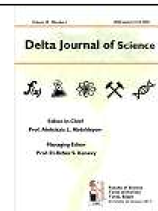
DJS Vol. 39 (2018) 9-17

Delta Journal of Science

Available online at

https://djs.journals.ekb.eg/



Research Article

MATHEMATICS

Probabilistic Mixture Shortage Multi-Source Inventory Model with Varying Holding Cost Under Constraint

Hala A. Fergany and Magda A. Gomaa

Mathematics Department, Faculty of Science, Tanta University, Tanta 31527, Egypt.

* ***Correspondence****: Magda A. Gomaa; Mathematics Department, Faculty of Science, Tanta University, Tanta 31527, Egypt; Email:* magda.farag@science.tanta.edu.eg

Key words:

Probabilistic inventory model; Mixture shortage; Lagrange multiplier approach; Varying holding cost; Trapezoidal fuzzy number and Continuous distributions.

.

Abstract:

This paper proposed a general probabilistic continuous review multi-item, multi-source inventory model with constraint for crisp and fuzzy environment. This constraint on the expected varying holding cost. The demand is a continuous random variable, the distribution of the lead time demand is known and the holding cost is varying. This model is formulated to analyze how the firm can deduce the optimal order quantity and the optimal reorder point for the item and source to reach the main goal of minimizing the expected total cost using a Lagrange multiplier technique. The lead time demand is follow some continuous distributions. Also, an application with real data is analyzed and the goal of minimization the expected total cost is achieved.

Introduction:

The multi-item, multi-source inventory system is the most general procurement system which may be described as follows; an inventory of i-items is maintained to meet the average demand rates designated , , ,..., .Most of the probabilistic inventory models assume that the units of cost are constant or one of these units is varying. Hundreds of papers and books have been published presenting models for doing this under a wide variety of conditions and assumptions [12] discussed analysis of inventory system [5] explained procurement and inventory system: theory and analysis. An application of the system-point method to inventory models under continuous review has been studied by [2]. [14] presented inventory model with a mixture of backorders and lost sales. [11] Explained unconstrained probabilistic inventory problems with constant units of cost. [1] Studied the probabilistic multi-item single- source inventory model with varying order cost under two restrictions. [6] Deduced probabilistic single-item inventory problem with varying order cost under two linear constraints. Constrained periodic review probabilistic inventory model with continuous distributions and varying holding cost have been studied by [7]. [13] Discussed the optimal and near -optimal policies for lost sales inventory models with at most one replenishment order outstanding. [8] Examined Probabilistic multi-item inventory model with varying mixture shortage cost under restrictions. [3] Deduced a multi-item single-source mixture inventory model involving random lead time and demand with budget constraint and surprise function. [16] Studied a two-demand-class inventory system with lost sales and backorders. [10] Studied Multi-item EOQ model with varying holding cost: a geometric programming approach.

The cost parameters in real inventory systems and other parameters such as price marketing and service elasticity to demand are imprecise and uncertain in nature. So, the notion of fuzziness can be applied to cope with this uncertainly. Since the proposed model is in a fuzzy environment, a fuzzy decision should be made to meet the decision criteria, and the results should be fuzzy. Fuzzy sets introduced by many researchers as a mathematical way of representing impreciseness or vagueness in everyday life. [4] Introduced fuzzy inventory with backorder defuzzification by signed distance method. [9] Deduced Probabilistic periodic review <Q,r> inventory model using LaGrange technique and fuzzy adapting particle swarm optimization. [15] Presented deterministic inventory models with a mixture of backorders and lost sales under fuzzy cost. In this paper, we investigate a probabilistic multi-item, multi-source (MIMS) continuous review inventory model with mixture shortage, varying holding unit cost, under the expected varying holding cost constraint for crisp and fuzzy environment. The expected total cost of inventory system is composed of three components (the expected order cost, the expected varying holding cost and the expected mixture shortage cost). The optimal solutions of the order quantity (Q^*_{is}), the reorder point (r^*_{is}), which minimize the expected total cost, E(TC(Q_{is}, r_{is})), using Lagrange technique, are obtained mathematically. Also, the model will be studied when the lead time demand follows Gamma, Weibull, Chi-square, Erlang and Exponential distributions and an application is added to illustrate the model for crisp and fuzzy environment.

2. Model Notions

The following notations are adopted for developing our model:

$\overline{D}_i$	The average annual demand for the i^{th} item.
Q_{is}	The decision variable representing the order quantity per cycle for the i^{th} item and s^{th} source.
r_{is}	The decision variable representing the reorder point per cycle for the i^{th} item and s^{th} source.
C_{ois}	The inventory order cost per unit per cycle for the i^{th} item and s^{th} source.
C_{hi}	The inventory holding cost per unit per cycle for the i^{th} item.
$C_{hi}(Q_{is})$	The varying holding cost per unit per cycle for the i^{th} item and s^{th} source.
C_{bi}	The inventory backorder cost per unit per cycle for the i^{th} item.
C_{li}	The inventory lost sales cost per unit per cycle for the i^{th} item.
k_{hi}	The limitation on the expected annual holding cost for the i^{th}item.
λ_{his}	Lagrange multiplier for the i^{th} item and s^{th} source.
$E_{is}(OC)$	The expected order cost for the i^{th}item and s^{th} source.
$E_{is}(HC)$	The expected varying holding cost for the i^{th} item and s^{th}source.
$E_{is}(BC)$	The expected backorder cost for the i^{th} item and s^{th} source.
$E_{is}(LC)$	The expected lost sales cost for the i^{th} item and s^{th} source.
$E_{is}(SC)$	The expected shortage cost = E_{is}(BC) + E_{is}(LC) for the i^{th} item and s^{th} source.
Min E(TC)	The minimum expected annual total cost for the i^{th} item and s^{th}source $= \sum_{i=1}^{m} \min_s \mathrm{ETC}_{is}, \quad s = 1,2,\dots,\upsilon$

3. Model Development

The following assumptions are made for developing the mathematical model:

(1) The expected order cost is given by:

$E_{is}(OC) = C_{ois}\frac{\overline{D}_i}{Q_{is}}$

(2) The varying expected holding cost is given by:

$E_{is}(HC) = C_{hi}(Q_{is})n_i\overline{H}_i$

$= C_{hi}\left(\alpha + \beta Q_{is}{}^{-1}\right)[\frac{Q_{is}}{2} + r_{is} - \mu + (1-\gamma_i)\overline{S}(r_{is})]$

Where, the varying holding cost per unit per cycle is a function of the order quantity which takes the following form:

$C_{hi}(Q_{is}) = C_{hi}(\alpha + \beta Q_{is}{}^{-1})$, α, β > 0

Also, H_i represents the average inventory level during the cycle. $\overline{H}_i = \frac{H_i}{n_i}$, n_i is average of cycle per year and $n_i = \frac{\overline{D}_i}{Q_{is}}$

(3) The expected shortage cost is the mixture of the expected backorder cost and the expected lost sales cost as follows:

$E_{is}(SC) = E_{is}(BC) + E_{is}(LC)$

Where,

$E_{is}(BC) = C_{bi}\gamma_i\frac{\overline{D}_i}{Q_{is}}\overline{S}(r_{is})$ and

$E_{is}(LC) = C_{li}(1-\gamma_i)\frac{\overline{D}_i}{Q_{is}}\overline{S}(r_{is}),\ 0 \le \gamma_i \le 1$

The objective is to minimize the relevant expected total cost function (i.e. the sum of the expected order cost, the expected varying holding cost, the expected backorder cost and the expected lost sales cost) which, according to the previous assumptions of the model is

$E(TC(Q_{is}, r_{is})) = E(TC_{is})$

$$= \sum_{i=1}^{m}[E_{is}(OC) + E_{is}(HC) + E_{is}(SC)]$$

$$\min E(TC) = \min \sum E(TC_{is})$$

$$= \min \sum_{i=1}^{m}\left[\begin{array}{c}\frac{C_{ois}\overline{D}_i}{Q_{is}} + C_{hi}\left(\alpha + \beta Q_{is}{}^{-1}\right)[\frac{Q_{is}}{2} + r_{is} \\ -\mu + (1-\gamma_i)\overline{S}(r_{is})] + C_{bi}\frac{\gamma_i\overline{D}_i}{Q_{is}}\overline{S}(r_{is}) \\ +C_{li}(1-\gamma_i)\frac{\overline{D}_i}{Q_{is}}\overline{S}(r_{is})\end{array}\right] \quad (1)$$

Subject to the following expected varying holding cost constraint:

$$\sum_{i=1}^{m}[E_{is}(HC) \le k_{hi}] \quad (2)$$

Then to solve this function, under the above constraint, the Lagrange multipliers technique should be used as follows:

$$L = \sum_{i=1}^{m}[E(TC_{is}) + \lambda_{his}\{E_{is}(HC) - k_{hi}\}], \lambda_{his} > 0$$

$$= \sum_{i=1}^{m}\left[\begin{array}{c}\frac{C_{ois}\overline{D}_i}{Q_{is}} + C_{hi}\left(\alpha + \beta Q_{is}{}^{-1}\right)[\frac{Q_{is}}{2} + r_{is} - \mu \\ +(1-\gamma_i)\overline{S}(r_{is})] + C_{bi}\frac{\gamma_i\overline{D}_i}{Q_{is}}\overline{S}(r_{is}) \\ +C_{li}(1-\gamma_i)\frac{\overline{D}_i}{Q_{is}}\overline{S}(r_{is}) \\ +\lambda_{his}[C_{hi}\left(\alpha + \beta Q_{is}{}^{-1}\right)\{\frac{Q_{is}}{2} + r_{is} \\ -\mu + (1-\gamma_i)\overline{S}(r_{is})\} - K_{hi}]\end{array}\right] \quad (3)$$

The optimal values of the order quantity ($Q_{is}{}^*$) and reorder point ($r_{is}{}^*$) which are minimizing the expected total costfor the i^{th} item and s^{th} source, can be calculated by setting each of the corresponding first partial derivatives of (3) with respect to the two decision variables equal to zero, then the following is obtained:

$$\frac{\partial L}{\partial Q_{is}} = -\frac{C_{ois}\overline{D}_i}{Q_{is}{}^2} - \frac{C_{hi}\beta}{Q_{is}{}^2}\left[\begin{array}{c}\frac{Q_{is}}{2} + r_{is} - \mu \\ +(1-\gamma_i)\overline{S}(r_{is})\end{array}\right]$$

$$+ \frac{C_{hi}(\alpha + \beta Q_{is}{}^{-1})}{2} - \frac{C_{bi}\gamma_i\overline{D}_i}{Q_{is}{}^2}\overline{S}(r_{is})$$

$$-\frac{C_{li}(1-\gamma_i)\overline{D}_i}{Q_{is}{}^2}\overline{S}(r_{is}) - \lambda_{his}[C_{hi}\frac{\beta}{Q_{is}{}^2}\left[\begin{array}{c}\frac{Q_{is}}{2} + r_{is} - \mu \\ +(1-\gamma_i)\overline{S}(r_{is})\end{array}\right]$$

$$+ \frac{\lambda_{his}C_{hi}(\alpha + \beta Q_{is}{}^{-1})}{2},$$

Then, we can obtain the optimal order quantity and the optimal reorder point from the following equations:

$$\alpha A\, Q_{is}^{*2} = B + 2A\beta\ \{ r_{is}^{*} - \mu + (1-\gamma_i)\bar{S}(r_{is}^{*})\} + 2M\bar{S}(r_{is}^{*})$$

Where $A = (1+\lambda_{his})C_{hi}$, $B = 2C_{ois}\bar{D}_i$,

$$M = [C_{bi}\gamma_i\bar{D}_i + C_{li}(1-\gamma_i)\bar{D}_i]$$

i.e.

