Delta J. Sci. 2008, 32: 21-33

MATHEMATICS

Some Set Matrix Systems

F.A.A. GHOURABA*
Ministry of Education (retired), Egypt
and M.A. SEOUD
Dept. of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo, Egypt

Received: 10 october 2008

Abstract Here we discuss the solutions of some particular set matrix systems. We found out that the solutions of any system are not necessarily always the same solutions of its conjugate system, also we found out that dividing any system into partitions called components is a reliable method to find out if this system has solutions or not.

1. Introduction

Throughout, all the matrices considered here have sets as their entries.
First, we mention here the material we need from [1] and [2].

Definition 1.1: Let $A := [a_{ij}]$ and $B := [b_{ij}]$ be two *matrices* of the same type, i.e. they have the same number of rows and columns.

(a) We define the union and the intersection of A and B in the obvious way:

$$A \bigcup B := [a_{ij} \bigcup b_{ij}] \qquad , \qquad A \bigcap B := [a_{ij} \bigcap b_{ij}]$$

(b) A is said to be a subset of B and we write $A \subset B$ if a_{ij} is a subset of b_{ij} for every i and j.

(c) The difference between A and B, in symbols, $A - B$ is defined as usual:

$$A - B = [a_{ij} - b_{ij}].$$

* Email: faa_ghouraba@yahoo.com

(d) $\bigcup A := \bigcup_{i,j} a_{ij}$

Definition 1.2: Let A and B be two matrices of types $m \times n$ and $n \times s$ respectively.

We define the first type of multiplication of A and B, denoted by $A \dot{\times} B$, which is a matrix of the type $m \times s$ as follows:

$$A \dot{\times} B = C := [c_{ij}] \quad , \quad \text{where} \quad c_{ij} := \bigcup_{r=1}^{n} (a_{ir} \cap b_{rj})$$

Definition 1.3: The intersection of a given set S with a matrix $A := [a_{ij}]$ is a matrix whose entries are the intersections of the given set with the entries of the matrix. So

$$S \cap A := S \cap [a_{ij}] := [S \cap a_{ij}].$$

Definition 1.4: If $\underline{D}_n$ is a solution of the system $A \times \underline{X} = \underline{B}$ or the homogeneous system $A \dot{\times} \underline{X} = \Phi$, where Φ is the matrix whose entries are the empty set, then we say that it is an internal solution for the system if $\bigcup \underline{D} \subset \bigcup A$. Otherwise, we call it an external solution.

All solutions we consider in this paper are the internal ones.

Theorem 1.5: For any three matrices A, B and C, whenever the operations are defined:

$$\text{(i)} \quad A \dot{\times} (B \cup C) = (A \dot{\times} B) \cup (A \dot{\times} C)$$

$$\text{(ii)} \quad A \times (B \cap C) \subset (A \dot{\times} B) \cap (A \times C)$$

Theorem 1.6: A necessary but not sufficient condition for the system $A \dot{\times} \underline{X} = \underline{B}$ to have a solution is: "the union of all members of any row of A is a superset of the "**corresponding**" row in B".

2. Some Theorems

Let $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, where a_{ij} is a set as mentioned above.

We define the matrix $\underline{\alpha} := \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$, $r_i := \bigcup_{j=1}^{n} a_{ij}$, $i = 1,2,\cdots,m$,

and the matrix $\underline{\gamma} := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, $c_j := \bigcup_{i=1}^{m} a_{ij}$, $j = 1,2,\cdots,n$.

We consider the system $A \dot{\times} \underline{X} = \underline{B}$, where $\underline{X} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\underline{B} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, and we show that solutions exist for two special cases of the system.

Theorem 2.1: The system $A \dot{\times} \underline{X} = \underline{\alpha}$ has a solution.

Proof: Let $A \dot{\times} \underline{\gamma} = \underline{B}$, i.e. $b_i = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \dot{\times} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \bigcup_{s=1}^{n} (a_{is} \cap c_s)$, $i = 1,2,\cdots,m$.

But, $a_{is} \subset c_s$ for every $s \in \{1,2\cdots,m\}$, hence $b_i = \bigcup_{s=1}^{n} a_{is} = r_i$, for every $i \in \{1,2\cdots,m\}$.

Thus $\underline{B} = \underline{\alpha}$, $A \dot{\times} \underline{\gamma} = \underline{\alpha}$. This means that $\underline{\gamma}$ is a solution for the given system.

