Stability Analysis of Plane Vibrations of a Satellite in a Circular Orbit

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Abstract

In this paper, we analyze for stability the problem of planar vibrational motion of a satellite about its centre of mass. The satellite is dynamically symmetric whose center of mass is moving in a circular orbit. The in-plane motion is a simple pendulum-like motion in which the axis of symmetry of the satellite remains in the orbital plane. It is expressed in terms of elliptic functions of time. Using Routh's equations we study the orbital stability of planar vibrations of the satellite, in the sense that the stable in-plane motions remain under perturbation very near to the orbital plane. The linearized equation for the out-of-plane motion takes the form of a Hill's equation. Detailed analysis of stability using Floquet theory is performed analytically and numerically. Zones of stability and instability are illustrated graphically in the plane of the two parameters of the problem: the ratio of moments of inertia and the amplitude of the unperturbed motion.

Keywords: Satellite, Plane vibrations, Stability, Floquet theory, Periodic solutions

Introduction

The field of dynamics of a rigid body is one of the most complicated fields of theoretical mechanics. Even in its simplest case, that of motion of a body about a fixed point in a uniform gravity field, it belongs to the type of non-integrable dynamics (see e.g. [1], [2]). Nevertheless, it is an inevitable component of the study of many important systems in science and technology. An example is the problem of study of the translational-rotational motion of a satellite. If the last problem is considered in its full generality, the equations of translational motion of the satellite and its rotational motion are coupled together. It was shown that for an artificial satellite a possible solution represents motion in a circular orbit (e.g. [3]). In that case the rotational motion of the satellite about its centre of mass relative to the orbital coordinate system can be described as motion about a fixed point of a rigid body acted upon by non-symmetric potential and gyroscopic forces [4].

Suppose that the satellite is a dynamically symmetric rigid body and its center of mass moves in a circular orbit. A possible solution of the equations of motion represents what is called in-plane motion. That is a pendulum-like motion in which the axis of symmetry of the satellite remains all the time in the orbital plane (see e.g. [3]). The aim of this paper is to study the stability of the planar periodic motion, in particular, the

stability of vibration motion. Markeev [5] was the first to study the problem in the above formulation. He studied the orbital stability of the planar periodic motion for dynamically symmetric satellite whose polar axis of the ellipsoid of inertia is shorter than the equatorial ones. But, diagrams of stability contained some imperfections because of low accuracy of numerical calculations. Akulenko, Nesterov and Shmatkov [6] pointed out these imperfections. Markeev and Bardin [7] studied the problem of stability of the planer periodic motions in the nonlinear setting using the normal forms of Birkhoff. Such analysis can be performed only numerically.

In this paper, we study the stability of plane vibrational motions of a satellite in a circular orbit. The problem is reduced to a form of Hill's equation, i.e. a single second-order linear differential equation with a periodic coefficient. It turned out that this equation is a generalization of Lame's equation of the first rank. To deal with this equation we combine Floquet theory [8] with the method devised by Ince for Lame's equation, which consists in finding primitive periodic solutions in the form of Fourier series expansions. Point sets in the space of parameters, corresponding to those solutions separate zones of stability and instability. We obtain a detailed picture of those zones, analytically and in a purely numerical treatment. Both analyses are in complete agreement.

Mathematical Formulation

Description of motion

Denote by O the center of the Earth and by O' the current position of the center of mass of the satellite describing a circular orbit of radius R with centre at O see Fig.1. Let $OXYZ$ be an inertial frame of reference, $O'x_1y_1z_1$ be the orbital system with $x_1 -$ axis along OO', $y_1 -$ along the tangent to the orbit in the direction of motion of the satellite and $z_1 -$ in the direction orthogonal to the orbital plane and let $O'xyz$ be the system of central principal axes of the satellite with moments of inertia A,B and C, respectively. Let also α , β, γ be the three unit vectors in the directions of x_1, y_1, z_1 and ω the angular velocity vector of the satellite relative to the orbital system, all components being

referred to the body system $O'xyz$. To describe the orientation of the satellite relative to the orbital frame we shall use Euler's angle ψ the angle of precession around the $z_1 -$ axis, θ the angle between the axis of the body and the orbital plane (so that the nutation angle between z and $z_1 -$ axis is $\frac{\pi}{2} - \theta$) and φ the angle of proper rotation of the satellite around its $z -$ axis.
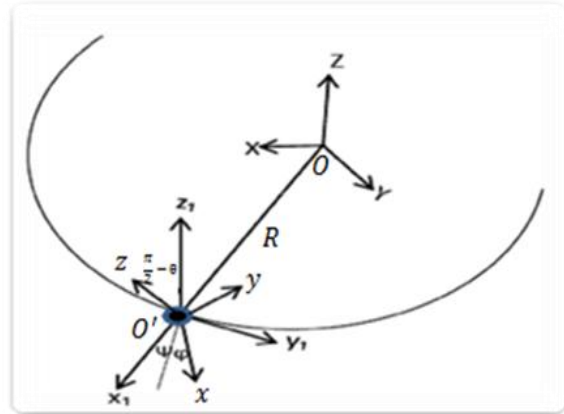

Fig. 1 Euler's angles for describing the orientation of the satellite

In those variables we can write

$$\alpha = (cos\psi cos\phi - sin\theta sin\phi sin\psi ,$$
$$-cos\psi sin\phi - sin\theta cos\phi sin\psi ,$$
$$cos\theta sin\psi)$$

$$\beta = (sin\psi cos\phi + sin\theta sin\phi cos\psi,$$
$$-sin\psi cos\phi + sin\theta sin\phi cos\psi,$$
$$-cos\theta cos\psi)$$

$$\gamma = (cos\theta sin\phi, cos\theta cos\phi, sin\theta)$$

$$\omega = (\dot{\psi} cos\theta sin\phi - cos\phi,$$
$$\dot{\psi} cos\theta cos\phi + sin\phi, \dot{\psi} sin\phi + \dot{\phi})$$

