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Associated graphs and chain maps

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Abstract:

In this paper, we defined the associated graph constructed to a cellular folding defined on regular CW-complexes. These graphs declare the effect of a cellular folding on the complex.

Besides we studied the properties of this graph and we proved that it is connected and vertex transitive if the cellular folding is neat.

Finally, by using chain maps and homology groups we obtained the necessary and sufficient conditions for a cellular map to be cellular folding and neat cellular folding respectively.

Key words:

Cellular folding, chain map, regular CW-complexes, vertex transitive, neat folding.

1-Introduction:

The study of foldings of a manifold into anther manifold began with S.A. Robertson's work on isometric folding of Riemannian manifolds [10]. After several attempts of generalizing the notion of isometric foldings, regular foldings were first studies by S.A. Robertson, H.R. Forran and E.El-Kholy [2]. The notion of cellular foldings is invented by E.El-Kholy and H.A.AL-Khurassani [1]. Different types of foldings are introduced by E.EL-Kholy and others [3, 4, 2].

(a) A cell decomposition of a topological space X is a is closed in X for each $e \in \zeta$, [8].

A CW-complex is said to be regular if all its attaching maps are homemorphisms. If each closed n-cell is homeomorphic to a closed Euclidean n-cell [8]. A topological space that admits the structure of a regular CW-complex is termed a regular CW-space.

(b) Let K and L be cellular complexes and $f: |K| \to |L|$ a continuous map. Then $f: K \to L$ is a cellular map if

(i) for each cell $\sigma \in K$, $f(\sigma)$ is a cell in L,

(ii) $\dim(f(\sigma)) \le \dim(\sigma)$, [7].

(c) Let K and L be regular CW-complexes of the same dimension and K be equipped with finite cellular subdivision such that each closed n-cell is homeomorphic to a closed Euclidean n-cell. A cellular map $f: K \to L$ is a cellular folding

decomposition of X into disjoint open cells such that for each cell e of the decomposition, the boundary $\partial e = \bar{e} - e$ is a union of lower dimensional cells of the decomposition. The set of cells of a cell decomposition of a topological space is called cell complex, [9].

A pair (X, ζ) consisting of a Hausdorff space X and a cell-decomposition ζ of X is called a CW-complex if the following three axioms are satisfied:

1- (Characteristic Maps): For each n-cell $e \in \zeta$ there is a continuous map $\Phi_e : D_n \to X$ restricting to a homeomorphism $\Phi_e|_{\text{int}(D_n)} : \text{int}(D_n) \to e$ and taking S^{n-1} into X^{n-1}.

2-(Closure Finiteness): For any cell $e \in \zeta$ the closure $\bar{e}$ intersects only a finite number of other cells in ζ.

3-(Weak Topology): A subset $A \subseteq X$ is closed iff $A \cap \bar{e}$

This set associates a cell decomposition C_f of M. If M is a surface, then the edges and vertices of C_f form a graph Γ_f embedded in M, [6].

(e) Let $f:|K|\to|L|$ be a continuous function. If, for each k-chain C in K, f(C) is a k-chain in L and if the diagram

$$\begin{array}{ccc} C_k(K) & \xrightarrow{f} & C_k(L) \\ \partial & & \partial\downarrow \\ C_{k-1}(K) & \xrightarrow{f} & C_{k-1}(L) \end{array}$$

commutes, then $f:K\to L$ is a chain function from K to L, [7].

(f) The set S_n of all permutations on n objects forms a group of order $n!$, called the symmetric group of degree n, the law of composition being that for maps of the objects onto themselves. A group of permutations is said to be transitive if, given any pair of letters a, b (which need not be distinct), iff : (i) for each i-cell $\sigma^i\in K$, $f(\sigma^i)$ is an i-cell in L, i.e., f maps i-cells to i-cells,

(ii) if $\overline{\sigma}$ contains n vertices, then $\overline{f(\sigma)}$ must contains n distinct vertices.

