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On Stability of Nonlinear Differential System Via Cone-Perturbing Laipunov Function Method

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Abstract

Totally equistable, totally ϕ_o – equistable, practically – equistable ,practically Φ_o – equistable of system of differential equations are studied, Cone valued perturbing Liapunov functions method and comparison methods are our technique , Some results of these properties are given.

Keywords: Totally equistable, totally ϕ_o – equistable, practically – equistable ,practically Φ_o – equistable- Cone valued perturbing Liapunov functions method.

1. Introduction

Consider the non linear system of ordinary differential equations

$$x' = f(t,x), \qquad x(t_0) = x_0 \qquad (1.1)$$

and the perturbed system

$$x' = f(t,x) + R(t,x), \qquad x(t_0) = x_0. \qquad (1.2)$$

Let R^n be Euclidean n –dimensional real space with any convenient norm $\|\cdot\|$, and scalar product $(.,.) \le \|\cdot\|\|\cdot\|$ .Let for some $\rho > 0$

$$S_\rho = \{x \in R^n, \|x\| < \rho\}.$$

where

$f, R \in C[J \times S_\rho, R^n], J = [0,\infty)$ and $C[J \times S_\rho, R^n]$ denotes the space of continuous mappings $J \times S_\rho$ *into* R^n .

Consider the scalar differential equations with an initial condition

$$u' = g_1(t,u) \qquad u(t_0) = u_0, \qquad (1.3)$$

$$\omega' = g_2(t,\omega) \qquad \omega(t_0) = \omega_0 \qquad (1.4)$$

and the perturbing equations

$$u' = g_1(t.u) + \varphi_1 \qquad u(t_0) = u_0 \qquad (1.5)$$

$$\omega' = g_2(t.\omega) + \varphi_2 \qquad \omega(t_0) = \omega_0 \qquad (1.6)$$

where $g_1, g_2 \in C[J \times R, R]$, $\varphi_1, \varphi_2 \in C[J, R]$ respectively.

The following definitions [1] will be needed in the sequal .

Definition 1.1

A proper subset K of R^n is called a cone if
(i) $\lambda K \subset K, \lambda \ge 0.$ (ii) $K + K \subset K$, (iii) $\bar{K} = K$, (iv) $K^0 \ne \emptyset$, (v) $K \cap (-K) = \{0\}$.
where K and K^0 denotes the closure and interior of K respectively and ∂K denote the boundary of K.

Definition 1.2

The set $K^* = \{\phi \in R^n, (\phi, x) \ge 0, x \in K\}$ is called the adjoint cone if it satisfies the properties of the definition 3.1.

$$x \in \partial K \textit{ if } (\phi, x) = 0 \textit{ for some } \phi \in K_0^*, K_0 = \frac{K}{\{0\}}.$$

Definition 1.3

A function $g: D \to K, D \subset R^n$ is called quasimonotone relative to the cone K if $x, y \in D, y - x \in \partial K$ then *there exists* $\phi_0 \in K_0^*$ *such that* $[(\phi]_0, y - x) = 0$ *and* $(\phi_0, g(y) - g(x)) > 0$.

Definition 1.4

A function $a(.)$ is said to belong to the class $\mathcal{K}$ if
$a \in [R^+, R^+]$, $a(0) = 0$ and $a(r)$ is strictly monotone *increasing in r* .

2. Totally equistable

In this section we discuss the concept of totally equistable of the zero solution of (1.1) using perturbing Liapuniv functions method and Comparison principle method.

We define for

$V \in C[J \times S_\rho, R^n]$, the function $D^+V(t,x)$ by

$$\sup \frac{1}{h\left(V\left(t+h, x+h(f(t,x)+R(t,x)\right)-V(t,x)\right).]}$$

The following definition [2-10] will be needed in the sequal.

Definition 2.1

The zero solution of the system (1.1) is said to be $T_1 -$ totally equistable (stable with respect to permanent perturbations) , if for every $\epsilon > 0, t_0 \in J$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that for every solution of perturbed equation (1.2), the inequality

$$\|x(t, t_0, x_0)\| < \epsilon \quad \text{for } t \ge t_0$$

holds ,provided that $\|x\_0\| < \delta\_1$ and $\|R(t,x)\| < \delta_2$.