In what follows we study the rotational motion of the satellite about its centre of mass. It is supposed that the rotational motion is independent of the orbital motion of the satellite with uniform angular speed Ω (say), given by the

expression $\Omega^2 = \frac{\mu}{R^3}$ where μ is Gauss' constant of the Earth [3]. The Lagrangian of the rotational motion can be written as

$$L = \frac{1}{2} wI \cdot w - V \tag{1}$$

$$w = \omega + \Omega\gamma \tag{2}$$

where w the absolute angular velocity of the satellite and V its potential function in the gravitational field of Earth. A normally acceptable approximate form (see e.g. Beletskii [3]) is:

$$V = \frac{3\mu}{2R^3} \alpha I \cdot \alpha = \frac{3}{2} \Omega^2 \alpha I . \alpha \tag{3}$$

Substituting (2) and (3) in (1) we get

$$L = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2)$$

$$+ \Omega(Ap\gamma^1 + Bq\gamma^2 + Cr\gamma^3)$$

$$+ \frac{\Omega^2}{2}(\gamma I \cdot \gamma + 3\alpha I \cdot \alpha) \tag{4}$$

The equation of motion of the satellite relative to the orbital frame, corresponding to the Lagrangian can be written in the Euler-

$$\dot{G} + \omega \times (G - 2\Omega\gamma\bar{I}) =$$

$$3\Omega^2 \alpha \times \alpha I - \Omega^2 \gamma \times \gamma I$$

$$\dot{\alpha} + \omega \times \alpha = 0,$$

$$\dot{\beta} + \omega \times \beta = 0$$

$$\dot{\gamma} + \omega \times \gamma = 0 \tag{5}$$

where $\bar{I} = \frac{1}{2}(TrI)\delta - I.$ This system admits Jacobi's integral

$$\frac{1}{2}\omega I \cdot \omega + \frac{\Omega^2}{2}(3\alpha I \cdot \alpha - \gamma I \cdot \gamma) = h \tag{6}$$

It was noted in [4] that this system of equations (6) of motion of the satellite on a circular orbit is form-equivalent to the equations of motion of an electrically charged rigid body about a fixed point of it, while acted upon by gravitational, electric and magnetic fields. The system (5) admits a simple solution, which represents motion of the satellite with two of its principal axes lying always in the orbital plane and the third orthogonal to it. Let the last axis be the $x -$ axis. It is not hard to see that this solution is

$$\omega = \omega\gamma, \omega = \dot{\psi},$$

$$\alpha = (0, \cos\psi, \sin\psi),$$

$$-\sin\psi, \cos\psi), \gamma = (1,0,0)$$

When $\dot{\psi} = 0$ the satellite takes the relative equilibrium position with one of its axes directed to the Earth's centre and another directed along the tangent to the orbit. In general we have two types of motion: vibration and rotation. Each is centered about one of the equilibrium positions. This occurs according to the ratio $\alpha = \frac{C}{A}$ of moment of inertia of the satellite, restricted by the triangle inequality to the interval [0, 2]. The subinterval

i) $0 < \alpha < 1$ corresponds to the case $C < A = B$

ii) $1 < \alpha < 2$ to the case $C > A = B$.

For the first case the satellite is moving around the tangent of the orbital plane and the second cases the satellite is moving around OO'.
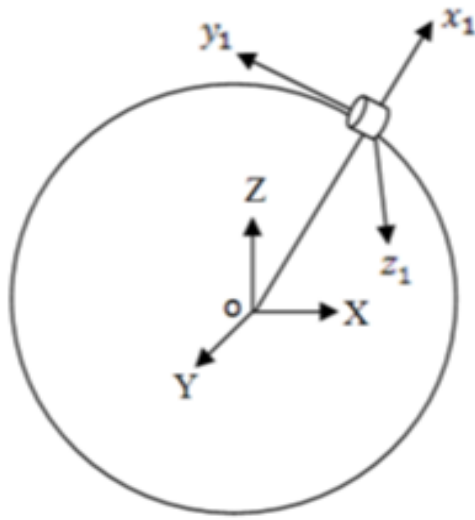
Equation of motion
We may express the Lagrangian of the problem in the form:

$$L = \frac{1}{2}A(p^2 + q^2) + \frac{1}{2}Cr^2$$

$$+ \Omega[A(p\gamma_1 + q\gamma_2) + Cr\gamma_3]$$

$$+ \frac{\Omega^2}{2}[A(\gamma_1^2 + \gamma_2^2)$$

$$+ C\gamma_3^2 + (C - A)(1 - 3\alpha_3^2)] \tag{7}$$

Case i ($0 < \alpha < 1$)
Under the condition i the Lagrangian is ($\Psi = \frac{\pi}{2} + \psi$)

Fig. 2 Direction of the satellite axes for the case I $(0 < \alpha < 1)$

$$L = \left[\frac{A}{2}\cos^2\theta + \frac{C}{2}\sin^2\theta\right]\dot{\Psi}^2$$

$$+[\Omega(A\cos^2\theta + C\sin^2\theta) + C\dot{\varphi}\sin\theta]\dot{\Psi} + \frac{A}{2}\dot{\theta}^2$$

$$+\Omega\dot{\varphi}C\sin\theta + \frac{C}{2}\dot{\varphi}^2$$

$$+\frac{\Omega^2}{2}(A-C)[\cos^2\theta + 3\cos^2\Psi\cos^2\theta] \qquad (8)$$

Now, we note that φ is a cyclic coordinate where:

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = C[(\dot{\Psi}+\Omega)\sin\theta + \dot{\varphi}] = C_1$$

$$\dot{\varphi} = C_1 - (\dot{\Psi}+\Omega)\sin\theta \qquad (9)$$

Next, we ignore φ and construct the Routhian:

$$R = \frac{A}{2}[\dot{\theta}^2 + \dot{\Psi}^2\cos^2\theta]$$

$$+[\Omega A\cos^2\theta + CC_1\sin\theta]\dot{\Psi}$$

$$+\frac{\Omega^2}{2}[C-A+$$

$$A\cos^2\theta + 3(A-C)\cos^2\Psi\cos^2\theta]$$

$$+\frac{CC_1}{2}(2\Omega\sin\theta - C_1) \qquad (10)$$

The satellite can perform the in-plane motions only if the real constant $C_1 = 0$.

According to this condition the Routhian takes

the form:

$$R = \frac{A}{2}[\dot{\theta}^2 + \dot{\Psi}^2\cos^2\theta]$$

$$+\Omega A\cos^2\theta\,\dot{\Psi} + \frac{\Omega^2}{2}[C-A+$$

$$A\cos^2\theta + 3(A-C)\cos^2\Psi\cos^2\theta] \qquad (11)$$

The equations of motion of the satellite in this case become

$$A\ddot{\Psi}\cos^2\theta - 2A\dot{\theta}(\dot{\Psi}+\Omega)\sin\theta\cos\theta$$

$$+3\Omega^2(A-C)\cos^2\theta\sin\Psi\cos\Psi = 0 \qquad (12)$$

$$A\ddot{\theta} + A\dot{\Psi}^2\sin\theta\cos\theta + 2A\Omega\dot{\Psi}\sin\theta\cos\theta$$

$$+3\Omega^2(A-C)\cos^2\Psi\sin\theta\cos\theta +$$

$$\Omega^2 A\sin\theta = 0 \qquad (13)$$

Defining the non-dimensional time $\tau = \Omega t$ and denoting derivatives with respect to τ by $()'$ the non-dimensional equations of motion become:

$$\Psi''\cos^2\theta - 2\theta'(\Psi'+1)\sin\theta\cos\theta$$

$$+3(1-\alpha)\cos^2\theta\sin\Psi\cos\Psi = 0 \qquad (14)$$

$$\theta'' + \Psi'^2\sin\theta\cos\theta + 2\Psi'\sin\theta\cos\theta$$

$$+(1-\alpha)\cos^2\Psi\sin\theta\cos\theta$$

$$+\sin\theta = 0 \qquad (15)$$

Case ii $(1 < \alpha < 2)$

In this case the equatorial axis remains approximately orthogonal to the plane of the orbit and the polar axis performs small angular oscillations about this plane so we shall use Euler's angle ψ the angle of precession around the z_1 −axis (see Fig. 3).

Now, the Lagrangian function takes the form:

$$L = (\frac{A}{2}\cos^2\theta + \frac{C}{2}\sin^2\theta)\dot{\psi}^2 +$$

$$[\Omega(A\cos^2\theta + C\sin^2\theta) + C\dot{\varphi}\sin\theta]\dot{\psi} +$$

$$\frac{A}{2}\dot{\theta}^2 + \Omega\dot{\varphi}C\sin\theta + \frac{C}{2}\dot{\varphi}^2 -$$

$$\frac{\Omega^2}{2}\begin{bmatrix} 4(C-A)\cos^2\theta - 2C + \\ A - 3(C-A)\cos^2\psi\cos^2\theta \end{bmatrix} \qquad (16)$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = C\left[(\dot\psi + \Omega)\sin\theta + \dot\varphi\right] = C_2 \qquad (17)$$
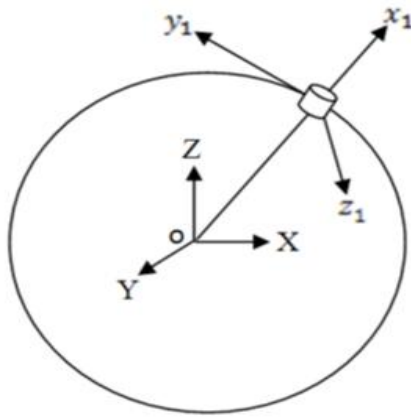
and we get the Routhian

$$R = \frac{A}{2}\left[\dot\theta^2 + \dot\psi^2\cos^2\theta\right] + \left[\Omega A\cos^2\theta + CC_2\sin\theta\right]\dot\psi$$

$$+ \frac{\Omega^2}{2}\left[C - A + (4A - 3C)\cos^2\theta + 3(C - A)\cos^2\psi\cos^2\theta\right]$$

$$+ \frac{CC_2}{2}(2\Omega\sin\theta - C_2) \qquad (18)$$

Then the equations of motion are:

$$A\ddot\psi\cos^2\theta + \left[CC_2\cos\theta - 2A(\dot\psi + \Omega)\sin\theta\cos\theta\right]\dot\theta$$
$$+ 3\Omega^2(C - A)\cos^2\theta\sin\psi\cos\psi = 0 \qquad (19)$$

$$A\ddot\theta + A\dot\psi^2\sin\theta\cos\theta - \left[CC_2 - 2A\Omega\sin\theta\right]\dot\psi\cos\theta$$
$$+ 3\Omega^2(C - A)\cos^2\psi\sin\theta\cos\theta - CC_2\Omega\cos\theta$$
$$+ \Omega^2(4A - 3C)\sin\theta\cos\theta = 0 \qquad (20)$$

