1) when $\{\mathfrak{k}\}$ is γ-open, put $\mathcal{V}_{\mathfrak{k}}=\{\mathfrak{k}\}$, then $\mathcal{V}_{\mathfrak{k}}\in\delta\gamma O(\square,\tau)$.

(2) when $\{\mathfrak{k}\}$ is γ-closed, $\mathfrak{l}\notin\{\mathfrak{k}\}=\gamma\mathfrak{C}(\{\mathfrak{k}\})$ and $\mathfrak{l}\notin\delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$. By (1), $\mathfrak{k}\notin\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$. Now, put $\mathcal{V}_{\mathfrak{k}}=\square\backslash\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$. Then, $\mathfrak{l}\notin\mathcal{V}_{\mathfrak{k}}$, $\mathfrak{k}\in\mathcal{V}_{\mathfrak{k}}$ and $\mathcal{V}_{\mathfrak{k}}\in\delta\gamma O(\square,\tau)$. Therefore, we get $\forall\ \mathfrak{k}\in\square\backslash\{\mathfrak{l}\}$, $\exists\ \mathcal{V}_{\mathfrak{k}}\in\delta\gamma O(\square,\tau)$ s.t. $\mathfrak{l}\notin\mathcal{V}_{\mathfrak{k}}$, $\mathfrak{k}\in\mathcal{V}_{\mathfrak{k}}\in\delta\gamma O(\square,\tau)$. Hence, $\square\backslash\{\mathfrak{l}\}=\bigcup_{\mathfrak{k}\in\square\backslash\{\mathfrak{l}\}}\mathcal{V}_{\mathfrak{k}}\in\delta\gamma O(\square,\tau)$. Therefore, $\{\mathfrak{l}\}\in\delta\gamma\mathcal{C}(\square,\tau)$.

[(ii)⇒ (iii):] Consider $\{p\}\in\delta\gamma\mathcal{C}(\square,\tau)$, $\forall\ p\in\square$. Let $\mathfrak{l},\mathfrak{k}\in\square$ with $\mathfrak{l}\neq\mathfrak{k}$. Now, $\mathfrak{l}\neq\mathfrak{k}$ implies $\mathfrak{k}\in\square\backslash\{\mathfrak{l}\}$. Hence, $\square\backslash\{\mathfrak{l}\}\in\delta\gamma O(\square,\tau)$ has $\mathfrak{k}$ but has not $\mathfrak{l}$. Similarly, $\square\backslash\{\mathfrak{k}\}\in\delta\gamma O(\square,\tau)$ has $\mathfrak{l}$ but has not $\mathfrak{k}$.

[(iii)⇒ (i):] Let $y\notin\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$. Then, since $\mathfrak{l}\neq\mathfrak{k}$, by (iii),

$\exists\ \mathcal{U} \in \delta\gamma O(\square,\tau)$ has x s.t. $\mathfrak{k} \notin \mathcal{U}$ and so $\mathfrak{l} \notin \delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$. So, $\mathfrak{l} \in \delta\gamma\mathfrak{C}(\{y\})$ indicates $\mathfrak{k} \in \delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$.

THEOREM 6.11

A $\boldsymbol{Top}_{(\square,\tau)}$ *is* (δ,γ)-$\mathcal{R}_0$ *iff* (δ,γ)-T_1.

Proof. Necessity. Let $\mathfrak{l} \neq \mathfrak{k}$ in $\square$. $\{\mathfrak{l}\}$ is γ-open or γ-closed, by Proposition 6.7. There are two cases:

(1) when $\{\mathfrak{l}\} \in \gamma(\square,\tau)$, let $\mathcal{V} = \{\mathfrak{l}\}$. Then, $\mathfrak{l} \in \mathcal{V}$, $\mathfrak{k} \notin \mathcal{V}$ and $\mathcal{V} \in \delta\gamma O(\square,\tau)$. Moreover, since $(\square,\tau)$ is (δ,γ)-$\mathcal{R}_0$, we get $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\} \subseteq \mathcal{V}$. Hence, $\mathfrak{l} \notin \square\backslash\mathcal{V}$, $\mathfrak{k} \in \square\backslash\mathcal{V}$ and $\square\backslash\mathcal{V} \in \delta\gamma O(\square,\tau)$;