In the case of directed complexes it is also required that f maps directed i-cells of K to i-cells of L but of the same direction, [5].

A cellular folding $f:K\to L$ is neat if $L^n - L^{n-1}$ consists of a single n-cell, interior L. The set of all cellular foldings of K into L is denoted by $C(K, L)$ and the set of all neat foldings of K into L by $\mathcal{N}(K, L)$.

(d) If $f\in C(K,L)$, then $x\in K$ is said to be a singularity of f iff f is not a local homeomorphism at x. The set of all singularities of f corresponds to the "folds" of the map.

can join v to v' by an arc e in R^3 that runs from v through σ and σ' to v' crossing E transversely at a single point. The correspondence $\sigma \leftrightarrow v$, $E \leftrightarrow e$ is trivially a graph isomorphism from G_f to $\widetilde{G}_f$.

It should be noted that the graph G_f has no multiple edges, no loops and generally disconnected.

In this paper by a a complex we mean a regular CW-complex.

Examples(2-1):

(a) Let K be a complex with the cellular subdivisions given in Fig.(1-a). Let $f: K \to K$ be a cellular folding defined by $f(v_2, v_5, v_8, v_{11}) = (v_4, v_7, v_{10}, v_{13})$, $f(e_1, e_4, e_6, e_9, e_{11}, e_{14}, e_{16}, e_{19}, e_{21}) = (e_3, e_5, e_8, e_{10}, e_{13}, e_{15}, e_{18}, e_{20}, e_{23})$ and $f(\sigma_i)=\sigma_{i+1}$, $i = 1, 3, 5, 7, 9$, where the omitted 0, 1, 2-cells through this paper will be mapped to themselves. The graph G_f in this case has ten vertices and five edges as shown in Fig.(1-b).

there exists at least one permutation in the group which transforms a into b, [11]. Otherwise the group is called in transitive. And is said to be 1-transitive if for any pair of letters a, b, there exists a unique element x of the group such that $a * x = b$.

2-The associated graph:

Let $f: K \to L$ be a cellular folding. By using the cell subdivision C_f of K we can define the associated graph G_f constructed from the n-cells of K and the cellular folding f as follows:

The vertices of G_f are just the n-cells of K and if σ and σ' are distinct n-cells of K such that $f(\sigma) = f(\sigma')$, then there exists an edge E with end points σ and σ'. We then say that E is an edge in G_f with end points σ, σ'.

The graph G_f can be realized as a graph $\widetilde{G}_f$ embedded in R^3 as follows. For each n-cells σ, σ' choose any points $v \in \sigma$, $v' \in \sigma'$. If σ and σ' are end points of an edge E, then we a cellular subdivision consists of eight 0-cells, sixteen 1-cells and eight 2-cells, see Fig.(3). Let $f:K\to K$ be a cellular folding defined by: $f(v_5, v_6, v_7, v_8) = (v_1, v_3, v_3, v_3)$,

$, e_5, e_6, e_8, e_{11}, e_{12}, e_{13}, e_{14}) = (e_9, e_9, e_9, e_9,$

$e_{15}, e_7, e_9, e_{10}, e_{16}, e_{15}, e_{16})$ and

$f(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_8)=(\sigma_6,\sigma_6,\sigma_7,\sigma_7,\sigma_6,\sigma_7)$.

This can be done by the composition of the following two cellular foldings: $f_1(v_5, v_8) = (v_1, v_3)$,

$f_1(e_1,e_2,e_6,e_8,e_{11},e_{13},e_{14}) = (e_3,e_4,e_7,e_9,e_{10},e_{15},e_{16})$ and

$f_1(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=(\sigma_5,\sigma_6,\sigma_7,\sigma_8)$.

$f_2(v_6,v_7)=(v_3,v_3)$,

$f_2(e_3,e_4,e_5,e_{12})=(e_9,e_9,e_{15},e_{16})$ and

$f_2(\sigma_5,\sigma_8)=(\sigma_6,\sigma_7)$.