Fig. 3 Direction of the satellite axes for the case II ($1 < \alpha < 2$)

And in terms of the non-dimensional time

$$\psi''\cos^2\theta - \left[\alpha C_2\cos\theta - 2(\psi' + 1)\sin\theta\cos\theta\right]\theta'$$
$$+ 3(\alpha - 1)\cos^2\theta\sin\psi\cos\psi = 0 \qquad (21)$$

$$\theta'' + \psi'^2\sin\theta\cos\theta -$$
$$(\alpha C_2 - 2\sin\theta)\psi'\cos\theta + 3(\alpha - 1)\cos^2\psi\sin\theta\cos\theta$$
$$+ (4 - 3\alpha)\sin\theta\cos\theta - \frac{\alpha C_2}{\Omega}\cos\theta = 0 \qquad (22)$$

The equations (14), (15), (21) and (22) describe the motion of the satellite relative to the orbital frame.

Solution for in-plane motion
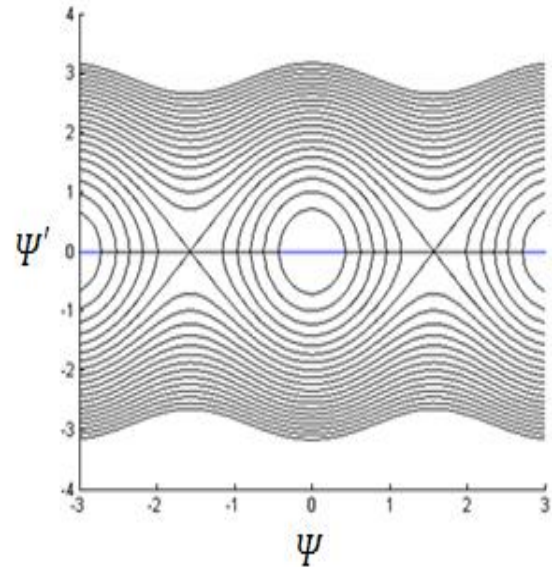
Assume that $0 < \alpha < 1$, by backing to equation (14) the in-plane motion $\theta = 0$ is described by

$$\Psi'' + 3(1 - \alpha)\sin\Psi\cos\Psi = 0 \qquad (23)$$

Integrating this equation we obtain

$$\Psi'^2 + 3(1 - \alpha)\sin^2\Psi = h \qquad (24)$$

Where h is a measure of the energy of the in-plane motion. We will show that the motion is classified as follow (see Fig. 4):

$h < 0$	no motion
$h = 0$	equilibrium
$0 < h < 3(1 - \frac{C}{A})$	periodic oscillation
$h > 3(1 - \frac{C}{A})$	continuous rotation and the direction of the rotation determined by the sign of the initial rotation rate
$h = 3(1 - \frac{C}{A})$	separate the zones of oscillations and liberations

Fig. 4 Phase portrait for in-plane motion   assuming zero out-of-plane inclination

From equation (24) under the condition ($0 < \alpha < 1$)

$$\Psi' = \pm\sqrt{h - 3(1 - \alpha)\sin^2\Psi} \qquad (25)$$

and on separation of variables

$$\int \frac{d\Psi}{\sqrt{1 - k_1^2\sin^2\Psi}} = \pm\sqrt{h}\int d\tau$$

$$k_1^2 = \frac{3}{h}(1 - \alpha)$$

If $|k_1| > 1, h < 3(1 - \alpha)$

we define $v_1 = \dfrac{1}{k_1}$

$\Psi(\tau) = \sin^{-1}\left(v_1 sn\left(\sqrt{3(1-\alpha)}\tau, v_1\right)\right),$

$\Psi'(\tau) = \sqrt{h}\, cn(\sqrt{3(1-\alpha)}\tau, v_1) \qquad (26)$

$if \ |k_1| < 1, \qquad h > 3(1 - \alpha)$
$\Psi(\tau) = am(\pm\sqrt{h}\,\tau, k_1),$
$\Psi'(\tau) = \pm\sqrt{h}dn(\sqrt{h}\tau, k_1) \qquad (27)$

For second case when we study the in-plane motion $\theta = 0$ and $C_2 = 0$ then

$\psi'' + 3(\alpha - 1)\sin\psi\cos\psi = 0 \qquad (28)$

$\psi'^2 + 3(\alpha - 1)\sin^2\psi = h$
$\psi' = \pm\sqrt{h - 3(\alpha - 1)\sin^2\psi} \qquad (29)$

$k_2^2 = \dfrac{3}{h}(\alpha - 1)$
$\displaystyle\int \frac{d\psi}{\sqrt{1 - k_2^2\sin^2\psi}} = \pm\sqrt{h}\int d\tau$

$if \ |k_2| < 1, h > 3(\alpha - 1)$
$\psi(\tau) = am(\pm\sqrt{h}\,\tau, k_2),$
$\psi'(\tau) = \pm\sqrt{h}dn(\sqrt{h}\tau, k_2) \qquad (30)$

$if \ |k_2| > 1, h < 3(\alpha - 1), v_2 = \dfrac{1}{k_2}$
$\psi(\tau) = \sin^{-1}\left(v_2 sn\left(\sqrt{3(\alpha - 1)}\tau, v_2\right)\right),$
$\psi'(\tau) = \sqrt{h}cn\left(\sqrt{3(\alpha - 1)}\tau, v_2\right) \qquad (31)$

Studying of Vibration motion

Stability of in-plane motion

The linear stability equation w. r. to θ is
$\theta'' + [\Psi'^2 + 1 + 2\Psi' + 3(1 - \alpha) - 3(1 - \alpha)\sin^2\Psi]\theta = 0 \quad (0 < \alpha < 1) \qquad (32)$