$$\alpha A Q_{is}^{*2} - B - 2A\beta\{r_{is}^{*} - \mu + (1-\gamma_i)\bar{S}(r_{is}^{*})\} - 2M\bar{S}(r_{is}^{*}) = 0 \quad (4)$$

also,

$$\frac{\partial L}{\partial r_{is}} = c_{hi}(\alpha + \beta Q_{is}^{-1})[1-(1-\gamma_i)R(r_{is}) - \frac{c_{bi}\gamma_i\bar{D}_i}{Q_{is}}R(r_{is}) - \frac{C_{li}(1-\gamma_i)\bar{D}_i}{Q_{is}}R(r_{is}) + \lambda_{his}[C_{hi}(\alpha + \beta Q_{is}^{-1})\{(1-(1-\gamma_i)R(r_{is})\}]$$

$$R(r_{is}^{*}) = \frac{A(\alpha Q_{is}^{*} + \beta)}{M + A(1-\gamma_i)(\alpha Q_{is}^{*} + \beta)} \quad (5)$$

and we can prove that:

$$\begin{cases} \left[\frac{\partial^2 L}{\partial Q_i^2}\right]\left[\frac{\partial^2 L}{\partial r_i^2}\right] - \left[\frac{\partial^2 L}{\partial Q_i \partial r_i}\right]^2 > 0 \\ \frac{\partial^2 L}{\partial Q_i^2} \quad or, \quad \frac{\partial^2 L}{\partial r_i^2} > 0 \end{cases}$$

It is clearly that there is no closed form solution, then an iterative method must be used to determine Q_{is}^{*} and r_{is}^{*} (as the following algorithm) which are used to determine the minimum expected total cost.

Algorithm

Step 1: Input all the inventory model data for example, expected demand value, order unit cost, holding unit cost, mean, etc. at one β value and assumption value of λ and put, $r_0 = \mu$ as an initial value so, $S_0 = 0$, then calculate the first order quantity Q_1.

Step 2: Use the calculated order quantity in step 1 to calculate $r_1$andS_1.

Step 3: Use the calculated r_1 and S_1 in step 2 to calculate a new order quantity Q_2.

Step 4: Repeat steps 1 and 2. If two values of respectively calculated order quantity are equaled, then it is the optimal $Q^{*}$and r^{*}.

Step 5: Using the calculated optimal order quantity Q^{*} and optimal reorder point r^{*} to calculate the expected total cost.

Step 6: Repeat all steps at changes values of λ to be the condition is active. If the condition is active, then it is the minimum expected total cost at this value of β.

Step 7: Repeat all steps at other values of β.

4. The Model When all Parameters are Trapezoidal Fuzzy Numbers

Consider inventory cost coefficients in the model are fuzzy in nature. Therefore, the decision variables and the objective function should be fuzzy as well, and we are interested in driving the membership functions of E(TC) by solving the model via Lagrange multipliers technique according to the assumptions, we can investigate the following formulation for the fuzzy inventory model:

$$E\left(\widetilde{TC}_{is}\right) = \sum_{i=1}^{m} \begin{bmatrix} \frac{\tilde{C}_{ois}\bar{D}_i}{Q_{is}} + \tilde{C}_{hi}(\alpha + \beta Q_{is}^{-1})[\frac{Q_{is}}{2} + r_{is} - \mu \\ +(1-\gamma_i)\bar{S}(r_{is})] \\ +\tilde{C}_{bi}\frac{\gamma_i\bar{D}_i}{Q_{is}}\bar{S}(r_{is}) \\ +\tilde{C}_{li}(1-\gamma_i)\frac{\bar{D}_i}{Q_{is}}\bar{S}(r_{is}) \end{bmatrix} \quad (6)$$

Subject to the following varying holding cost constraint:

$$\sum_{i=1}^{m} \tilde{C}_{hi}(\alpha + \beta Q_{is}^{-1})\,[\frac{Q_{is}}{2} + r_{is} - \mu + (1-\gamma_i)\bar{S}(r_{is})] \le \sum_{i=1}^{m} k_{hi} \quad (7)$$

To solve this inventory model using Lagrange multipliers technique, we should find the left and the right shape functions of the objective function and decision variables, by find the upper bound and the lower bound of the objective function, i.e. $\tilde{L}_v(\alpha)$ and $\tilde{L}_u(\alpha)$ respectively. Recall that $\tilde{L}_v(\alpha)$ and $\tilde{L}_u(\alpha)$ represent the smallest and largest values (The left and right α cuts) of the optimal objective function$\tilde{L}(\alpha)$.

Consider the model when all parameters are trapezoidal fuzzy numbers as given below

$\tilde{C}_{ois} = (C_{ois} - \delta_{1i}, C_{ois} - \delta_{2i}, C_{ois} + \delta_{3i}, C_{ois} + \delta_{4i})$,

$\tilde{C}_{hi} = (C_{hi} - \delta_{5i}, C_{hi} - \delta_{6i}, C_{hi} + \delta_{7i}, C_{hi} + \delta_{8i})$,

$\tilde{C}_{bi} = (C_{bi} - \delta_{9i}, C_{bi} - \delta_{10i}, C_{bi} + \delta_{11i}, C_{bi} + \delta_{12i})$,

and $\tilde{C}_{li} = (C_{li} - \delta_{13i}, C_{li} - \delta_{14i}, C_{li} + \delta_{15i}, C_{lr} + \delta_{16i})$.

Where δ_i, i = 1,2, …, 16 are arbitrary positive numbers satisfy the following restrictions:

$\tilde{C}_{ois} > \delta_{1i} > \delta_{2i}, \delta_{3i} < \delta_{4i}$,

$\tilde{C}_{hi} > \delta_{5i} > \delta_{6i}, \delta_{7i} < \delta_{8i}$,

$\tilde{C}_{bi} > \delta_{9i} > \delta_{10i}, \delta_{11i} < \delta_{12i}$ and

$\tilde{C}_{li} > \delta_{13i} > \delta_{14i}, \delta_{15i} < \delta_{16i}$.