(2) when $\{\mathfrak{l}\} \in \gamma\mathcal{C}(\square,\tau)$, $\mathfrak{k} \in \square\backslash\{\mathfrak{l}\}$ and $\{\mathfrak{l}\} \in \delta\gamma O(\square,\tau)$. Since $(\square,\tau)$ is (δ,γ)-$\mathcal{R}_0$, then $\delta\gamma\mathfrak{C}(\{\mathfrak{k}\}) \subseteq \square\backslash\{\mathfrak{l}\}$. Now, let $\mathcal{V} = \delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$. Then, $\mathfrak{l} \in \mathcal{V}$, $\mathfrak{k} \notin \mathcal{V}$ and $\mathcal{V} \in \delta\gamma O(\square,\tau)$. Sufficiency. Let $\mathcal{V} \in \delta\gamma O(\square,\tau)$ and $\mathfrak{l} \in \mathcal{V}$. $\forall\ \mathfrak{k} \in \square\backslash\mathcal{V}$, $\exists\ \mathcal{V}_{\mathfrak{k}} \in \delta\gamma O(\square,\tau)$ s.t. $\mathfrak{l} \notin \mathcal{V}_{\mathfrak{k}}$ and $\mathfrak{k} \notin \mathcal{V}_{\mathfrak{k}}$. Then, we get $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \cap \mathcal{V}_{\mathfrak{k}} = \phi$, $\forall\ \mathfrak{k} \in \square\backslash\mathcal{V}$ and so $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \cap (\cup_{\mathfrak{k}\in\square\backslash\mathcal{V}} \mathcal{V}_{\mathfrak{k}}) = \phi$. Since $\mathfrak{k} \in \mathcal{V}_{\mathfrak{k}}$, $\square\backslash\mathcal{V} \subseteq \cup_{\mathfrak{k}\in\square\backslash\mathcal{V}} \mathcal{V}_{\mathfrak{k}}$ and $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \cap (\square\backslash\mathcal{V}) = \phi$. Therefore, $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \subseteq V$.

COROLLARY 6.12

For a $\boldsymbol{Top}_{(\square,\tau)}$, *we get the implication:*

(δ,γ)-$\mathcal{R}_0 \Leftrightarrow (\delta,\gamma)$-$T_1 \Leftrightarrow (\delta,\gamma)$-symmetric

Proof. This is an immediate consequence of Theorems 6.10 and 6.11.

REMARK 6.13

Observe by using Theorem 6.10 and Corollary 6.12 that $(\square,\tau)$ *is* (δ,γ)-$\mathcal{R}_0$ *iff* $\forall\ \{\mathfrak{l}\} \in \delta\gamma\mathcal{C}(\square,\tau)$, *for* $\mathfrak{l} \in \square$.

COROLLARY 6.14

Let $\square \neq \phi$ *and* $|\square| \geq 2$, *where* $|\square|$ *is the cardinality of* $\square$. *Then, every* (δ,γ)-$\mathcal{R}_0$*-space is sober* (δ,γ)-$\mathcal{R}_0$.

Proof. Let $\mathfrak{l} \neq \mathfrak{k}$ of $\square$. Since $(\square,\tau)$ is (δ,γ)-$\mathcal{R}_0$, by Theorem 6.11, (δ,γ)-T_1. Hence, by Theorem 6.10, $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) = \{\mathfrak{l}\}$ and $\delta\gamma\mathfrak{C}(\{\mathfrak{k}\}) = \{\mathfrak{k}\}$ and so $\cap_{p\in\square} \delta\gamma\mathfrak{C}(\{p\}) \subseteq \delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) = \{\mathfrak{l}\} \cap \delta\gamma\mathfrak{C}(\{\mathfrak{k}\}) = \{\mathfrak{l}\} \cap \{\mathfrak{k}\} = \phi$.

QUESTION 6.15

Is there any example showing that the converse of Corollary 6.14 is not true?