$\theta'' + [\psi'^2 + 1 + 2\psi' - 3(\alpha - 1)\sin^2\psi]\theta = 0$
$(1 < \alpha < 2) \qquad (33)$

Let $w = \sqrt{3(1 - \alpha)}\tau$ and $\Psi = am(w)$. Using (26) and (32) we get

$\dfrac{d^2\theta}{dw^2} + \begin{bmatrix} 1 + v_1^2 + \dfrac{1}{3(1-\alpha)} - \\ 2v_1^2 sn^2(w, v_1) + \\ \dfrac{2v_1}{\sqrt{3(1-\alpha)}}cn(w, v_1) \end{bmatrix}\theta = 0 \qquad (34)$

In trigonometric form with $\sin\Psi = v_1\sin\beta$
$(1 - v_1^2\sin^2\beta)\dfrac{d^2\theta}{d\beta^2} -$
$v_1^2\sin\beta\cos\beta\dfrac{d\theta}{d\beta} +$
$(1 + v_1^2 + \dfrac{1}{3(1 - \alpha)} - 2v_1^2\sin^2\beta$
$+\dfrac{2v_1}{\sqrt{3(1-\alpha)}}\cos\beta)\theta = 0 \qquad (35)$

This equation can be transformed to hill's equation as follow:
Let $\theta = gf$

$g'' + \left[\dfrac{\dfrac{3v_1^4\sin^2 2\beta}{16(1 - v_1^2\sin^2\beta)^2} +}{\dfrac{1 + 3(2 + 3v_1^2\cos 2\beta)(1 - \alpha) + 4v_1\sqrt{3(1-\alpha)}\cos\beta}{6(1-\alpha)(1 - v_1^2\sin^2\beta)}}\right]g = 0 \qquad (36)$

When $w = \sqrt{3(\alpha - 1)}\tau$ and $\psi = am(w)$ then use equations (30) and (33) to get

$\dfrac{d^2\theta}{dw^2} + \begin{bmatrix} v_2^2 + \dfrac{1}{3(\alpha-1)} - 2v_2^2 sn^2(w, v_2) \\ +\dfrac{2v_2}{\sqrt{3(\alpha-1)}}cn(w, v_2) \end{bmatrix}\theta = 0 \qquad (37)$

We will use the same previous method to get the trigonometric form

$\sin\psi = v_2\sin\beta \quad \& \quad \psi' = \sqrt{h}\cos\beta$
$(1 - v_2^2\sin^2\beta)\dfrac{d^2\theta}{d\beta^2} - v_2^2\sin\beta\cos\beta\dfrac{d\theta}{d\beta}$
$+(v_2^2 + \dfrac{1}{3(\alpha - 1)} - 2v_2^2\sin^2\beta$
$+\dfrac{2v_2}{\sqrt{3(\alpha-1)}}\cos\beta)\theta = 0 \qquad (38)$

Transform it to Hill's equation

$g'' + \left[\dfrac{\dfrac{3v_2^4\sin^2 2\beta}{16(1 - v_2^2\sin^2\beta)^2} +}{\dfrac{1 + 3v_2^2(\alpha-1)\cos 2\beta + 4v_2\sqrt{3(\alpha-1)}\cos\beta}{6(\alpha-1)(1 - v_2^2\sin^2\beta)}}\right]g = 0 \qquad (39)$

The equations (36 & 39) are periodic with periods 2π and 4π and depend on two parameters (α, ψ_0). In the following subsection we deduce the equations of the boundaries between zones of stability and instability in the plane of those parameters.

The stability of vibration motion

Using Floquet's theorem we will study the stability of equation (35) both analytically and numerically. For simplicity we write the equation in the form

$$(1 - v_1^2 \sin^2 \beta)\frac{d^2\theta}{d\beta^2} -$$

$$v_1^2 \sin \beta \cos \beta \frac{d\theta}{d\beta} +$$

$$(b - 2v_1^2 \sin^2 \beta + d\cos\beta)\theta = 0 \qquad (40)$$

$$b = 1 + v_1^2 + \frac{1}{3(1-\alpha)} \quad , \quad d = \frac{2v_1}{\sqrt{3(1-\alpha)}}$$

and equation (38) in the form

$$(1 - v_2^2 \sin^2 \beta)\frac{d^2\theta}{d\beta^2} -$$

$$v_2^2 \sin \beta \cos \beta \frac{d\theta}{d\beta} +$$

$$(b_1 - 2v_2^2 \sin^2 \beta + d_1\cos\beta)\theta = 0 \qquad (41)$$

$$b_1 = v_2^2 + \frac{1}{3(\alpha - 1)} \quad , \quad d_1 = \frac{2v_2}{\sqrt{3(\alpha-1)}}$$

The primitive periodic solutions of equations (40) and (41) can be expressed as a Fourier series in one of the forms:

Odd 2π periodic solution

$$\theta = \sum_{n=0}^{\infty} A_n \sin n\beta \qquad (42)$$

Substitution this solution in (40), yields the following recurrence relations for coefficients A_n

$$(b - 1 - v_1^2)A_1 + \frac{d}{2}A_2 - v_1^2 A_3 = 0$$

$$\frac{d}{2}A_1 + (b - 4 + v_1^2)A_2 + \frac{d}{2}A_3 - \frac{5}{2}v_1^2 A_4 = 0$$

$$\frac{d}{2}A_2 + \left(b - 9 + \frac{7}{2}v_1^2\right)A_3 + \frac{d}{2}A_4 - \frac{9}{2}v_1^2 A_5 = 0$$