Remark 2.2: There may be more than one solution for the system $A \dot{\times} \underline{X} = \underline{\alpha}$.

Consider:

$$A := \begin{bmatrix} \{1,2\} & \{2\} \\ \{2,3\} & \{1,2\} \end{bmatrix}, \quad \underline{\alpha} := \begin{bmatrix} \{1,2\} \\ \{1,2,3\} \end{bmatrix}$$

The following are solutions of the system:

$$\begin{bmatrix} \{1,2,3\} \\ \{1,2\} \end{bmatrix} = \underline{\gamma}, \quad \begin{bmatrix} \{1,3\} \\ \{1,2\} \end{bmatrix}, \quad \begin{bmatrix} \{1,3\} \\ \{1,2,3\} \end{bmatrix}, \quad \begin{bmatrix} \{1,2,3\} \\ \{1\} \end{bmatrix}, \quad \begin{bmatrix} \{1,2,3\} \\ \{1,3\} \end{bmatrix} \text{ and } \begin{bmatrix} \{1,2,3\} \\ \{1,2,3\} \end{bmatrix}.$$

Theorem 2.3: If $\underline{B}$ is the following column of m entries:

$\begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix} \subset \underline{\alpha}$, then the system $A \dot{\times} \underline{X} = \underline{B}$ has a solution.

Proof: Let $\underline{X}$ be the following column of n entries $\begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix}$. Then:

$$A \dot{\times} \underline{X} = \underline{H} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix}, \text{ where } [h_i] = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \dot{\times} \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix},$$

$$h_i = \bigcup_{j=1}^{n} (a_{ij} \cap s) = \left(\bigcup_{j=1}^{n} a_{ij} \right) \cap s = r_i \cap s = s, \text{ since } \underline{B} \subset \underline{\alpha}.$$

Then $\underline{H} = \underline{B}$, and the system has a solution

Remark 2.4: The solution of the system in Theorem 2.3 is not necessarily unique, e.g. consider the system:

$$\begin{bmatrix} \{1,2\} & \{2\} \\ \{2,3\} & \{1,2\} \end{bmatrix} \dot{\times} \underline{X} = \begin{bmatrix} \{2\} \\ \{2\} \end{bmatrix}.$$

Notice that $\underline{\alpha} = \begin{bmatrix} \{1,2\} \\ \{1,2,3\} \end{bmatrix} \supset \begin{bmatrix} \{2\} \\ \{2\} \end{bmatrix}$. The following are solutions of the system:

$\begin{bmatrix} \{2\} \\ \{2\} \end{bmatrix}$(= the solution given by the theorem), $\begin{bmatrix} \phi \\ \{2,3\} \end{bmatrix}$, $\begin{bmatrix} \{2\} \\ \phi \end{bmatrix}$, $\begin{bmatrix} \{2\} \\ \{3\} \end{bmatrix}$ and $\begin{bmatrix} \{2\} \\ \{2,3\} \end{bmatrix}$.

Now we consider the usual system: $A \dot{\times} \underline{X} = \underline{B}$, which is:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad (*)$$

i.e. $b_i = \bigcup_{j=1}^{n} (a_{ij} \cap x_j)$, $i = 1,2,\cdots m$.

It follows that:

$$b_i = b_i \cap b_i = b_i \cap \left(\bigcup_{j=1}^{n} (a_{ij} \cap x_j) \right) = \bigcup_{j=1}^{n} (b_i \cap a_{ij} \cap x_j) \;, \quad i = 1,2,\cdots,m.$$

Hence, we have the "conjugate system":

$$\begin{bmatrix} a_{11} \cap b_1 & a_{12} \cap b_1 & \cdots & a_{1n} \cap b_1 \\ a_{21} \cap b_2 & a_{22} \cap b_2 & \cdots & a_{2n} \cap b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \cap b_m & a_{m2} \cap b_m & \cdots & a_{mn} \cap b_m \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad (**),$$

which may be written in the form

$$A^{b} \dot{\times} \underline{X} = \underline{B},$$

where $A^{b} = \left[a_{ij}^{b}\right]$, $a_{ij}^{b} = a_{ij} \cap b_i$, $i = 1,2,\cdots m$, $j = 1,2,\cdots n$. $\qquad (**)$

Theorem 2.5: Every solution of the system $(*)$ is a solution of the conjugate system $(**)$.