The left and right limits α cuts of $C_{ois}, C_{hi}, C_{bi}, C_{li}$ are given by:

$\tilde{C}_{ois_v}(\alpha) = C_{ois} - \delta_{1i} + (\delta_{1i} - \delta_{2i})\alpha$, $\tilde{C}_{ois_u}(\alpha) = C_{ois} + \delta_{4i} - (\delta_{4i} - \delta_{3i})\alpha$,

$\tilde{C}_{hi_v}(\alpha) = C_{hi} - \delta_{5i} + (\delta_{5i} - \delta_{6i})\alpha$, $\tilde{C}_{hi_u}(\alpha) = C_{hi} + \delta_{8i} - (\delta_{8i} - \delta_{7i})\alpha$,

$\tilde{C}_{bi_v}(\alpha) = C_{bi} - \delta_{9i} + (\delta_{9i} - \delta_{10i})\alpha$, $\tilde{C}_{bi_u}(\alpha) = C_{bi} + \delta_{12i} - (\delta_{12i} - \delta_{11i})\alpha$ and

$\tilde{C}_{li_v}(\alpha) = C_{li} - \delta_{13i} + (\delta_{13i} - \delta_{14i})\alpha$, $\tilde{C}_{li_u}(\alpha) = C_{li} + \delta_{16i} - (\delta_{16i} - \delta_{15i})\alpha$.

Where $\tilde{C}_{ois} = \frac{1}{4}[4C_{ois} - \delta_{1i} - \delta_{2i} + \delta_{3i} + \delta_{4i}]$, $\tilde{C}_{hi} = \frac{1}{4}[4C_{hi} - \delta_{5i} - \delta_{6i} + \delta_{7i} + \delta_{8i}]$, $\tilde{C}_{bi} = \frac{1}{4}[4C_{bi} - \delta_{9i} - \delta_{10i} + \delta_{11i} + \delta_{12i}]$ and $\tilde{C}_{li} = \frac{1}{4}[4C_{li} - \delta_{13i} - \delta_{14i} + \delta_{15i} + \delta_{16i}]$.

Using approximated value of trapezoidal fuzzy numbers which observe in Figure 1.

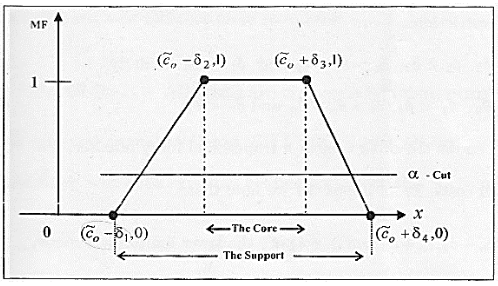


Figure (1): Order cost as trapezoidal fuzzy number.

By using the Lagrange multiplier technique, the above fuzzy system of equations (6) and (7) reduces to

$$\tilde{L} = \sum_{i=1}^{m} \begin{bmatrix} \frac{\tilde{C}_{ois}\overline{D}_i}{Q_{is}} + \tilde{C}_{hi}(\alpha + \beta Q_{is}^{-1})[\frac{Q_{is}}{2} + r_{is} \\ -\mu + (1 - \gamma_i)\overline{S}(r_{is})] \\ +\tilde{C}_{bi}\frac{\gamma_i \overline{D}_i}{Q_{is}}\overline{S}(r_{is}) \\ +\tilde{C}_{li}(1 - \gamma_i)\frac{\overline{D}_i}{Q_{is}}\overline{S}(r_{is}) \\ +\lambda_{his}[\tilde{C}_{hi}(\alpha + \beta Q_{is}^{-1})\{\frac{Q_{is}}{2} + r_{is} \\ -\mu + (1 - \gamma_i)\overline{S}(r_{is})\} - K_{hi}] \end{bmatrix} \quad (8)$$

Then, we can obtain the optimal order quantity and the optimal reorder point from the following equations:

$$\alpha A1 Q_{is}^{*2} - B1 - 2A1\beta\,\{r_{is}^{*} - \mu + (1 - \gamma_i)\overline{S}(r_{is}^{*})\} - 2M1\overline{S}(r_{is}^{*}) = 0 \quad (9)$$

Where $A1 = (1 + \lambda_{his})\tilde{C}_{hi}$, $B1 = 2\tilde{C}_{ois}\overline{D}_i$, $M1 = [\tilde{C}_{bi}\gamma_i\overline{D}_i + \tilde{C}_{li}(1 - \gamma_i)\overline{D}_i]$

and

$$R(r_{is}^{*}) = \frac{A1(\alpha Q_{is}^{*} + \beta)}{M1 + A1(1 - \gamma_i)(\alpha Q_{is}^{*} + \beta)} \quad (10)$$

By using the algorithm, we can deduce the optimal valuesQ_{is}^{*},r_{is}^{*}

Then, the model will be studied for crisp and fuzzy environment when the lead time demand follows some continuous distributions such as Gamma, Weibull, Chi-square, Erlang and Exponential distributions.

5. The Model with Continuous Distributions

Assume that the lead time demand follows some continuous distributions as follows:

5.1.The Gamma Distribution:

If the lead time demand follows the Gamma distribution with parameters p, θ, the density function is given by:

$$f(x) = \frac{\theta e^{-\theta x}(\theta x)^{p-1}}{\Gamma(p)}, \quad 0 < x < \infty \quad (11)$$

and the reliability function is given by:
$R(r) = \int_r^{\infty} f(x)dx = \frac{\Gamma(p,\theta r)}{\Gamma(p)}$, where
$\Gamma(p, \theta r)$: upper incomplete Gamma

$$R(r) = e^{-\theta r}\sum_{j=0}^{p-1}\frac{(\theta r)^j}{j!}, \quad (12)$$

and the expected shortage quantity is given by:

$$\bar{S}(r) = \int_r^{\infty}(x-r)f(x)dx = e^{-\theta r}\left[\frac{p}{\theta}\frac{(\theta r)^p}{p!} - r\right]\sum_{j=0}^{p-1}\frac{(\theta r)^j}{j!} \quad (13)$$