For $n \geq 2$

$$-\frac{n(2n-3)}{2}v_1^2 A_{2n-2} + \frac{d}{2}A_{2n-1} +$$

$$(b - 4n^2 + (2n^2 - 1)v_1^2)A_{2n} + \frac{d}{2}A_{2n+1}$$

$$-\frac{n(2n+3)}{2}v_1^2 A_{2n+2} = 0$$

$$-\frac{n(2n-1) - 1}{2}v_1^2 A_{2n-1} + \frac{d}{2}A_{2n}$$

$$+\left(b - (2n+1)^2 + \frac{(2n+1)^2 - 2)}{2}v_1^2\right)A_{2n+1}$$

$$+\frac{d}{2}A_{2n+2} - \frac{n(2n+5)+2}{2}v_1^2 A_{2n+3} = 0 \qquad (43)$$

For the second case, using (41) gives for coefficients A_n the recurrence relations

$$(b_1 - 1 - v_2^2)A_1 + \frac{d_1}{2}A_2 - v_2^2 A_3 = 0$$

$$\frac{d_1}{2}A_1 + (b_1 - 4 + v_2^2)A_2 + \frac{d_1}{2}A_3 - \frac{5}{2}v_2^2 A_4 = 0$$

$$\frac{d_1}{2}A_2 + \left(b_1 - 9 + \frac{7}{2}v_2^2\right)A_3 + \frac{d_1}{2}A_4 - \frac{9}{2}v_2^2 A_5 = 0$$

For $n \geq 2$

$$-\frac{n(2n-3)}{2}v_2^2 A_{2n-2} + \frac{d_1}{2}A_{2n-1}$$

$$+(b_1 - 4n^2 + (2n^2 - 1)v_2^2)A_{2n} + \frac{d_1}{2}A_{2n+1}$$

$$-\frac{n(2n+3)}{2}v_2^2 A_{2n+2} = 0$$

$$-\frac{n(2n-1) - 1}{2}(2n^2 - 1)v_2^2 A_{2n-1} + \frac{d_1}{2}A_{2n}$$

$$+\left(b_1 - (2n+1)^2 + \frac{(2n+1)^2 - 2)}{2}v_2^2\right)A_{2n+1}$$

$$+\frac{d_1}{2}A_{2n+2} - \frac{n(2n+5)+2}{2}v_2^2 A_{2n+3} = 0 \qquad (44)$$

Even 2π -periodic solution

Substituting

$$\theta = \sum_{n=0}^{\infty} B_n \cos n\beta \qquad (45)$$

in equation (40) yields the recurrence relations for coefficients B_n:

$$(b - v_1^2)B_0 + \frac{d}{2}B_1 = 0$$

$$dB_0 + (b - 1)B_1 + \frac{d}{2}B_2 - v_1^2 B_3 = 0$$

$$v_1^2 B_0 + \frac{d}{2}B_1 + (b - 4 + v_1^2)B_2 + \frac{d}{2}B_3 - \frac{5}{2}v_1^2 B_4 = 0$$

$$\frac{d}{2}B_2 + \left(b - 9 + \frac{7}{2}v_1^2\right)B_3 + \frac{d}{2}B_4 - \frac{9}{2}v_1^2 B_5 = 0$$

For $n \geq 2$

$$-\frac{n(2n-3)}{2}v_1^2 B_{2n-2} + \frac{d}{2}B_{2n-1} +$$

$$(b - 4n^2 + (2n^2 - 1)v_1^2)B_{2n} + \frac{d}{2}B_{2n+1}$$

$$-\frac{n(2n+3)}{2}v_1^2 B_{2n+2} = 0$$

$$-\frac{n(2n-1)-1}{2}v_1^2 B_{2n-1} + \frac{d}{2}B_{2n} +$$

$$\left(b - (2n+1)^2 + \frac{(2n+1)^2 - 2}{2}v_1^2\right)B_{2n+1}$$

$$+\frac{d}{2}B_{2n+2} - \frac{n(2n+5)+2}{2}v_1^2 B_{2n+3} = 0 \qquad (46)$$

And in (41) leads to the relations

$$(b_1 - v_2^2)B_0 + \frac{d_1}{2}B_1 = 0$$

$$d_1 B_0 + (b_1 - 1)B_1 + \frac{d_1}{2}B_2 - v_2^2 B_3 = 0$$

$$v_2^2 B_0 + \frac{d_1}{2}B_1 + (b_1 - 4 + v_2^2)B_2 + \frac{d_1}{2}B_3 - \frac{5}{2}v_2^2 B_4 = 0$$

$$\frac{d_1}{2}B_2 + \left(b_1 - 9 + \frac{7}{2}v_2^2\right)B_3 + \frac{d_1}{2}B_4 - \frac{9}{2}v_2^2 B_5 = 0$$

For $n \geq 2$

$$-\frac{n(2n-3)}{2}v_2^2 B_{2n-2} + \frac{d_1}{2}B_{2n-1}$$

$$+(b_1 - 4n^2 + (2n^2 - 1)v_2^2)B_{2n} + \frac{d_1}{2}B_{2n+1}$$

$$-\frac{n(2n+3)}{2}v_2^2 B_{2n+2} = 0$$

$$-\frac{n(2n-1)-1}{2}v_2^2 B_{2n-1} + \frac{d_1}{2}B_{2n} +$$

$$\left(b_1 - (2n+1)^2 + \frac{(2n+1)^2 - 2}{2}v_2^2\right)B_{2n+1}$$

$$+\frac{d_1}{2}B_{2n+2} - \frac{n(2n+5)+2}{2}v_2^2 B_{2n+3} = 0 \qquad (47)$$

Even 4π -periodic solution

Substitution a solution in the form
$$\theta = \sum_{n=0}^{\infty} C_n \cos\left(n + \tfrac{1}{2}\right)\beta \qquad (48)$$