Proof: Let $\underline{X} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a solution of the system $(*)$, i.e.

$b_i = \bigcup_{j=1}^{n} (a_{ij} \cap x_j)$, $i = 1,2,\cdots,m$. Let:

$$\begin{bmatrix} a_{11} \cap b_1 & a_{12} \cap b_1 & \cdots & a_{1n} \cap b_1 \\ a_{21} \cap b_2 & a_{22} \cap b_2 & \cdots & a_{2n} \cap b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \cap b_m & a_{m2} \cap b_m & \cdots & a_{mn} \cap b_m \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_m \end{bmatrix}.$$

Then: $b'_i = \bigcup_{j=1}^{n} ((a_{ij} \cap b_i) \cap x_j) = b_i \cap \left(\bigcup_{j=1}^{n} (a_{ij} \cap x_j) \right) = b_i \cap b_i = b_i$, and the theorem follows.

Remark 2.6: The converse of Theorem 2.5 is not true. The following example shows that the system $(**)$ may have a solution, which is not a solution of the system $(*)$.

Let $A := \begin{bmatrix} \{1,2,3\} & \{2,5,6\} & \{3,4,6\} \\ \{2,5,7\} & \{1,2,3\} & \{4,7\} \\ \{3,5,6\} & \{2,6,7\} & \{4,5,7\} \end{bmatrix}$, $\underline{B} := \begin{bmatrix} \{2,3,5,6\} \\ \{2,7\} \\ \{3,5,6,7\} \end{bmatrix}$

$\underline{X} := \begin{bmatrix} \{2,3,7\} \\ \{5,6\} \\ \{2,5,7\} \end{bmatrix}$ is a solution for the system $(*)$: $A \dot{\times} \underline{X} = \underline{B}$.

Then $A^{\#} = \begin{bmatrix} \{2,3\} & \{2,5,6\} & \{3,6\} \\ \{2,7\} & \{2\} & \{7\} \\ \{3,5,6\} & \{6,7\} & \{5,7\} \end{bmatrix}$, $\alpha^{\#} = \begin{bmatrix} \{2,3,5,6\} \\ \{2,7\} \\ \{3,5,6,7\} \end{bmatrix} = \underline{B}$

$\underline{\gamma}^{\#} := \begin{bmatrix} \{2,3,5,6,7\} \\ \{2,5,6,7\} \\ \{3,5,6,7\} \end{bmatrix}$, is a solution for the system $(**)$: $A^{\#} \dot{\times} \underline{X} = \underline{B}$, according to Theorem 2.1.

But $A \dot{\times} \underline{\gamma}^{\#} = \begin{bmatrix} \{1,2,3\} & \{2,5,6\} & \{3,4,6\} \\ \{2,5,7\} & \{1,2,3\} & \{4,7\} \\ \{3,5,6\} & \{2,6,7\} & \{4,5,7\} \end{bmatrix} \dot{\times} \begin{bmatrix} \{2,3,5,6,7\} \\ \{2,5,6,7\} \\ \{3,5,6,7\} \end{bmatrix} = \begin{bmatrix} \{2,3,5,6\} \\ \{2,5,7\} \\ \{2,3,5,6,7\} \end{bmatrix} \neq \begin{bmatrix} \{2,3,5,6\} \\ \{2,7\} \\ \{3,5,6,7\} \end{bmatrix} = \underline{B}$,

i.e. $\underline{\gamma}^{\#}$ is not a solution of the system $(*)$.

We give an example in which the system $(*)$ has no solution, while the system $(**)$ has more than one solution:

Example 2.7: The system $\begin{bmatrix} \{1,2\} & \{2\} \\ \{2,3\} & \{1,2\} \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \{2\} \\ \phi \end{bmatrix}$ $(*)$

has no solution, since:

$(\{1,2\} \cap x_1) \cup (\{2\} \cap x_2) = \{2\} \Rightarrow 2 \in x_1 \cup x_2$,

$(\{2,3\} \cap x_1) \cup (\{1,2\} \cap x_2) = \phi \Rightarrow 2 \notin x_1 \cup x_2$.