So, the expected total cost can be minimized mathematically by substituting from (13) and (12) in to (4) and (5), respectively, for any i^{th} item and s^{th} source, it is found that the optimal values $Q_{is}{}^*$ and $r_{is}{}^*$ are given by:

$$\alpha A Q_{is}{}^{*2} - B - 2A\beta\{r_{is}{}^* - \mu + (1-\gamma_i)[e^{-\theta r}(\frac{p}{\theta}\frac{(\theta r)^p}{p!} - r)\sum_{j=0}^{p-1}\frac{(\theta r)^j}{j!}]\} - 2M\left[e^{-\theta r}\left(\frac{p}{\theta}\frac{(\theta r)^p}{p!} - r\right)\sum_{j=0}^{p-1}\frac{(\theta r)^j}{j!}\right] = 0 \quad (14)$$

$$R(r_{is}{}^*) = \frac{A(\alpha Q_{is}{}^* + \beta)}{M + A(1-\gamma_i)(\alpha Q_{is}{}^* + \beta)} = e^{-\theta r}\sum_{j=0}^{p-1}\frac{(\theta r)^j}{j!} \quad (15)$$

Hence,

If we put $p = \frac{n}{2}$, $\theta = \frac{1}{2}$ in equation (11), we found the lead time demand follows the **Chi-square** distribution, then equations (12) and (13) will be has the following form:

$$R(r) = \int_r^{\infty} f(x)dx = e^{-r/2}\sum_{j=0}^{\frac{n}{2}-1}\frac{(r/2)^j}{j!} \quad (16)$$

$$\bar{S}(r) = \int_r^{\infty}(x-r)f(x)dx = e^{-r/2}\left[n\frac{(r/2)^{\frac{n}{2}}}{(n/2)!} - r\right]\sum_{j=0}^{\frac{n}{2}-1}\frac{(r/2)^j}{j!} \quad (17)$$

The optimal values of the order quantity $(Q_{is}{}^*)$ and reorder point $(r_{is}{}^*)$ which are minimizing the expected total cost for the i^{th} item and s^{th} source, can be calculated by substituting from (17) and (16) in to (4) and (5), respectively, it is found that the optimal values $Q_{is}{}^*$ and $r_{is}{}^*$ are given by:

$$\alpha A Q_{is}{}^{*2} - B - 2A\beta[r_{is}{}^* - \mu + (1-\gamma_i\left\{e^{-r/2}\left(n\frac{(r/2)^{\frac{n}{2}}}{(n/2)!} - r\right)\sum_{j=0}^{\frac{n}{2}-1}\frac{(r/2)^j}{j!}\right\} - 2M\left[e^{-r/2}\left(n\frac{(r/2)^{\frac{n}{2}}}{(n/2)!} - r\right)\sum_{j=0}^{\frac{n}{2}-1}\frac{(r/2)^j}{j!}\right] = 0 \quad (18)$$

$$R(r_{is}{}^*) = \frac{A(\alpha Q_{is}{}^* + \beta)}{M + A(1-\gamma_i)(\alpha Q_{is}{}^* + \beta)} = e^{-r/2}\sum_{j=0}^{\frac{n}{2}-1}\frac{(r/2)^j}{j!} \quad (19)$$

which represent the optimal values when the lead time demand follows the **Chi-square** distribution with parameter n.

Also,

If we put p = q, where q is a positive integer number, in equation (11), we found the lead time demand follows the **Erlang** distribution and it is found that the optimal values $Q_{is}{}^*$ and $r_{is}{}^*$ are given by:

$$\alpha A Q_{is}{}^{*2} - B - 2A\beta\{r_{is}{}^* - \mu + (1-\gamma_i)\left[e^{-\theta r}\left(\frac{q}{\theta}\frac{(\theta r)^q}{q!} - r\right)\sum_{j=0}^{q-1}\frac{(\theta r)^j}{j!}\right]\} - 2M\left[e^{-\theta r}\left(\frac{q}{\theta}\frac{(\theta r)^q}{q!} - r\right)\sum_{j=0}^{q-1}\frac{(\theta r)^j}{j!}\right] = 0 \quad (20)$$

$$R(r_{is}{}^*) = \frac{A(\alpha Q_{is}{}^* + \beta)}{M + A(1-\gamma_i)(\alpha Q_{is}{}^* + \beta)} = e^{-\theta r}\sum_{j=0}^{q-1}\frac{(\theta r)^j}{j!} \quad (21)$$

which represent the optimal values when X follows the **Erlang** distribution with parameter θ, q.

And,

If we putp = 1, in equation (11), we found the lead time demand follows the **Exponential** distribution and it is found that the optimal values $Q_{is}{}^*$ and $r_{is}{}^*$ are given by:

$$\alpha A Q_{is}{}^{*2} - B - 2A\beta\{r_{is}{}^* - \mu + (1-\gamma_i)[\frac{1}{\theta}e^{-\theta r}]\} - 2M[\frac{1}{\theta}e^{-\theta r}] = 0,$$

$$R(r_{is}{}^*) = \frac{A(\alpha Q_{is}{}^* + \beta)}{M + A(1-\gamma_i)(\alpha Q_{is}{}^* + \beta)} = e^{-\theta r}$$

Which represent the optimal values when X follows the **Exponential** distribution with parameter θ.