Definition 2.2

The zero solution of the equation (1.3) is said to be $T_1 -$ totally equistable (stable with respect to permanent perturbations) , if for every $\epsilon > 0, t_0 \in J$, there exist two positive numbers

$\delta_1^* = \delta_1^*(\epsilon) > 0$ and $\delta_2^* = \delta_2^*(\epsilon) > 0$ such that for every solution of perturbed equation (1.5).the inequality

$u(t, t_0, u_0) < \epsilon, \quad t \geq t_0$

holds , provided that $u_0 < \delta_1^*$ and $\varphi_1(t) < \delta_2^*$.

Theorem 2.1

Suppose that there exist two functions $g_1, g_2 \in C[J \times R, R]$ with $g_1(t,0) = g_2(t,0) = 0$ and there exist two Liapunov functions

$V_1 \in C\left[J \times S_\rho, R^n\right]$ and $V_{2\eta} \in C\left[J \times S_\rho \cap S_\eta^C, R^n\right]$

with $V_1(t,0) = V_{2\eta}(t,0) = 0$ where $S_\eta = \{x \in R^n, \|x\| < \eta\}$ for $\eta > 0$ and S_η^C denotes the complement of S_η satisfying the following conditions:

(H_1) $V_1(t,x)$ is locally Lipschitzian in x .

$D^+V_1(t,x) \leq g_1(t, V_1(t,x)) \quad \forall (t,x) \in J \times S_\rho.$

(H_2) $V_{2\eta}(t,x)$ is locally Lipschitzian in x

$b(\|x\|) \leq V_{2\eta}(t,x) \leq a(\|x\|) \quad \forall (t,x) \in J \times S_\rho \cap S_\eta^C.$

where $a, b \in \mathcal{K}$ are increasing functions.

(H_3)

$D^+V_1(t,x) + D^+V_{2\eta}(t,x) \leq g_2\left(t, V_1(t,x) + V_{2\eta}(t,x)\right) \quad \forall (t,x) \in J \times S_\rho \cap S_\eta^C.$

(H_4) If the zero solution of (1.3) is equistable , and the zero solution of (1.4) is totally equistable .

Then the zero solution of (1.1) is totally equistable.

Proof

Since the zero solution of the system (1.4) is totally equistable , given $b(\epsilon) > 0$, there exist two positive numbers $\delta_1^* = \delta_1^*(\epsilon) > 0$ and $\delta_2^* = \delta_2^*(\epsilon) > 0$ such that for every solution $\omega(t, t_0, \omega_0)$ of perturbed equation (1.6) the inequality

$$\omega(t, t_0, \omega_0) < \epsilon, \quad t \geq t_0 \tag{2.1}$$

holds , provided that $\omega_0 < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$.

Since the zero solution of (1.3) is equistable given $\frac{\delta_0(\epsilon)}{2}$ and $t_0 \in J$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that

$$u(t, t_0, u_0) < \frac{\delta_0(\epsilon)}{2} \tag{2.2}$$

holds ,provided that $u_0 \leq \delta$

From the condition (H_2) we can find $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$a(\delta_1) + \frac{\delta_0}{2} < \delta_1^* \tag{2.3}$$

To show that the zero solution of (1.1) is T_1- totally equistable , it must show that for every $\epsilon > 0$, $t_0 \in J$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that for every solution $x(t, t_0, x_0)$ of perturbed equation (1.2).the inequality

$\| x(t, t_0, x_0)\| < \epsilon$ for $t \geq t_0$

holds ,provided that $\|x_0\| < \delta_1$ and $\|R(t,x)\| < \delta_2$.

Suppose that this is false, then there exists a solution $x(t, t_0, x_0)$ of (1.2) with $t_1 > t_0$ such that

$$\|x(t_0, t_0, x_0)\| = \delta_1 , \quad \|x(t_1, t_0, x_0)\| = \epsilon \tag{2.4}$$

(2.4)

$\delta_1 \leq \|x(t, t_0, x_0)\| \leq \epsilon$ for $t \in [t_0, t_1]$.

Let $\delta_1 = \eta$ and setting $m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$

Since $V_1(t,x)$ and $V_{2\eta}(t,x)$ are Lipschitzian in x for constants M_1 and M_2 respectively.