From the above, the following properties hold:

$\delta\gamma O(\square,\tau) = \gamma\mathcal{S}O(\square,\tau)$, $\delta\gamma\mathcal{C}(\square,\tau) = \gamma\mathcal{S}\mathcal{C}(\square,\tau)$, $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) = s\gamma\mathfrak{C}(\{\mathfrak{l}\})$ and $\{\mathfrak{l}\}^{\wedge\delta\gamma} = \{\mathfrak{l}\}^{\wedge s\gamma}$ $\forall$ $\mathfrak{l}$ of a $\mathbf{Top}_{(\square,\tau)}$. Therefore, we obtain the following important characterizations of (δ,γ)-$\mathcal{R}_0$-spaces which are modifications of following theorems.

THEOREM 6.16

The followings are equivalent: (i) A $\boldsymbol{Top}_{(\square,\tau)}$ *is a* (δ,γ)-$\mathcal{R}_0$;

(ii) For $\mathfrak{A} \neq \phi$ and $\mathcal{G} \in \delta\gamma O(\square,\tau)$ s.t. $\mathfrak{A} \cap \mathcal{G} \neq \phi$, $\exists\ \mathfrak{F} \in \delta\gamma\mathcal{C}(\square,\tau)$ s.t. $\mathfrak{A} \cap \mathfrak{F} \neq \phi$ and $\mathfrak{F} \subseteq \mathcal{G}$;

(iii) Any $\mathcal{G} \in \delta\gamma O(\square,\tau)$, $\mathcal{G} = \cup\{\mathfrak{F} \in \delta\gamma\mathcal{C}(\square,\tau): \mathfrak{F} \subseteq \mathcal{G}\}$;

(iv) Any $\mathfrak{F} \in \delta\gamma\mathcal{C}(\square,\tau)$, $\mathfrak{F} = \cap\{\mathcal{G} \in \delta\gamma O(\square,\tau): \mathfrak{F} \subseteq \mathcal{G}\}$;

(v) $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \subseteq \{\mathfrak{l}\}^{\wedge\delta\gamma}$, for $\mathfrak{l} \in \square$, .

THEOREM 6.17

The followings are equivalent:

(i) $\mathbf{Top}_{(\square,\tau)}$ is a (δ,γ)-$\mathcal{R}_0$; (ii) If $\mathfrak{F} \in \delta\gamma\mathcal{C}(\square,\tau)$, then $\mathfrak{F} = \mathfrak{F}^{\wedge\delta\gamma}$;

(iii) If $\mathfrak{F} \in \delta\gamma\mathcal{C}(\square,\tau)$ and $\mathfrak{l} \in \mathfrak{F}$, then $\{\mathfrak{l}\}^{\wedge\delta\gamma} \subseteq \mathfrak{F}$;

(iv) If $\mathfrak{l} \in \square$, then $\{\mathfrak{l}\}^{\wedge\delta\gamma} \subseteq \delta\gamma\mathfrak{C}(\{\mathfrak{l}\})$.

7. $(\boldsymbol{\delta},\boldsymbol{\gamma})$-$\boldsymbol{\mathcal{R}_1}$-SPACES

DEFINITION 7.1

A $\boldsymbol{Top}_{(\square,\tau)}$ *is* (δ,γ)-$\mathcal{R}_1$ *if for any* $\mathfrak{l},\mathfrak{k} \in \square$ *with* $\delta\gamma\mathfrak{C}(\mathfrak{l}\}) \neq \delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$, $\exists$ $\mathcal{U},\mathcal{V} \in \delta\gamma O(\square,\tau)$, $\mathcal{U} \cap \mathcal{V} = \phi$ *s.t.* $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \subseteq \mathcal{U}$ *and* $\delta\gamma\mathfrak{C}(\{\mathfrak{k}\}) \subseteq \mathcal{V}$.

DEFINITION 7.2

A $\boldsymbol{Top}_{(\square,\tau)}$ *is* γ-$\mathcal{R}_1$ *if for any* $\mathfrak{l},\mathfrak{k} \in \square$ *with* $\gamma\mathfrak{C}(\mathfrak{l}\}) \neq \gamma\mathfrak{C}(\{\mathfrak{k}\})$, $\exists\ \mathcal{U},\mathcal{V} \in \gamma O(\square,\tau)$, $\mathcal{U} \cap \mathcal{V} = \phi$ *s.t.* $\gamma\mathfrak{C}(\{\mathfrak{l}\}) \subseteq \mathcal{U}$ *and* $\gamma\mathfrak{C}(\{\mathfrak{k}\}) \subseteq \mathcal{V}$.