The graph G_f in this case has eight vertices and twelve edges see Fig.(3-b).

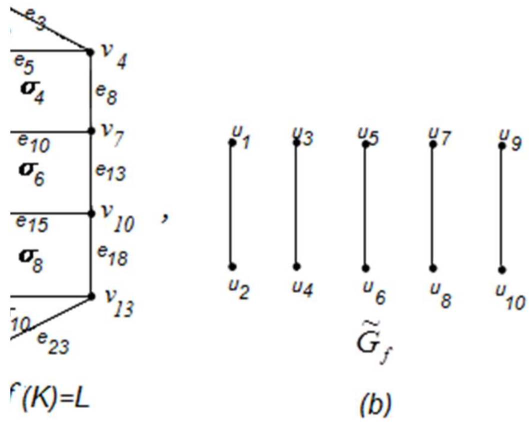


Fig.(1)

b) Consider the complex K **(**shown in Fig.(2), which consists of one 2-cell, seven 1-cells and seven 0-cells. Let $f:K\to K$ be a cellular folding defined as follow: $f(v_5, v_6, v_7) = (v_2, v_3, v_2)$, $f(e_i) = e_2$, $i = 5,6,7$ and $f(\sigma) = \sigma$. The graph G_f in this case consists of a vertex only with no edges.

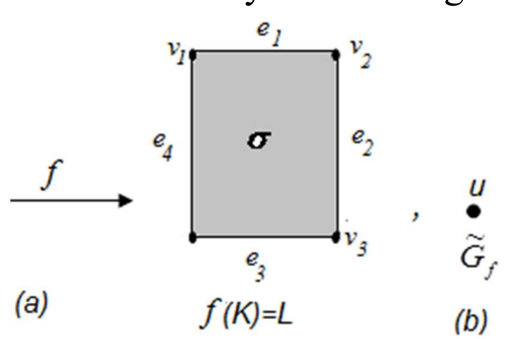


Fig.(2)

(c) Let K be a complex such that $|K|$ is a cylindrical surface with

$f(e_3, e_4) = (e_2, e_1)$ and $f(\sigma_2, \sigma_4) = (\sigma_1, \sigma_3)$. The graph G_f

in this case has four vertices and two edges, see Fig.(4-b).

Fig.(4)

3-Properties of the associated graph:

Some of the properties of the associated graph can be characterized by the following theorems:

Theorem (3-1):

Let K and L be complexes of the same dimension n, $f \in C(K,L)$. The associated graph G_f is disconnected unless f is a neat cellular folding.

Proof:

Let σ_1 and σ_2 be distinct n-cells of $K^{(n)}$, and let $\sigma_1 \sim \sigma_2$ means $f(\sigma_1) = f(\sigma_2)$. It is clear that the relation $\sim$ is an equivalence relation. Hence the quotient set $K^{(n)}/\sim = \{[\sigma], \sigma \in K^{(n)}\}$ is a partition on $K^{(n)}$, where $[\sigma]$ is the equivalence class of any n-cell σ. It follows that G_f has more than one

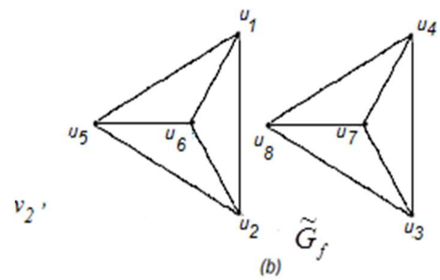
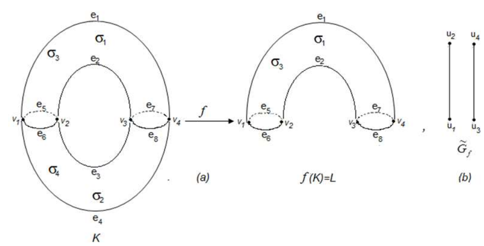