5.2.The Weibull Distribution:

If the lead time demand follows the Weibull distribution with parameters σ, η, the density function is given by:

$$f(x) = \frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}e^{-\left(\frac{x}{\sigma}\right)^{\eta}}, \quad x > 0 \quad (22)$$

and the reliability function is given by:

$$R(r) = \int_r^\infty f(x)dx = e^{-(\frac{r}{\sigma})^{\eta}} \quad (23)$$

and the expected shortage quantity is given by:

$$\bar{S}(r) = \int_r^\infty (x-r)f(x)dx$$

$$= \sigma \left(\frac{1}{\eta}\right)!\ e^{-(\frac{r}{\sigma})^{\eta}} \sum_{j=0}^{1/\eta} \frac{[(r/\sigma)^{\eta}]^j}{j!} - r\, e^{-(\frac{r}{\sigma})^{\eta}} \quad (24)$$

The optimal values of the order quantity ($Q_{is}{}^*$) and reorder point ($r_{is}{}^*$) which are minimizing the expected total costfor the i^{th} item and s^{th} source, can be calculated by substituting from (24) and (23) in to (4) and (5), respectively, it is found that the optimal values $Q_{is}{}^*$ and $r_{is}{}^*$ are given by:

$$\alpha A Q_{is}{}^{*2} - B - 2A\beta\, [r_{is}{}^* - \mu + (1-\gamma_i)\left\{\sigma \left(\frac{1}{\eta}\right)!\ e^{-(\frac{r}{\sigma})^{\eta}} \sum_{j=0}^{1/\eta} \frac{[(r/\sigma)^{\eta}]^j}{j!} - re^{-(\frac{r}{\sigma})^{\eta}}\right\} re^{-(\frac{r}{\sigma})^{\eta}}\}] - 2M\ [\sigma \left(\frac{1}{\eta}\right)!\ e^{-(\frac{r}{\sigma})^{\eta}} \sum_{j=0}^{1/\eta} \frac{[(r/\sigma)^{\eta}]^j}{j!} - r\, e^{-(\frac{r}{\sigma})^{\eta}}] = 0 \quad (25)$$

$$R(r_{is}{}^*) = \frac{A(\alpha Q_{is}{}^* + \beta)}{M + A(1-\gamma_i)(\alpha Q_{is}{}^* + \beta)} = e^{-(\frac{r}{\sigma})^{\eta}} \quad (26)$$

Hence,

If we put $\eta = 1$, $\lambda = \frac{1}{\sigma}$, in equation (22), we found the lead time demand follows the **Exponential** distribution and it is found that the optimal values $Q_{is}{}^*$ and $r_{is}{}^*$ are given by:

$$\alpha A Q_{is}{}^{*2} - B\ -2A\beta\ \{\ r_{is}{}^* - \mu + (1-\gamma_i)[\frac{1}{\theta}\ e^{-\theta r}]\ \} - 2M[\frac{1}{\theta}\ e^{-\theta r}] = 0,$$

$$R\ (r_{is}{}^*) = \frac{A(\alpha Q_{is}{}^* + \beta)}{M + A(1-\gamma_i)(\alpha Q_{is}{}^* + \beta)} = e^{-\theta r}$$

Which represent the optimal values when X follows the **Exponential** distribution with parameter θ.

6. Applications

A businessman manages his import and export company in Egypt. He decided to import three electronic appliances (three products) from three different vendors. Table 8 in Appendix shows the values of the random variable of the lead time demand for 50 sample. The parameters are given in Table 1, Table 2 and Table 3. Hence, a businessman wishes to get an optimal policy to minimize the expected total cost.

Table (1): The crisp parameters for multi-item

	Item 1	Item 2	Item 3
$\bar{D}$	1000	900	810
C_{hi}	10	25	35
C_{bi}	20	30	40
C_{li}	30	35	45
p_i	50	40	60
θ_i	0.5	0.4	0.8
γ_i	0.7	0.67	0.56
K_{hi}	1000	2000	2300

Table (2): The fuzzy parameters for multi-item

	Item 1	Item 2	Item 3
$\tilde{C}_{hi}$	(1,2,11,12)	(1,2,26,27)	(1,2,36,37)
$\tilde{C}_{bi}$	(1,3,21,23)	(1,3,31,33)	(1,3,41,43)
$\tilde{C}_{li}$	(2,5,32,33)	(2,5,37,38)	(2,5,47,48)
$\tilde{K}_{hi}$	780	1100	1200

7. Solution:

By using One-Sample Kolmogorov–Smirnov Test, the data is fitted to Gamma distribution, where Table 4 shows the K-S statistic with their P value. If β is a constant real number selected to provide the best fit of estimated expected total cost function and using the Mathematica program 9**.** Using the parameters of Tables 1, 2, 3 and 5in equations (4), (5), (9) and (10) to obtain the optimal solutions, λ^*, Q^*, r^* and the minimum expected total cost for each item and source min (E TC_{is}) is given by Table 7 at some different values of β, where the best fit of β here is in 0< β ≤ 1 as shown in Table 6.

Table (4): one-sample Kolmogrov-Smirnov test of the lead time demand.

	X1	X2	X3
Sample size	50	50	50
Statistic	0.08098	0.09251	0.11868
p-value	0.872	0.75071	0.44764
Level significance (α)	0.05	0.05	0.05

Table (5): The value of λ^* for Item 1 and the first source at β = 0.1

λ	E(HC)	E(TC_i)
0	1237.08	10.9564
0.5	1036.85	13.5533
0.6304	1000.01	14.2241
0.6305	**999.981**	**14.2247**