THEOREM 7.3

A $\boldsymbol{Top}_{(\square,\tau)}$ *is* (δ,γ)-$\mathcal{R}_1$ *iff* $(\square,\tau_s)$ *is* γ-$\mathcal{R}_1$.

Proof. Follows from that $\delta\gamma O(\square,\tau) = \gamma O(\square,\tau_s)$ and $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) = \gamma\mathfrak{C}_{\tau_s}(\{\mathfrak{l}\})$, $\forall\ \mathfrak{l} \in \square$.

THEOREM 7.4

A $\boldsymbol{Top}_{(\square,\tau)}$ *is* (δ,γ)-$\mathcal{R}_1$ *iff it is* (δ,γ)-T_2.

Proof. Necessity. Let $\mathfrak{l} \neq \mathfrak{k}$ and y in $\square$. By Proposition 6.7, $\forall\ \mathfrak{l} \in \square$, $\{\mathfrak{l}\}$ is γ-open or γ-closed. Then, there are two cases

(i) when $\{\mathfrak{l}\} \in \gamma O(\square,\tau)$, since $\{\mathfrak{l}\} \cap \{\mathfrak{k}\} = \phi$, $\{\mathfrak{l}\} \cap \delta\gamma\mathfrak{C}(\{\mathfrak{k}\}) \subseteq \{\mathfrak{l}\} \cap \gamma\mathfrak{C}(\{\mathfrak{k}\}) = \phi$ and so $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \neq \delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$.

(ii) when $\{\mathfrak{l}\} \in \gamma\mathcal{C}(\square,\tau)$, $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \cap \{\mathfrak{k}\} \subseteq \gamma\mathfrak{C}(\{\mathfrak{l}\}) \cap \{\mathfrak{k}\} = \{\mathfrak{l}\} \cap \{\mathfrak{k}\} = \phi$ and so $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \neq \delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$. Since $(\square,\tau)$ is (δ,γ)-$\mathcal{R}_1$, $\exists\ \mathcal{U},\mathcal{V} \in \delta\gamma O(\square,\tau)$, $\mathcal{U} \cap \mathcal{V} = \phi$ s.t. $\mathfrak{l} \in \delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \subseteq \mathcal{U}$ and $\mathfrak{k} \in \delta\gamma\mathfrak{C}(\{\mathfrak{k}\}) \subseteq \mathcal{V}$.

Sufficiency. Let $\mathfrak{l},\mathfrak{k} \in \square$ s.t. $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) \neq \delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$. By Remark 6.3, every (δ,γ)-T_2 space is (δ,γ)-T_1. Therefore, by Theorem 6.10, $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) = \{\mathfrak{l}\}$ and $\delta\gamma\mathfrak{C}(\{\mathfrak{k}\}) = \{\mathfrak{k}\}$ and so $\mathfrak{l} \neq \mathfrak{k}$. Since $(\square,\tau)$ is (δ,γ)-T_2, $\exists$ $\mathcal{U},\mathcal{V} \in \delta\gamma O(\square,\tau)$, $\mathcal{U} \cap \mathcal{V} = \phi$ s.t. $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\}) = \{\mathfrak{l}\} \subseteq \mathcal{U}$ and $\delta\gamma\mathfrak{C}(\{\mathfrak{k}\}) = \{\mathfrak{k}\} \subseteq \mathcal{V}$.

COROLLARY 7.5

Every (δ,γ)-$\mathcal{R}_1$ *space* $(\square,\tau)$ *is* (δ,γ)- $\mathcal{R}_0$.

Proof. Since every (δ,γ)-T_2 space is (δ,γ)-T_1, then this is an immediately of Theorems 6.11 and Theorem 7.4.

The converse of Corollary 7.5 is not true, in general.