Fig.(3)



(d) Consider a complex K such that $|K|$ is a tours with four 0-cells, eight 1-cells and four 2-cells, see Fig.(4-a). Let $f: K \to K$ be a cellular folding given by: $f(v_i) = v_i$, $i = 1, 2, 3, 4$, vertices in the component, then any permutation of the set $V(G_f^i)$ is an automorphism of G_f^i. Thus the set of all permutations (automorphisms) form a group which is the symmetric group S_r acting on the set $V(G_f^i)$. The orbit of any $\sigma \in V(G_f^i)$ under S_r is the whole set $V(G_f^i)$, i.e., $V(G_f^i)$ has a single orbit and hence the automorphism group S_r is transitive on $V(G_f^i)$.

Results(3-3):

Let $f:K \to L$ be a neat cellular folding:

1) The symmetric group S_r, $r = \left|K^{(n)}\right|$ acts 1-transitively on the graph G_f.

2) G_f is vertex transitive.

3) From the above results we conclude that the graph G_f of a neat cellular folding is a complete graph.

Example (3-4):

Consider the complex K shown in Fig.(5-a), which consists of four 2-cells, eight 1-cells and five 0-cells. Let $f:K \to K$ be a cellular folding defined as follows: f component otherwise all the n-cells of K will be mapped to the same n-cell of L which in fact is the case of cellular neat folding. In the last case there will be a unique equivalence class $[\sigma]$ and hence the graph G_f is connected.

It follows from the above theorem that the components of the graph G_f is equal to the number of the equivalence classes generated by the relation ~.

Theorem (3-2):

Let K and L be complexes of the same dimension n, $f \in C(K, L)$ a cellular folding. Then each component of G_f is vertex transitive on itself.

Proof:

From Theorem(3.1) the equivalence relation defined on the n-cells $K^{(n)}$ of K defines a partition $\{[\sigma], \sigma \in K^{(n)}\}$ on $K^{(n)}$, where each equivalence class represents a component of G_f. Now, consider one of these components G_f^i, with say r vertices, i.e., $\left|V(G_f^i)\right| = r$. Each vertex of G_f^i is adjacent to the other

$\sigma \in K$ we can define a homomorphism $f_p : C_p(K) \rightarrow C_p(L)$ by:

$$f_p = \begin{cases} f(\sigma), & \text{if } f(\sigma) \text{ is a } p-\text{cell} \\ \varphi, & \text{if } \dim(f(\sigma)) < p \end{cases}$$

And since cellular foldings map p-cells to p-cells [5], $f_p(\sigma_\lambda)$ is a p-cell in L for all λ. Thus for a p-chain $C = a_1\sigma_1^p + a_2\sigma_2^p + \ldots + a_k\sigma_k^p \in C_p(K)$, where a_λ'$s \in Z$ and σ_λ's are p-cells in M,

$$f_p(C) = f_p(a_1\sigma_1^p + a_2\sigma_2^p + \ldots + a_k\sigma_k^p) = a_1 f_p(\sigma_1^p) + a_2 f_p(\sigma_2^p) + \ldots + a_k f_p(\sigma_k^p) \in C_p(L).$$

Now, since the closures of both σ_λ^p and $f(\sigma_\lambda^p)$ have the same number of distinct vertices, then $f_{p-1} o \partial_p = \partial'_p o f_p$, where $\partial_p : C_p(K) \rightarrow C_{p-1}(K)$ and $\partial'_p : C_p(L) \rightarrow C_{p-1}(L)$ are the boundary operators, that is to say the following diagram commutes

$$\begin{array}{ccc} C_p(K) & \xrightarrow{f_p} & C_p(L) \\ & & \downarrow \partial'_p \; \partial_p \\ C_{p-1}(K) & \xrightarrow{f_{p-1}} & C_{p-1}(L) \end{array}$$

$(v_4, v_5) = (v_3, v_2)$, $f(e_4, e_5, e_6, e_7, e_8) = (e_3, e_1, e_2, e_2, e_2)$ and $f(\sigma_i) = \sigma_1$, $i = 1,2,3,4$. The graph G_f in this case is complete, see Fig(5-b).