While the conjugate system $(**)$ is the following:

$\begin{bmatrix} \{2\} & \{2\} \\ \phi & \phi \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \{2\} \\ \phi \end{bmatrix}$,

which has the following solutions: $\begin{bmatrix} \{2\} \\ \{2\} \end{bmatrix}$, $\begin{bmatrix} \{2\} \\ \phi \end{bmatrix}$ and $\begin{bmatrix} \phi \\ \{2\} \end{bmatrix}$.

Another example to show that the system $(*)$ has solutions, which are the same solutions of the system $(**)$ is the following:

Example 2.8: The system $\begin{bmatrix}\{1,2\} & \{2\}\\ \{2,3\} & \{1,2\}\end{bmatrix}\dot{\times}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\phi\\ \{3\}\end{bmatrix}$ $(*)$

has the solutions: $\begin{bmatrix}\{3\}\\ \{3\}\end{bmatrix}$ and $\begin{bmatrix}\{3\}\\ \phi\end{bmatrix}$, which are the same solutions of the conjugate system:

$$\begin{bmatrix}\phi & \phi\\ \{3\} & \phi\end{bmatrix}\dot{\times}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\phi\\ \{3\}\end{bmatrix} \quad (**)$$

Now, we introduce the following theorem:

Theorem 2.9: A solution of the system $A\dot{\times}\underline{X}=\underline{B}$ (1) is a solution of the system $D\dot{\times}\underline{X}=\underline{B}$ (2) if:

$$\text{(i) } A\subset D \qquad \text{(ii) } \left(\bigcup(D\setminus A)\right)\cap\left(\bigcup A\right)=\phi$$

Proof: Let $\underline{Z}$ be a solution of the system (1), then $A\dot{\times}\underline{Z}=\underline{B}$ (3), and $\bigcup\underline{Z}\subset\bigcup A$ (4). It follows from (ii) and (4) that:

$$\left(\bigcup(D\setminus A)\right)\cap\left(\bigcup\underline{Z}\right)=\phi \quad (5)$$

Since $D=(D-A)\cup A$, it follows that:

$$\begin{aligned}D\dot{\times}\underline{Z}&=((D-A)\cup A)\dot{\times}\underline{Z}\\ &=((D-A)\dot{\times}\underline{Z})\cup(A\dot{\times}\underline{Z}) && \text{(by Theorem 1.5)}\\ &=\Phi\cup\underline{B} && \text{(from (5) and (3))}\\ &=\underline{B}\end{aligned}$$

i.e. $\underline{Z}$ is a solution of the system (2)

Remark 2.10: Let A and B be two matrices of the same type. It is clear that if $\left(\bigcup A\right)\cap\left(\bigcup B\right)=\phi$, then $A\cap B=\Phi$. The converse is not true. Let $A=\begin{bmatrix}\{1,2\} & \{3\}\\ \{2\} & \{1,3\}\end{bmatrix}$ and $B=\begin{bmatrix}\{3\} & \{2\}\\ \{3\} & \{2\}\end{bmatrix}$. Then $\bigcup A=\{1,2,3\}$, $\bigcup B=\{2,3\}$ and $\left(\bigcup A\right)\cap\left(\bigcup B\right)=\{2,3\}\neq\phi$.

Now, we consider the matrix system $A\dot{\times}\underline{X}=\underline{B}$ $(*)$, and let $\bigcup A\left(:=\bigcup_{ij}a_{ij}\right)=K$.

Let $K_1,K_2,\cdots,K_s$ be a partition of K (i.e. $\bigcup_{i=1}^{s}K_i=K$, and for every $i\neq j$: $K_i\cap K_j=\phi$).

Define $A_i:=K_i\cap A$ and $B_i:=K_i\cap B$, $i=1,2,\cdots,s$. It follows from $(*)$ that

$A_i\cap\underline{X}=B_i$, $i=1,2,\cdots,s$ $(*_i)$. (We identify the matrix system with its equation). We call the systems $(*_i)$ $,i=1,2,\cdots s$, the components of the system $(*)$.