In equation (40) yields the recurrence relations for coefficients C_n

$$\left(b + \frac{d}{2} - \frac{1}{4} - \frac{7}{8}v_1^2\right)C_0 + \left(\frac{5}{16}v_1^2 + \frac{d}{2}\right)C_1 - \frac{7}{16}v_1^2 C_2 = 0$$

$$\left(\frac{d}{2} + \frac{9}{16}v_1^2\right)C_0 + \left(b - \frac{9}{4} + \frac{1}{8}v_1^2\right)C_1 + \frac{d}{2}C_2 - \frac{27}{16}v_1^2 C_3 = 0$$

For $n \geq 2$

$$-\frac{(2n-5)(2n+1)}{16}v_1^2 C_{n-2} + \frac{d}{2}C_{n-1} +$$

$$\left(b - \frac{(2n+1)^2}{4} + \frac{(2n+1)^2 - 8}{8}v_1^2\right)C_n$$

$$+\frac{d}{2}C_{n+1} - \frac{(2n+1)(2n+7)}{16}v_1^2 C_{n+2} = 0 \qquad (49)$$

and for second case the recurrence relations are

$$\left(b_1 + \frac{d}{2} - \frac{1}{4} - \frac{7}{8}v_2^2\right)C_0 +$$

$$\left(\frac{5}{16}v_2^2 + \frac{d_1}{2}\right)C_1 - \frac{7}{16}v_2^2 C_2 = 0$$

$$\left(\frac{d_1}{2} + \frac{9}{16}v_2^2\right)C_0 + \left(b_1 - \frac{9}{4} + \frac{1}{8}v_2^2\right)C_1$$

$$+\frac{d_1}{2}C_2 - \frac{27}{16}v_2^2 C_3 = 0$$

For $n \geq 2$

$$-\frac{(2n-5)(2n+1)}{16}v_2^2 C_{n-2} + \frac{d_1}{2}C_{n-1} +$$

$$\left(b_1 - \frac{(2n+1)^2}{4} + \frac{(2n+1)^2 - 8}{8}v_2^2\right)C_n$$

$$+\frac{d_1}{2}C_{n+1} - \frac{(2n+1)(2n+7)}{16}v_2^2 C_{n+2} = 0 \qquad (50)$$

Odd 4π -periodic solution

Assuming such solution in the form

$$\theta = \sum_{n=0}^{\infty} D_n \sin\left(n + \tfrac{1}{2}\right)\beta \qquad (51)$$

and substituting in (40) yields the following recurrence relations for coefficients D_n

$$\left(b - \frac{d}{2} - \frac{1}{4} - \frac{7}{8}v_1^2\right)D_0 +$$

$$\left(\frac{d}{2} - \frac{5}{16}v_1^2\right)D_1 - \frac{7}{16}v_1^2 D_2 = 0$$

$$\left(\frac{d}{2} - \frac{9}{16}v_1^2\right)D_0 + \left(b - \frac{9}{4} + \frac{1}{8}v_1^2\right)D_1$$

$$+\frac{d}{2}D_2 - \frac{27}{16}v_1^2 D_3 = 0$$

For $n \geq 2$:

$$-\frac{(2n-5)(2n+1)}{16}v_1^2 D_{n-2} + \frac{d}{2}D_{n-1} +$$

$$\left(b - \frac{(2n+1)^2}{4} + \frac{(2n+1)^2 - 8}{8}v_1^2\right)D_n$$

$$+\frac{d}{2}D_{n+1} - \frac{(2n+1)(2n+7)}{16}v_1^2 D_{n+2} = 0 \qquad (52)$$

While for the second case the recurrence relations are

$$\left(b_1 - \frac{d_1}{2} - \frac{1}{4} - \frac{7}{8}v_2^2\right)D_0 +$$

$$\left(\frac{d_1}{2} - \frac{5}{16}v_2^2\right)D_1 - \frac{7}{16}v_2^2 D_2 = 0$$

$$\left(\frac{d_1}{2} - \frac{9}{16}v_2^2\right)D_0 + \left(b_1 - \frac{9}{4} + \frac{1}{8}v_2^2\right)D_1 +$$

$$\frac{d_1}{2}H_2 - \frac{27}{16}v_2^2 D_3 = 0$$

For $n \geq 2$:

$$-\frac{(2n-5)(2n+1)}{16}v_2^2 D_{n-2} + \frac{d_1}{2}D_{n-1} +$$

$$\left(b_1 - \frac{(2n+1)^2}{4} + \frac{(2n+1)^2 - 8}{8}v_2^2\right)D_n$$

$$+\frac{d_1}{2}D_{n+1} - \frac{(2n+1)(2n+7)}{16}v_2^2 D_{n+2} = 0 \quad (53)$$

The systems of equations (43,46,49,52) and also (44 , 47, 50, 53) are homogeneous infinite systems. The boundary curves can be drawn by solving the determinantal equations. We shall study the stability and instability zones in the plane of parameters α, ψ_0 where ψ_0 is the amplitude of vibrations ($v_{1,2} = \sin \psi_0$).

Numerical and analytical diagrams of stability

Let the symbols W_s , G_s denote the (s) zones of stability and instability, respectively, separated by the curves $\prod\limits_{2s}^{\pm}, \prod\limits_{2s+1}^{\pm}$ whose equations describe the distribution of eigen-values for boundary problems of periods 2π , 4π . We shall draw these curves by solving the last four determinants. We classify the curves as:

$\prod\limits_{2s}^{+}$ for even $2\pi -$ periodic solutions and

$\prod\limits_{2s}^{-}$ for odd $2\pi -$ periodic solutions.