Table (6): The results of crisp and fuzzy values for Gamma distribution

β	source	Item 1				Item 2				Item 3			
		λ_1	Q_1	r_1	ETC_{1i}	λ_2	Q_2	r_2	ETC_{2i}	λ_3	Q_3	r_3	ETC_{3i}
0.1	1	**0.6305**	**164.535**	**117.401**	**14.2247**	**0.349**	**124.847**	**117.074**	**32.865**	**0.2885**	**109.219**	**85.712**	**43.584**
	2	0.6966	165.291	117.002	14.526	0.4445	126.505	116.205	33.85	0.4592	110.858	84.837	46.259
	3	0.762	166.029	116.619	14.825	0.5193	127.734	115.556	34.632	-	-	-	-
0.2	1	0.634	164.433	117.382	14.239	0.3525	124.774	117.044	32.902	0.2923	109.132	85.693	43.643
	2	0.7	165.196	116.984	14.541	0.4481	126.428	116.177	33.889	0.4633	110.77	84.819	46.323
	3	0.7652	165.939	116.601	14.840	0.5229	127.659	115.528	34.672	-	-	-	-
0. 3	1	0.638	164.308	117.361	14.257	0.3561	124.695	117.014	32.942	0.2963	109.038	85.674	43.706
	2	0.7031	165.118	116.966	14.554	0.4531	126.298	116.142	33.944	0.4673	110.686	84.801	46.386
	3	0.7692	165.821	116.582	14.858	0.5269	127.571	115.498	34.715	-	-	-	-
0.4	1	0.642	164.182	117.341	14.275	0.3595	124.626	116.985	32.976	0.3	108.956	85.655	43.764
	2	0.7109	165.004	116.952	14.561	0.4553	126.278	116.119	33.965	0.4715	110.596	84.782	46.452
	3	0.7719	165.764	116.565	14.868	0.5302	127.508	115.472	34.750	-	-	-	-
0.5	1	0.646	164.058	117.32	14.293	0.364	124.519	116.952	33.023	0.3043	108.851	85.635	43.832
	2	0.711	164.882	116.926	14.589	0.4593	126.187	116.089	34.004	0.476	110.496	84.762	46.522
	3	0.7759	165.648	116.546	14.886	0.534	127.427	115.443	34.792	-	-	-	-
0.6	1	0.65	163.839	117.399	14.317	0.3667	124.473	116.923	33.052	0.3076	108.786	85.617	43.883
	2	0.7131	164.849	116.911	14.597	0.4625	126.128	116.003	34.042	0.4797	110.425	84.744	46.579
	3	0.7793	165.56	116.528	14.901	0.5375	127.359	115.415	34.829	-	-	-	-
0.7	1	0.653	163.857	117.282	14.323	0.3699	124.411	116.897	33.085	0.3116	108.695	85.597	43.945
	2	0.717	164.734	116.892	14.614	0.4665	126.038	116.032	34.086	0.484	110.332	84.726	46.647
	3	0.7841	165.411	116.506	14.922	0.5411	127.286	115.387	34.868	-	-	-	-
0.8	1	0.655	163.827	117.268	14.330	0.3738	124.322	116.866	33.127	0.3153	108.615	85.579	44.002
	2	0.72	164.662	116.874	14.626	0.47	125.969	116.003	34.123	0.487	110.255	84.708	46.706
	3	0.787	165.346	116.489	14.934	0.545	127.202	115.359	34.911	-	-	-	-
0.9	1	0.6591	163.702	117.246	14.348	0.3775	124.242	116.836	33.167	0.3192	108.528	85.560	44.063
	2	0.724	164.544	116.854	14.644	0.4744	125.864	115.972	34.171	0.4922	110.164	84.689	46.773
	3	0.791	165.233	116.469	14.951	0.5491	127.111	115.329	34.956	-	-	-	-
1	1	0.6615	163.653	117.231	14.358	0.381	124.17	116.807	33.204	0.3232	108.438	85.54	44.126
	2	0.727	164.472	116.837	14.656	0.4772	125.823	115.946	34.199	0.4963	110.08	84.67	46.837
	3	0.7939	165.167	116.453	14.963	0.5525	127.047	115.302	34.992	-	-	-	-
β		λ_1	$\tilde{Q}_1$	$\tilde{r}_1$	$E\widetilde{TC}_{1i}$	λ_2	$\tilde{Q}_2$	$\tilde{r}_2$	$E\widetilde{TC}_{2i}$	λ_3	$\tilde{Q}_3$	$\tilde{r}_3$	$E\widetilde{TC}_{3i}$
0.1	1	**0.1281**	**203.193**	**118.088**	**7.5030**	**0.6275**	**125.426**	**115.266**	**20.153**	**0.6543**	**106.429**	**84.469**	**26.975**
	2	0.1789	204.037	117.661	7.652	0.7286	126.928	114.471	20.757	0.8355	107.843	83.706	28.549
	3	0.2962	205.842	116.728	8.0001	0.8436	128.555	113.607	21.453	-	-	-	-
0.2	1	0.1298	203.098	118.075	7.5086	0.6315	125.348	115.238	20.1771	0.6593	106.337	84.449	27.0168
	2	0.1809	203.922	117.647	7.659	0.7327	126.849	114.443	20.7826	0.8406	107.756	83.686	28.5918
	3	0.2984	205.723	116.714	8.0073	0.8478	128.475	113.579	21.4798	-	-	-	-
0.3	1	0.1314	203.013	118.062	7.5139	0.6356	125.269	115.208	20.201	0.6641	106.243	84.439	27.057
	2	0.1828	203.818	117.632	7.6652	0.7366	126.777	114.416	20.8061	0.8444	107.705	83.67	28.5918
	3	0.301	205.575	116.698	8.0157	0.8518	128.401	113.552	21.5047	-	-	-	-
0.4	1	0.1333	202.903	118.047	7.5201	0.64	125.179	115.178	20.2284	0.669	106.164	84.410	27.0968
	2	0.1845	203.729	117.619	7.6707	0.7407	126.699	114.387	20.8312	0.8507	107.585	83.649	28.6757
	3	0.3024	205.529	116.686	8.0197	0.8559	128.323	113.526	21.5303	-	-	-	-
0.5	1	0.1354	202.776	118.032	7.5271	0.6444	125.088	115.148	20.2555	0.6741	106.071	84.390	27.139
	2	0.1865	203.616	117.605	7.6773	0.7448	126.621	114.359	20.8563	0.8557	107.502	83.630	28.7172
	3	0.3045	205.419	116.672	8.0265	0.8599	128.249	113.499	21.5552	-	-	-	-
0.6	1	0.1371	202.685	118.018	7.5326	0.6483	125.008	115.129	20.2797	0.6788	105.991	84.37	27.1769
	2	0.1882	203.529	117.591	7.6828	0.7488	126.547	114.331	20.8804	0.8609	107.415	83.61	28.7601
	3	0.3064	205.317	116.658	8.0329	0.8643	128.163	113471	21.5827	-	-	-	-
0.7	1	0.1392	202.559	118.003	7.5395	0.6522	124.946	115.091	20.3015	0.6844	105.884	84.349	27.224
	2	0.1909	203.360	117.574	7.6917	0.7541	126.430	114.299	20.914	0.8656	107.340	83.592	28.799
	3	0.3088	205.185	116.643	8.0407	0.8687	128.078	113.442	21.6103	-	-	-	-
0.8	1	0.1408	202.476	117.99	7.5448	0.6561	124.873	115.063	20.325	0.6886	105.819	84.331	27.2574
	2	0.1921	203.317	117.559	7.6954	0.7569	126.395	114.276	20.9298	0.8711	107.245	83.572	28.8448
	3	0.3105	205.108	116.63	8.0461	0.8726	128.007	113.416	21.6347	-	-	-	-
0.9	1	0.1429	202.350	117.975	7.5517	0.6608	124.776	115.031	20.3537	0.6944	105.699	84.319	27.3076
	2	0.1941	203.203	117.548	7.7020	0.761	126.318	114.248	20.9548	0.8759	107.169	83.554	28.8841
	3	0.3129	204.978	116.615	8.0538	0.8769	127.924	113.389	21.6619	-	-	-	-
1	1	0.1451	202.218	117.959	7.5589	0.6643	124.719	115.004	20.3739	0.6985	105.646	84.292	27.3385
	2	0.1962	203.084	117.533	7.7088	0.7658	126.219	114.217	20.9849	0.8814	107.075	83.534	28.9297
	3	0.3151	204.863	116.6	8.0609	0.8811	127.846	113.361	21.6879	-	-	-	-