Now we have the following lemma and theorem:

Lemma 2.11: Let A and B be two matrices of types $m\times n$ and $n\times l$ respectively and K be a set. Then:

$$K\cap\left(A\dot{\times}B\right)=\left(K\cap A\right)\dot{\times}B=A\dot{\times}\left(K\cap B\right)$$

Proof: Let $A\dot{\times}B=C=\left[c_{ij}\right]$, $c_{ij}=\bigcup_{s=1}^{n}\left(a_{is}\cap b_{sj}\right)$. Then:

$K\cap\left(A\dot{\times}B\right)=K\cap C=\left[K\cap c_{ij}\right]$, where

$$K\cap c_{ij}=K\cap\left(\bigcup_{s=1}^{n}\left(a_{is}\cap b_{sj}\right)\right)=\bigcup_{s=1}^{n}\left(K\cap\left(a_{is}\cap b_{sj}\right)\right)=\bigcup_{s=1}^{n}\left(\left(K\cap a_{is}\right)\cap b_{sj}\right)$$

$$=\bigcup_{s=1}^{n}\left(a_{is}\cap\left(K\cap b_{sj}\right)\right),$$ hence the result.

Theorem 2.12: The system $(*)$ has a solution if and only if every component $(*_i)$ has a solution.

Proof: Let η be a solution of the system $(*)$, i.e. $A \dot{\times} \underline{\eta} = \underline{B}$, and let $\underline{\eta}_i := K_i \cap \underline{\eta}$, $i = 1,2,\cdots s$, where $K_1, K_2, \cdots, K_s$ is a partition of K. It follows that:

$$A_i \dot{\times} \underline{\eta}_i = A_i \dot{\times} (K_i \cap \underline{\eta}) = K_i \cap (A_i \dot{\times} \underline{\eta}) \qquad \text{(by Lemma 2.11)}$$

$$= K_i \cap ((K_i \cap A) \dot{\times} \underline{\eta}) = K_i \cap (K_i \cap (A \dot{\times} \underline{\eta})) \qquad \text{(by Lemma 2.11)}$$

$$= K_i \cap (K_i \cap \underline{B}) = K_i \cap \underline{B} = \underline{B}_i \quad ,$$

i.e. $\underline{\eta}_i$ is a solution of the system $(*i)$.

Now, let every component $(*_i)$ has a solution $\underline{\eta}_i$, i.e.

$A_i \dot{\times} \underline{\eta}_i = \underline{B}_i$, for every $i = 1,2,\cdots,s$.

Since $\left(\bigcup(A - A_i)\right) \cap \left(\bigcup A_i\right) = \phi$ and $A_i \subset A$ for every $i = 1,2,\cdots,s$, it follows from Theorem 2.9 that:

$A \dot{\times} \underline{\eta}_i = \underline{B}_i$ for $i = 1,2,\cdots,s$, hence

$$\bigcup_{i=1}^{s} (A \dot{\times} \underline{\eta}_i) = \bigcup \underline{B}_i .$$

By Theorm 1.5, we have:

$$A \dot{\times} \left(\bigcup_{i=1}^{s} \underline{\eta}_i\right) = \underline{B}$$

i.e. $\bigcup_{i=1}^{s} \underline{\eta}_i$ is a solution of the system $(*)$.

Example 2.13: Consider the system:

$$\begin{bmatrix} \{1,2\} & \{2\} \\ \{2,3\} & \{1,2\} \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \{1\} \\ \{1,3\} \end{bmatrix} \qquad (*) \quad \text{of the form } A \dot{\times} \underline{X} = \underline{B}.$$

$K := \bigcup A = \{1,2,3\}$. Let $K_1 := \{1\}, K_2 := \{2\}, K_3 := \{3\}$

$A_1 := K_1 \cap A = \begin{bmatrix} \{1\} & \phi \\ \phi & \{1\} \end{bmatrix}$, $\underline{B}_1 := K_1 \cap \underline{B} = \begin{bmatrix} \{1\} \\ \{1\} \end{bmatrix}$, hence we have the component $(*_1)$:

$$\begin{bmatrix} \{1\} & \phi \\ \phi & \{1\} \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \{1\} \\ \{1\} \end{bmatrix} \qquad (*_1), \text{ which has the solution: } \begin{bmatrix} \{1\} \\ \{1\} \end{bmatrix}.$$

$A_2 := K_2 \cap A = \begin{bmatrix} \{2\} & \{2\} \\ \{2\} & \{2\} \end{bmatrix}$, $\underline{B}_2 := K_2 \cap \underline{B} = \begin{bmatrix} \phi \\ \phi \end{bmatrix}$, hence we have the component $(*_2)$:

$$\begin{bmatrix} \{2\} & \{2\} \\ \{2\} & \{2\} \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \phi \\ \phi \end{bmatrix} \qquad (*_2), \text{ which has the solution: } \begin{bmatrix} \phi \\ \phi \end{bmatrix}.$$