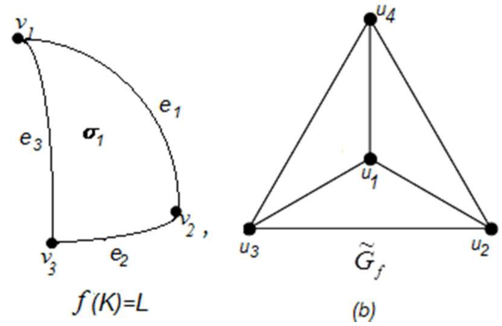


Fig.(5)

(4) Chain maps and cellular folding:

The following theorem gives the necessary and sufficient condition for a cellular map to be a cellular folding.

Theorem(4-1):

Let K and L be complexes of the same dimension n and $f : K \rightarrow L$ be a cellular map such that $f(K) = L \neq K$. Then f is a cellular folding if and only if the map $f_p : C_p(K) \rightarrow C_p(L)$, between chain complexes $(C_p(M), \partial_p)$, $(C_p(N), \partial'_p)$ is a chain map.

Proof:

Let $f : K \rightarrow L$ be a cellular folding, then it is a cellular map and for each p-cell

$f(e_i) = e_1^{/}, i = 1, 11, 21, \ldots, f(e_i) = e_2^{/}, i = 2, 12, 22, \ldots, f(e_i) = e_3^{/}, i = 3, 8, 13, \ldots, f(e_i) = e_4^{/}, i = 4, 9, 14, \ldots, f(e_i) = e_5^{/}, i = 5, 10, 15, \ldots, f(e_i) = e_6^{/}, i = 6, 16, 26, \ldots, f(e_i) = e_7^{/}, i = 7, 17, 27, \ldots$ *and* $f(\sigma_i) = \begin{cases} \sigma_1^{/}, & if\ i\ is\ odd, \\ \sigma_2^{/}, & if\ i\ is\ even \end{cases}.$

is a cellular folding.

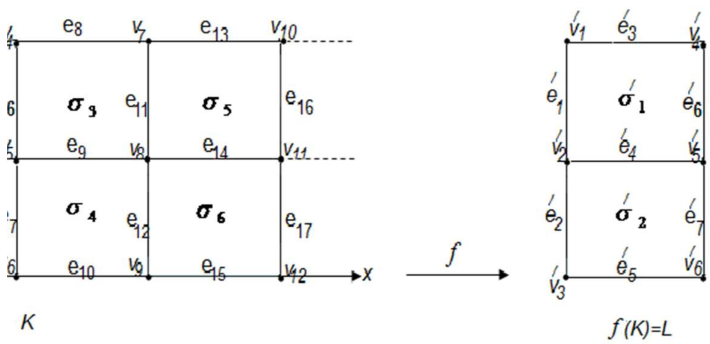


Fig.(6)

(b) Consider a complex K such that $|K| = S^2$, with cellular subdivision consisting of two 0-cells, four 1-cells and four 2-cells. Let $f : K \to K$ be a cellular map defined by: $f(e_2, e_4) = (e_1, e_3)$ and $f(\sigma_i) = \sigma_1$, $i = 1, \ldots, 4$.

This map is a cellular folding with image consisting of two and hence f_p is a chain map.