Table (7): The optimal policy of MIMS variables at β = 0.1

Item	$\lambda_{is}{}^*$	$Q_{is}{}^*$	$r_{is}{}^*$	ETC_{is}	Source	$\lambda_{is}{}^*$	$\tilde{Q}_{is}{}^*$	$\tilde{r}_{is}{}^*$	$\widetilde{ETC}_{is}$	Source
1	0.6305	164.535	117.401	14.2247	1	0.1281	203.193	118.088	7.5030	1
2	0.349	124.847	117.074	32.865	1	0.6275	125.426	115.266	20.153	1
3	0.2885	109.219	85.712	43.584	1	0.6543	106.429	84.469	26.975	1
Min TC				90.6737					54.631	

8. Conclusion and Future Work

Upon studying the probabilistic multi item multi-source inventory model with varying holding cost under constraint using the Lagrange multipliers technique, the optimal order quantity$Q^*$and the optimal reorder point r^* for the i^{th} item and s^{th} sourceare introduced. Then, the minimum expected total cost min E(TC_{is}) for crisp and trapezoidal fuzzy number are deduced for each item and source. Also, we studied the model when the lead time demand follows Gamma, Weibull, Chi-square, Erlang and Exponential distributions. Also for an application we can determine the optimal source for each item. Finally, we found that the results in case fuzzy numbers are better than crisp numbers. In the future we will apply this model with constraints on the expected holding cost and the expected mixture shortage cost. Also, we will apply rough numbers for this model.

References

[1] Abuo-El-Ata M.O., Fergany H.A. andElwakeel M.F., (2003) "Probabilistic multi-item inventory model with varying order cost under two restrictions". International Journal of Production Economics, 83, 223-231.

[2] Azoury,K. and Brill, P.H. (1986): " An Application of The System-Point Method ToInventory Models Under Continuous Review". J. App. Prob.,23, pp.778-789.

[3] Bera, U. K., Rong, M., Mahapatra, N. K., and Maiti, M. (2009): "Amulti-item mixture inventory model involving random lead time and demand with budget constraint and surprise function". Applied Mathematical Modelling 33, 4337-4344.

[4] Chiang, J., Yao, J. S., and Lee, H. M. (2005): "Fuzzy inventory with backorder defuzzification by signed distance method". Journal of Information Science and Engineering, 21, 673-694.

[5] Fabrycky W.J. and Banks J. (1967) "Procurement and Inventory System: Theory and Analysis". Reinhold Publishing Corporation, USA.

[6] Fergany H.A. and Elwakeel M.F., (2004) "Probabilistic single-item inventory problem with varying order cost under two linear constraints". Statistical Theory and MethodAbstracts,12(1),71-81.

[7] Fergany H.A. and El-Saadani, M.E., (2005) "Constrained Probabilistic inventory model with continuous distributions and varying holding cost". International Journal of Applied Mathematics, Vol 17, No. 1, pp. 53-67.

[8] Fergany H.A., (2016):"Probabilistic multi-item inventory model with varying mixture shortage cost under restrictions". Springer plus, 5(1), 1-13.

[9] Fergany H.A., Hefnawy N.A. and Hollah O.M., (2014):" Probabilistic periodic review $<Q,r>$ inventory model using lagrange technique and fuzzy adapting particle swarm optimization". Journal of Mathematics and Statistics.

[10] Kotb, K.A.M., Fergany, H.A., (2011):"Multi-item EOQ model with varying holding cost: a geometric programming approach". Int. Math. Forum 6.

[11] Gupta, P. K. and Hira, D. S. (1993): "Operations research" S. Chand & Company Ltd New Delhi.

[12] Hadley, G. and Whitin, T.M., (1963) "Analysis of Inventory System". Prentice Hall, Inc., Englewood Cliffs, New Jersey.

[13] Hill, R. M. and Johansen, S. G. (2006): "Optimal and neer-optimal policies for lost sales inventory models with at most one replenishment order outstanding". European Journal of Operational Research, VoL 169, pp. 111-132.

[14] Padmanabhan, G. and Vrat, P. (1990): "Inventory model with a mixture of backorders and lost sales ". International Journal of Systems Sciences, Vol. 21: 1721-1726.

[15] Vijayan, T. and Kumaran M. (2008): "Inventory models with a mixture of backorders and lost sales under fuzzy cost". European Journal of operational Research, VoL 189, pp. 105-119.

[16] Yun, Z. and Xiaobo, Z. (2010): "A two-demand-class inventory system with lost sales and backorders". European Journal of Operational Research, Vol. 38, Issue.4:261-266.