Then

$D^+V_1(t,x)_{1.2} + D^+V_{2\eta}(t,x)_{1.2} \leq D^+V_1(t,x)_{1.1} + D^+V_{2\eta}(t,x)_{1.1} + M\|R(t,x)\|$

where $M = M_1 + M_2$ From the condition (H_3) we obtain the differential inequality

$D^+V_1(t,x) + D^+V_{2\eta}(t,x) \leq g_2\left(t, V_1(t,x) + V_{2\eta}(t,x)\right) + M\|R(t,x)\|$

for $t \in [t_0, t_1]$ Then we have

$D^+m(t,x) \leq g_2(t, m(t,x)) + M\|R(t,x)\|$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0)$

Applying the comparison Theorem (1.4.1) of [7] , it yields

$m(t,x) \leq r_2(t, t_0, \omega_0)$ for $t \in [t_0, t_1]$.

where $r_2(t, t_0, \omega_0)$ is the maximal solution of the perturbed equation (1.6)

Define $\varphi_2(t) = M\|R(t,x)\|$

To prove that

$r_2(t, t_0, \omega_0) < b(\epsilon)$.

It must be show that

$\omega_0 < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$.

Choose $u_0 = V_1(t_0, x_0)$. From the condition (H_1) and applying the comparison Theorem of [7], it yields

$V_1(t,x) \leq r_1(t, t_0, u_0)$

where $r_1(t, t_0, u_0)$ is the maximal solution of (1.3).

From (2.2) at $t = t_0$

$$V_1(t_0, x_0) \leq r_1(t_0, t_0, u_0) < \frac{\delta_0(\epsilon)}{2} \tag{2.5}$$

From the condition (H_2) and (2.4) , at $t = t_0$

$$V_{2\eta}(t_0, x_0) \leq a(\|x_0\|) \leq a(\delta_1) \tag{2.6}$$

From (2.3), we get

$$\omega_0 = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0) \leq \frac{\delta_0(\epsilon)}{2} + a(\delta_1) < \delta_1^*.$$

Since $\varphi_2(t) = M\|R(t,x)\| \leq M\delta_2 = \delta_2^*$

From (2.1) ,we get

$$m(t,x) \le r_2(t,t_0,\omega_0) < b(\epsilon) \qquad (2.7)$$

Then from the condition (H_2), (2.4) and (2.7) we get $t = t_1$

$b(\epsilon) = b(\|x(t_1)\|) \le V_{2\eta}(t_1,x(t_1)) < m(t_1,x(t_1)) \le r_2(t_1,t_0,\omega_0) < b(\epsilon)$.

This is acontradiction ,then it must be $\| x(t,t_0,x_0)\| < \epsilon$ for $t \ge t_0$ holds ,provided that $\|x_0\| < \delta_1$ and $\|R(t,x)\| < \delta_2$.

Therefore the zero solution of (1.1) is totally equistable.

3. Totally $\Phi_0 -$ equistable.

In this section we discuss the concept of Totally $\Phi_0 -$ equistable of the zero solution of (1.1) using cone valued perturbing Liapunov functions method and Comparison principle method.

The following definition [3] will be needed in the sequal.

Definition 3.1

The zero solution of the system (1.1) is said to be totally $\Phi_0 -$ equistable ($\Phi_0 -$ equistable with respect to permanent perturbations) ,if for every $\epsilon > 0$, $t_0 \in J$ and $\phi_0 \in K_0^*$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that the inequality

$$(\phi_0, x(t,t_0,x_0)) < \epsilon \quad \text{for } t \ge t_0$$

holds ,provided that $(\phi_0,x_0) < \delta_1$ and $\|R(t,x)\| < \delta_2$ where $x(t,t_0,x_0)$ is the maximal solution of perturbed equation (1.2).