Conversely, suppose f is not a cellular folding then there exists a j-cell σ in K such that $f(\sigma)$ is an m-cell in L, where $j \neq m$. Since f_p is a homomorphism from the p^{th}-chain of K to the p^{th}-chain of L, then

$$f_j\left(\sum_{i=1}^{n-1} \lambda_i \sigma_i^{(j)} + \lambda_n \sigma\right) = \sum_{i=1}^{n-1} \lambda_i f_j(\sigma_i^{(j)}) + \lambda_n f(\sigma),$$

but $f(\sigma)$ is not a j-cell, then f_j cannot be a j-chain map and hence our assumption is false, and we have the result.

<u>Examples (4-2):</u>

(a) Let K be a complex such that $|K|$ is the infinite strip $\{(x, y) : 0 \leq x \leq \infty,$ $0 \leq y \leq l\}$ equipped with an infinite number of 2-cells such that the closure of each 2-cell consists of four 0-cells and four 1-cells, P_4. Let L be a complex with six 0-cells, seven 1-cells and two 2-cells, see Fig.(6). The cellular map $f : K \to L$ defined by:

$f(v_i) = v_i^{/}$ where $i = 1, 2, \ldots, 6$,

$f(v_i) = v_j^{/}$, where $j = 1, 2, \ldots, 6$ and $(i - j)$ is a multiple of 6, 1-cells and four 2-cells, see Fig.(9).

Let $f: K \to K$ be a cellular map defined by $f(v_i) = v_i$, $i = 1, \ldots, 4$, $f(e_2, e_3) = (e_1, e_4)$ and $f(\sigma_i) = \sigma_2$, $i = 1, \ldots, 4$. This map is not a cellular folding since $\overline{\sigma_1}$ and $\overline{f(\sigma_1)}$ do not contain the same number of vertices.

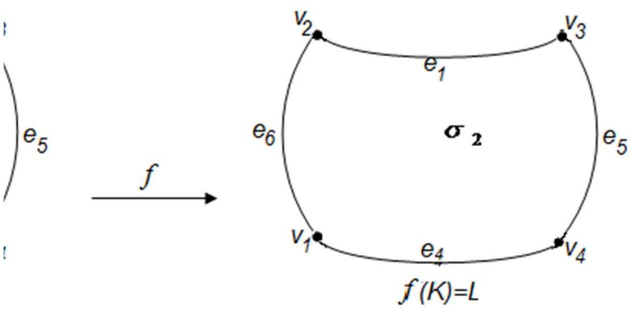


Fig.(9)

Result (4-3):

Let $f: K \to L$, be a cellular folding. Then the induced homomorphism $f_p^*: H_p(K) \to H_p(L)$ will maps the generators of $H_p(K)$ to either the generators of L or to zeros. This follows directly from the fact that the chain map $f_p: C_p(K) \to C_p(L)$ defines a homomorphism that has this property [5].

(5)Homology groups and neat cellular foldings:

The following theorem gives the necessary and sufficient condition for a

0-cells, two 1-cells and a single 2-cell, see Fig.(7).

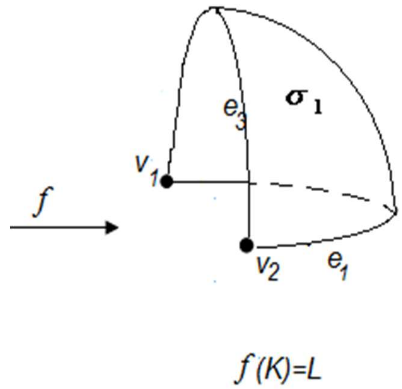


Fig.(7)

(c) Consider a complex K such that $|K|$ is a tours with cellular subdivision consisting of three 0-cells, six 1-cells and three 2-cells. Any cellular map $f: K \to K$ which has two vertices in the image is not a cellular folding since f_1 in this case is not a chain map, see Fig.(8).