$\prod\limits_{2s+1}^{+}$ for even $4\pi -$ periodic solutions and

$\prod\limits_{2s+1}^{-}$ for odd $4\pi -$ periodic solutions.

As a first step we now find the intersections of $\prod\limits_{2s}^{\pm}$, $\prod\limits_{2s+1}^{\pm}$ with the line $\psi_0 = 0$. On this line the equation of vibration (36) takes the form

$$g'' + \left[1 + \frac{1}{3(1-\alpha)}\right]g = 0 \quad (54)$$

let $1 + \frac{1}{3(1-\alpha)} = \omega^2$ where ω is constant, then $\alpha = 1 - \frac{1}{3(\omega^2 - 1)}$, for $\omega = 2,3,4,5,6, ...$

$$\alpha_1 = \frac{8}{9}, \alpha_2 = \frac{23}{24}, \alpha_3 = \frac{44}{45}, \alpha_4 = \frac{71}{72}, \alpha_5 = \frac{104}{105}, ...$$

and so on. There exist two curves with the same period one even and other odd passing through each of those values of α. We note that the stability zones occupy the lower part near to $\psi_0 = 0$. For example, two curves, one odd $\prod\limits_{2s+1}^{-}$ and the other even $\prod\limits_{2s+1}^{+}$. with the same period 4π at $\alpha = \frac{8}{9}$ and $\prod\limits_{2s}^{+}$, $\prod\limits_{2s}^{-}$ with period 2π at $\alpha = \frac{23}{24}$ (see Fig. 5).
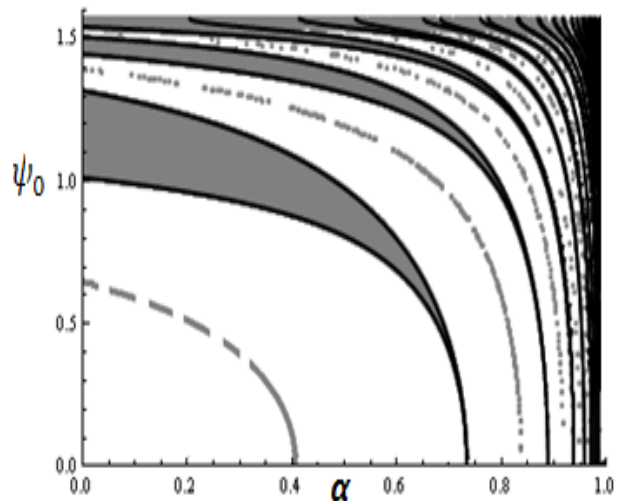

Fig. 5 stability zones for the case I

In the same manner as in the case of $(0 < \alpha < 1)$ we get the intersections of $\prod\limits_{2s}^{\pm}$, $\prod\limits_{2s+1}^{\pm}$ with $\psi_0 = 0$ to determine the zones of stability and instability for the case $(1 < \alpha < 2)$
Equation (39) becomes:

$$g'' + \left[1 + \frac{1}{3(\alpha-1)}\right] g = 0 \qquad (55)$$

Let $\frac{1}{3(\alpha-1)} = n^2$ where n (+) ve integer, then

$\alpha = \frac{1}{3n^2} + 1$ for $n = 1,2,3,4, \ldots$

$\alpha_1 = \frac{4}{3}, \alpha_2 = \frac{13}{12}, \alpha_3 = \frac{28}{27}, \alpha_4 = \frac{49}{48}, \ldots \ldots$

and so on. The zones of stability and instability are shown in Fig. 6.
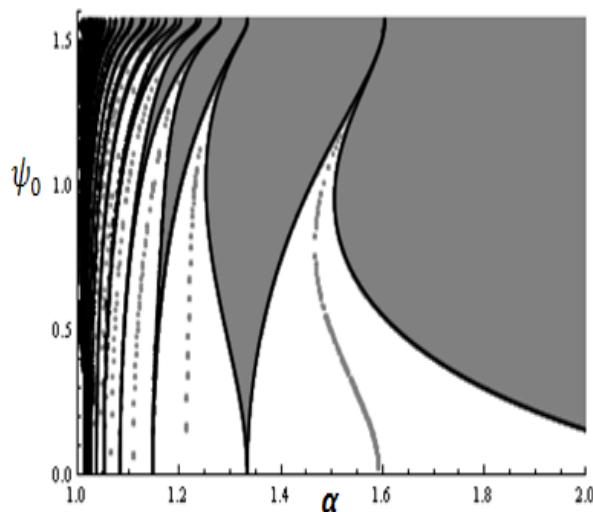

Fig. 6 stability zones for the case II

Conclusion

The stability of the vibrational in-plane motion of the satellite in a circular orbit is studied. The plane of parameter (α, ψ_0) is divided into distinct regions of stability and instability separated by certain curves corresponding to four types of periodic solutions of Hill's equations. Those zones are illustrated graphically.

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الملخص العربى

تحليل استقرار الحركه التذبذبيه لقمر صناعي علي مسار دائري

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في هذا البحث تمت دراسة استقرار الحركه التذبذبيه للقمر الصناعي علي مسار دائري في المستوي، وتم إيضاح النتائج بيانيا فى مستوي البارامترين للمسألة الذى انقسم الي مناطق استقرار ومناطق عدم استقرار تفصل بينها منحنيات ناتجه من اربعة انواع من الحلول الدوريه لمعادلة هيل فى التغيرات.