Let for some $\rho > 0$

$$S_\rho^* = \{x \in R^n, (\phi_0,x) < \rho, \phi_0 \in K_0^*\}$$

Theorem 3.1

Suppose that there exist two functions $g_1,g_2 \in C[J \times R, R]$ with $g_1(t,0) = g_2(t,0) = 0$ and let there exist two cone valued Liapunov functions

$V_1 \in C\left[J \times S_\rho^*, K\right]$ and $V_{2\eta} \in C\left[J \times S_\rho^* \cap S_\eta^{*C}, K\right]$ with $V_1(t,0) = V_{2\eta}(t,0) = 0$

where

$S_\eta^* = \{x \in K, (\phi_0,x) < \eta, \phi_0 \in K_0^*\}$ for $\eta > 0$ and S_η^{*C} denotes the complement of S_η^* satisfying the following conditions:

(h_1) $V_1(t,x)$ is locally Lipschitzian in x and $D^+[(\phi_0,V]_1(t,x)) \le g_1(t,V_1(t,x))$ for $(t,x) \in J \times S_\rho^*$.

(h_2) $V_{2\eta}(t,x)$ is locally Lipschitzian in x and $b(\phi_0,x) \le [(\phi_0,V]_{2\eta}(t,x)) \le a(\phi_0,x)$ for $(t,x_t) \in J \times S_\rho^* \cap S_\eta^{*C}$.

where $a, b \in \mathcal{K}$ are increasing functions.

(h_3) $D^+[(\phi_0,V]_1(t,x)) + D^+[(\phi_0,V]_{2\eta}(t,x)) \le g_2\left(t, V_1(t,x) + V_{2\eta}(t,x)\right)$ for $(t,x) \in J \times S_\rho^* \cap S_\eta^{*C}$.

(h_4) If the zero solution of (1.3) is $\phi_0 -$ equistable , and the zero solution of (1.4)is totally $\Phi_0 -$ equistable . then the zero solution of (1.1) is totally $\Phi_0 -$ equistable.

Proof

Since the zero solution of (1.4) is totally $\Phi_0 -$ equistable , given , given $b(\epsilon) > 0$ there exist two positive numbers $\delta_1^* = \delta_1^*(\epsilon) > 0$ and $\delta_2^* = \delta_2^*(\epsilon) > 0$ such that the inequality

$$(\phi_0, r_2(t,t_0,\omega_0)) < \epsilon, \qquad t \ge t_0 \qquad (3.1)$$

holds , provided that $(\phi_0,\omega_0) < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$. where $r_2(t,t_0,\omega_0)$ is the maximal solution of perturbed equation (1.6).

Since the zero solution of the system (1.3) is $\Phi_0 -$ equistable , given $\frac{\delta_0(\epsilon)}{2}$ and $t_0 \in J$ there exists $\delta = \delta(t_0,\epsilon) > 0$ such that

$$(\phi_0, r_1(t,t_0,u_0)) < \frac{\delta_0(\epsilon)}{2} \qquad (3.2)$$

holds ,provided that $[(\phi_0,u]_0) \le \delta$ where $r_1(t,t_0,u_0)$ is the maximal solution of(1.3)

From the condition (h_2) we can choose $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$a(\delta_1) + \frac{\delta_0}{2} < \delta_1^* \qquad (3.3)$$

To show that the zero solution of (1.1) is $T_1 -$ totally $\Phi_0 -$ equistable ,it must be prove that for every $\epsilon > 0$, $t_0 \in J$ and $\phi_0 \in K_0^*$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that the inequality $(\phi_0, x(t,t_0,x_0)) < \epsilon$ for $t \ge t_0$ holds ,provided that $(\phi_0,x_0) < \delta_1$ and $\|R(t,x)\| < \delta_2$ where $x(t,t_0,x_0)$ is the maximal solution of perturbed equation (1.2).

Suppose that is false, then there exists a solution $x(t,t_0,x_0)$ of (1.2) with $t_1 > t_0$ such that

$$(\phi_0,x(t_0,t_0,x_0)) = \delta_1, \quad (\phi_0,x(t_1,t_0,x_0)) = \epsilon \qquad (3.4)$$

$$\delta_1 \le (\phi_0, x(t,t_0,x_0)) \le \epsilon \quad \text{for } t \in [t_0,t_1].$$

Let $\delta_1 = \eta$ and setting $m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$

Since $V_1(t,x)$ and $V_{2\eta}(t,x)$ are Lipschitzian in x for constants M_1 and M_2 respectively.