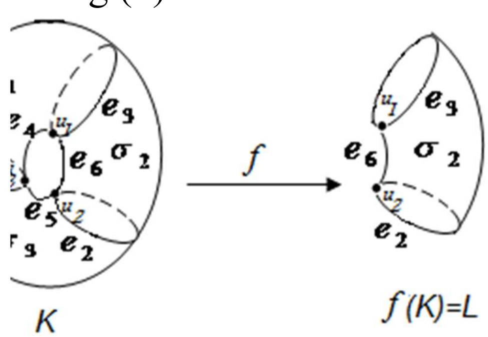


Fig.(8)

(d) Consider a complex K such that $|K| = S^2$, with cellular subdivision consisting of four 0-cells, six

The exactness of this sequence implies that $H_p(K) \cong \ker f_*$.

Conversely, suppose f is a chain map between chain complexes and $H_p(K) \cong \ker f_*$ but f is not neat, then $L^n - L^{n-1}$ consists of more than one n-cell. Thus $H_0(L) \cong Z^j, H_p(L) = 0$, for $p = 1,2,...,n$

and

$H_p(K) \cong H_p(L) \oplus \ker f_* \cong \ker f_*$ for $p = 0$, and hence the assumption is false and f is neat.

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cellular map to be a neat cellular folding.

Theorem (5-1):

Let K and L be complexes of the same dimension n.

If $f \in C(K, L)$, then f is neat if and only if the map $f_p : C_p(K) \to C_p(L)$ between chain complexes $(C_p(M), \partial_p)$, $(C_p(N), \partial'_p)$ is a chain map and $H_p(K) \cong \ker f_*$, where $f_* : H_p(K) \to H_p(L)$, $p \geq 1$ is the induced homomorphisms.

Proof:

Assuming that f is a neat folding, then it is a cellular folding and hence the map $f_p : H_p(K) \to H_p(L)$ between the chain complexes $(C_p(K), \partial_p), (C_p(L), \partial'_p)$ is a chain map. Now consider the induced homomorphism $f_* : H_p(K) \to H_p(L)$, there is a short exact sequence

$$0 \to \ker f_* \xrightarrow{i^*} H_p(K) \xrightarrow{f_*} \operatorname{Im} f_*$$

where i^* is the induced homomorphism by the inclusion. Since f surjective, we have $\operatorname{Im} f_* \cong H_p(L)$, but $H_p(L) = 0$ for neat cellular foldings, hence the above sequence will take the form

$$0 \to \ker f_* \xrightarrow{i^*} H_p(K) \to 0$$

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المخلص العربى

المخططات المنشأه والدوال السلسلية

فى هذا البحث تم تعريف المخطط المنشأ G_f والمرتبط بالطى الخلوى على التراكيب -CW المنتظمه. هذه المخططات توضح تأثير الطى الخلوى على المركب. بجانب ذلك قدمنا خواص هذا المخطط وأثبتنا إنه مخطط مترابط وله تأثير متعد على الرؤوس (vertex transitive) إذا كان الطى الخلوى صافى. وأخيرا بإستخدام

الدوال السلسلية والزمر الهومولوجية حصلنا على الشرط الكافى والضرورى لجعل الدالة الخلويه طى خلوى وطى خلوى صافى على التوالى.

أولا: تم تقديم تعريف المخطط المنشأ مع إعطاء بعض من الأمثلة التى توضح هذا التعريف.

ثانيا: تم توضيح خواص هذا المخطط للطى الخلوى وللطى الصافى على التوالى وأثبتنا التالى:

(1) المخطط المنشأ يكون غير مترابط إلا إذا كان الطى الخلوى هو طى صافى.

(2) لأى طى خلوى يكون كل مركب من مركبات المخطط المنشأ هو تأثير متعد على رؤوس المركبة.

ثالثا: درسنا حالة أن تكون الدالة الخلويه هى طى خلوى وحصلنا على الشروط المتحققه بواسطة المخططات المنشأة للحصول على الطى المتتابع.

رابعا: درسنا نفس المشكلة ولكن بالنسبه للطى الصافى ولقد حصلنا على الشروط المتحققه بدلالة الزمر الهومولوجية.