EXAMPLE 7.6

Let □ be the result of combining any infinite set $\mathbb{N}$ *with two distinct one-point sets* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$. *□ is* (δ,γ)-$\mathcal{R}_0$, *but not* (δ,γ)-$\mathcal{R}_1$ *if any subset of* $\mathbb{N}$ *is open and any set has* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *open iff it contains all but a limited number of points in* $\mathbb{N}$.

EXAMPLE 7.7

Let □ be $\mathcal{R}\times\mathcal{R}$, *where* $\mathcal{R}$ *is the set of real numbers. Let* τ *consists of* ϕ *and all subsets of □ whose complements are subsets of a finite number of lines parallel to the* x*-axis. Then, □ is* (δ,γ)-$\mathcal{R}_0$ *but not* (δ,γ)-$\mathcal{R}_1$.

Theorem 7.8 is a modification for Theorems 4.3 [5] and 6.5 [6].

THEOREM 7.8

The followings are equivalent: (i) $\boldsymbol{Top}_{(□,\tau)}$ *is* (δ,γ)-$\mathcal{R}_1$;

(ii) If $\mathfrak{l},\mathfrak{k}\in□$ s.t. $\delta\gamma\mathfrak{C}(\{\mathfrak{l}\})\neq\delta\gamma\mathfrak{C}(\{\mathfrak{k}\})$, then $\exists\ \mathfrak{F}_1,\mathfrak{F}_2\in\delta\gamma\mathcal{C}(□,\tau)$ s.t $\mathfrak{l}\in\mathfrak{F}_1$, $\mathfrak{k}\notin\mathfrak{F}_1$, $\mathfrak{k}\in\mathfrak{F}_2$, $\mathfrak{l}\notin\mathfrak{F}_2$ and $□=\mathfrak{F}_1\cup\mathfrak{F}_2$.

Proof. Follows directly from Definition 7.1 and Theorem 7.4.

8. APPLICATION (KERNELS OF SELF-SIMILAR FRACTALS)

Each self similar fractals which are shown in Fig. 1 can be represented graphically as shown in Figs 2,3 and 4. It cited by El Atik and Nasef [16]. It is noted that the $\mathbf{Top}_{(□_n,\tau_n)}$ is based on $□_n=\{0,1\}^n\cup(\cup_{k<n}\{0,1\}^k\times\{e\})$, where e denote to a connecting vertex in the graph and will be a closed point in τ_n, such that each $\mathfrak{l}\in\{0,1\}^n$ is a singleton open set. Also, each $v=v_1v_2\cdots v_ke\in\cup_{k<n}\{0,1\}^k\times\{e\}$ is a a singleton closed set with a minimal neighborhood. A continuous function is defined as $\mathbf{f}(u_1u_2\cdots u_n)=u_1u_2\cdots u_{n-1}$, $\mathbf{f}(v_1v_2\cdots v_me)=v_1v_2\cdots v_m$, $m=n-1$ and $\mathbf{f}(v_1v_2\cdots v_me)=v_1v_2\cdots v_me$, $m<n-1$. It is shown that $\mathbf{Top}_{(□_n,\tau_n)}$ is Alexandröff, since each singleton sets has a unique minimal base.

By Definition 3.1 and Theorems 3.2 and 3.10, we get the basis of $\mathbf{Top}_{(□_n,\tau_n)}$ will be $\mathcal{V}(\mathcal{G}_n)=\cup\{v_{i_1i_2i_3\cdots i_n}: i_1i_2i_3\cdots i_n\in\prod\ \{1,2\}^n\}$ such that $|E(\mathcal{G}_n)|=|\mathcal{V}(\mathcal{G}_n)|-1=2^n-1$ in figures 2,3 and 4. In this case, the topological kernels of $\mathbf{Top}_{(□_n,\tau_n)}$ induced by self similar fractals that shown in Figure 1 are the points which connect each self similar pieces. Equivalently, these points are closed in $\mathbf{Top}_{(□_n,\tau_n)}$, for each n.