Then

$$D^+(\phi_0, V_1(t,x))_1 . 2 + D^+ [(\phi_0,V]_{2\eta}(t,x))_1 . 2$$

$$\le D^+[(\phi_0,V]_1(t,x))_1 . 1 + D^+\left(\phi_0, V_{2\eta}(t,x)\right)_1 . 1 + M\|R(t,x)\|$$

where $M = M_1 + M_2$ From the condition (h_3) we obtain the differential inequality

$$D^+[(\phi_0, V]_1(t,x)) + D^+\left(\phi_0, V_{2\eta}(t,x)\right) \le g_2\left(t, V_1(t,x) + V_{2\eta}(t,x)\right) + M\|R(t,x)\|$$

for $t \in [t_0, t_1]$ Then we have

$$D^+(\phi_0, m(t,x)) \le g_2(t, m(t,x)) + M\|R(t,x)\|$$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0)$

Applying the comparison Theorem of [7] , yields

$(\phi_0, m(t,x)) \le (\phi_0, r_2(t, t_0, \omega_0))$ *for* $t \in [t_0, t_1]$.

Define $\varphi_2(t) = M\|R(t,x)\|$

To prove that

$[(\phi_0, r]_2(t, t_0, \omega_0)) < b(\epsilon)$.

It must be shown that

$[(\phi_0, \omega]_0) < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$.

Choose $u_0 = V_1(t_0, x_0)$. From the condition (h_1) and applying the comparison Theorem [7], it yields

$[(\phi_0, V]_1(t,x)) \le [(\phi_0, r]_1(t, t_0, u_0))$

From (3.2) at $t = t_0$

$$(\phi_0, V_1(t_0, x_0)) \le [(\phi_0, r]_1(t_0, t_0, u_0)) < \frac{\delta_0(\epsilon)}{2}$$

(3.5)

From the condition (h_2) and (3.4) , at $t = t_0$

$$\left(\phi_0, V_{2\eta}(t_0, x_0)\right) \le a(\phi_0, x_0) \le a(\delta_1)$$

(3.6)

From (3.3), we get

$$[(\phi_0, \omega]_0) = (\phi_0, V_1(t_0, x_0)) + \left(\phi_0, V_{2\eta}(t_0, x_0)\right) \le \frac{\delta_0(\epsilon)}{2} + a(\delta_1) < \delta_1^*.$$

Since $\varphi_2(t) = M\|R(t,x)\| \le M\delta_2 = \delta_2^*$

From (3.1) ,we get

$$(\phi_0, m(t,x)) \le (\phi_0, r_2(t, t_0, \omega_0)) < b(\epsilon)$$

(3.7)

Then from the condition (h_2), (3.4) and (3.7) we get at $t = t_1$

$b(\epsilon) = b(\phi\_0, x(t\_1))$

$\le (\phi\_0, V\_2\eta(t\_1, x(t\_1)) < (\phi\_0, m(t\_1, x(t\_1)) \le [(\phi\_0, r]\_2(t\_1, t\_0, \omega\_0)) < b(\epsilon)$.

This is a contradiction ,then

$(\phi_0, x(t, t_0, x_0)) < \epsilon$ **for** $t \ge t_0$

provided that $(\phi_0, x_0) < \delta_1$ and $\|R(t,x)\| < \delta_2$ where $x(t, t_0, x_0)$ is the maximal solution of perturbed equation (1.2).

Therefore the zero solution of (1.1) is totally $\phi_0 -$ equistable.

4. Practically equistable

In this section, we discuss the concept of practically equistable of the zero solution of (1.1) using perturbing Liapunov functions method and Comparison principle method.

The following definition [5] will be needed in the sequal.

Definition 4.1

Let $0 < \lambda < A$ be given . The system (1.1) is said to be practically equistable if for $t_0 \in J$ such that the inequality

$$\| x(t, t_0, x_0)\| < A \quad \text{for } t \ge t_0 \tag{4.1}$$

holds ,provided that $\|x\_0\| < \lambda$ where $x(t, t_0, x_0)$ is any solution of (1.1).

In case of uniformly practically equistable ,the inequality (4.1) holds for any t_0.

We define

$S(A) = \{x \in R^n: \|x\| \le A, \quad A > 0\}$.