$A_3 := K_3 \cap A = \begin{bmatrix} \phi & \phi \\ \{3\} & \phi \end{bmatrix}$, $\underline{B}_3 := K_3 \cap \underline{B} = \begin{bmatrix} \phi \\ \{3\} \end{bmatrix}$, and we have the component $(*_3)$:

$$\begin{bmatrix} \phi & \phi \\ \{3\} & \phi \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \phi \\ \{3\} \end{bmatrix} \qquad (*_3), \text{ which has the two solutions: } \begin{bmatrix} \{3\} \\ \{3\} \end{bmatrix}, \begin{bmatrix} \{3\} \\ \phi \end{bmatrix}.$$

It follows that the system $(*)$ has the two solutions:

$$\begin{bmatrix} \{1\} \\ \{1\} \end{bmatrix} \cup \begin{bmatrix} \phi \\ \phi \end{bmatrix} \cup \begin{bmatrix} \{3\} \\ \{3\} \end{bmatrix} = \begin{bmatrix} \{1,3\} \\ \{1,3\} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \{1\} \\ \{1\} \end{bmatrix} \cup \begin{bmatrix} \phi \\ \phi \end{bmatrix} \cup \begin{bmatrix} \{3\} \\ \phi \end{bmatrix} = \begin{bmatrix} \{1,3\} \\ \{1\} \end{bmatrix},$$

which one can verify directly by solving the system $(*)$.

Now, we give the following example for a system, which has no solution, and we show that one of its components has no solution.

Example 2.14: We consider the system:

$\begin{bmatrix} \{1,2\} & \{2\} \\ \{2,3\} & \{1,2\} \end{bmatrix} \dot{\times} \underline{X} = \begin{bmatrix} \{1,2\} \\ \{1\} \end{bmatrix}$, in the form $A \dot{\times} \underline{X} = \underline{B}$, which has no solution, as one can directly verify.

$K := \bigcup A = \{1,2,3\}$. First, let $K_1 := \{1\}, K_2 := \{2\}, K_3 := \{3\}$.

$A_2 := K_2 \cap A = \begin{bmatrix} \{2\} & \{2\} \\ \{2\} & \{2\} \end{bmatrix}$, $B_2 := \begin{bmatrix} \{2\} \\ \phi \end{bmatrix}$; the component $(*_2)$ will be:

$\begin{bmatrix} \{2\} & \{2\} \\ \{2\} & \{2\} \end{bmatrix} \dot{\times} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \{2\} \\ \phi \end{bmatrix}$, which has no solution.

Second, if we define $K_1 := \{1,2\}, K_2 := \{3\}$, then the component

$A_1 := K_1 \cap A = \begin{bmatrix} \{1,2\} & \{2\} \\ \{2\} & \{1,2\} \end{bmatrix}$, $B_1 := \begin{bmatrix} \{1,2\} \\ \{1\} \end{bmatrix}$, and the corresponding component in this case has no solution.

Finally, we define $K_1 := \{2,3\}, K_2 := \{1\}$. In this case we have:

$A_1 := K_1 \cap A = \begin{bmatrix} \{2\} & \{2\} \\ \{2,3\} & \{2\} \end{bmatrix}$, $B_1 := \begin{bmatrix} \{2\} \\ \phi \end{bmatrix}$, and the corresponding component in this case also has no solution. (The case $K_1 := \{1,3\}, K_2 := \{2\}$ is similar to the first one). So, always in all cases one of the components has no solution.

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Delta J. Sci. (Math) 32: 21-33, (2008)

Some Set Matrix Systems

فؤاد غرابه والاستاذ الدكتور محمد عيد العظيم سعود

قمنا فى هذا البحث بحل بعض انظمه معادلات الفئات فى حالات خاصه ثم وضحنا ان هناك على وجه العموم فروق فى مجموعه حل النظام والنظام المرافق له . ثم قمنا بعمل تجزىء للنظام الـى مجموعـه مـن الانظمـه المتباعده اطلقنا عليها مركبات النظام مما سهل كثيرا معرفه إن كان النظام الاصلى له حل ام لا ثم ايجاد الحلـول فى حاله وجود حل.