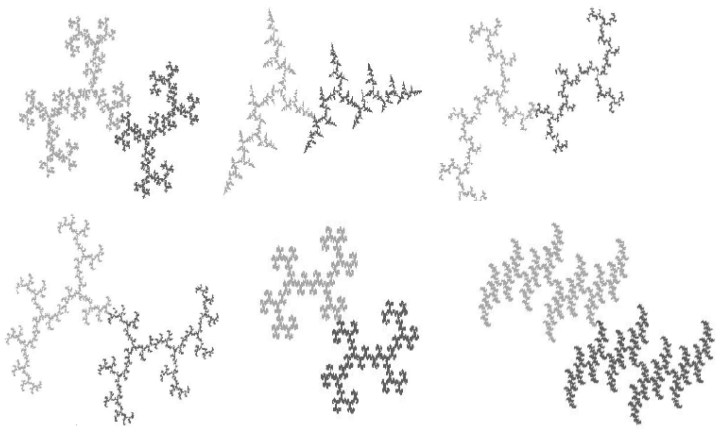

Fig. 1. Some types of Julia sets

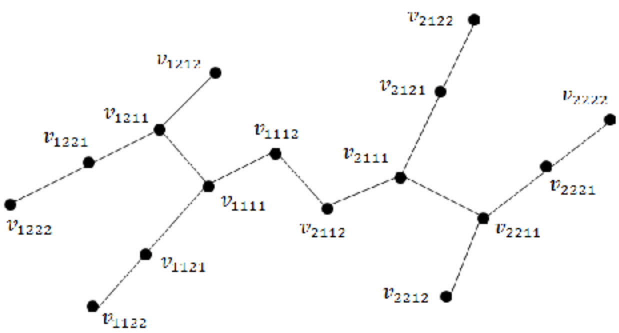

Fig. 2. Graph G_5

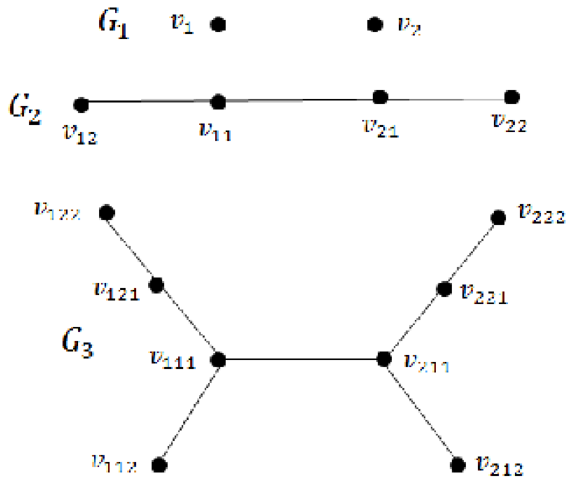

Fig. 3. Graph G_1, G_2 and G_3

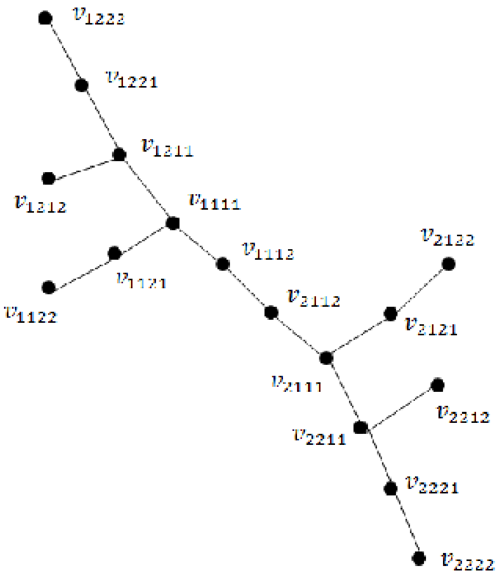

Fig. 4. Graph G_4

9. CONCLUSION

In various mathematical sciences, sets and functions in topology have been extensively developed and exploited. Some novel separation axioms have been discovered through studying generalizations of closures duo to closed sets, ∧ -sets, and V-sets. Moreover, the kernels of self-similar fractals are determined. In computer science, the concept of a set's kernel is useful. The majority of this paper is based on this concept.

DATA AVAILABILITY

No data were used to support this study.

CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

AUTHORS'CONTRIBUTION

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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