Theorem 4.1

Suppose that there exist two functions $g_1, g_2 \in C[J \times R, R]$ with $g_1(t,0) = g_2(t,0) = 0$ and there exist two Liapunov functions $V_1 \in C[J \times S(A), R^n]$ and $V_{2\eta} \in C[J \times S(A) \cap S(B)^C, R^n]$ with

$V_1(t,0) = V_{2B}(t,0) = 0$

where

$S(B) = \{x \in R^n, \|x\| < B, 0 < B < A\}$ and $S(B)^C$ denotes the complement of $S(B)$ satisfying the following conditions:

(I) $V_1(t,x)$ is locally Lipschitzian in x .

$D^+V_1(t,x) \le g_1(t, V_1(t,x)) \quad \forall (t,x) \in J \times S(A)$.

(II) $V_{2B}(t,x)$ is locally Lipschitzian in x .

$b(\|x\|) \le V_{2B}(t,x) \le a(\|x\|) \quad \forall (t,x) \in J \times S(A) \cap S(B)^C$.

where $a, b \in \mathcal{K}$ are increasing functions.

(III)

$D^+V_1(t,x) + D^+V_{2\eta}(t,x) \le g_2(t, V_1(t,x) + V_{2B}(t,x)) \quad \forall (t,x) \in J \times S(A) \cap S(B)^C$.

(IV) If the zero solution of (1.3) is equistable , and the zero solution of (1.4) is uniformly practically equistable .

Then the zero solution of (1.1) is practically equistable.

Proof

Since the zero solution of (1.4) is uniformly practically equistable, given $0 < \lambda_0 < A$ such that for every solution $\omega(t, t_0, \omega_0)$ of (1.4) the inequality

$$\omega(t, t_0, \omega_0) < b(A) \tag{4.2}$$

holds provided $\omega_0 \le \lambda_0$.

Since the zero solution of the system (1.3) is equistable , given $\frac{\lambda_0}{2}$ and $t_0 \in R_+$ there exist $\delta = \delta(t_0, \epsilon) > 0$ such that for every solution $u(t, t_0, u_0)$ of (1.3)

$$u(t, t_0, u_0) < \frac{\lambda_0}{2} \tag{4.3}$$

holds provided that $u_0 \le \delta$.

From the condition (II) we can find $\lambda > 0$ such that

$$a(\lambda) + \frac{\lambda_0}{2} \le \lambda_0 \tag{4.4}$$

To show that The zero solution of (1.1) practically equistable , it must be exist $0 < \lambda < A$ such that for for any solution $x(t, t_0, x_0)$ of (1.1) the inequality

$\| x(t, t_0, x_0)\| < A$ for $t \geq t_0$

holds ,provided that $\|x_0\| < \lambda$.

Suppose that this is false, then there exists a solution $x(t, t_0, x_0)$ of (1.1) with $t_1 > t_0$ such that

$\|x(t_0, t_0, x_0)\| = \lambda$, $\|x(t_1, t_0, x_0)\| = A$ (4.5)

$\lambda \leq \|x(t, t_0, x_0)\| \leq A$ for $t \in [t_0, t_1]$.

Let $\lambda = B$ and setting

$m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$

From the condition (III) we obtain the differential inequality for $t \in [t_0, t_1]$

$D^+ m(t,x) \leq g_2(t, m(t,x))$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2B}(t_0, x_0)$

Applying the comparison Theorem [7] , yields

$m(t,x) \leq r_2(t, t_0, \omega_0)$ for $t \in [t_0, t_1]$.

where $r_2(t, t_0, \omega_0)$ is the maximal solution of (1.4)

To prove that

$r_2(t, t_0, \omega_0) < b(A)$.

It must be show that $\omega_0 \leq \lambda_0$.

Choose $u_0 = V_1(t_0, x_0)$, from the condition (II) and applying the comparison Theorem of [7], yields

$V_1(t,x) \leq r_1(t, t_0, u_0)$

where $r_1(t, t_0, u_0)$ is the maximal solution of (1.3).

From (4.3) at $t = t_0$

$$V_1(t,x) \leq r_1(t, t_0, u_0) < \frac{\lambda_0}{2}$$

From the condition (II) and (4.5) , at $t = t_0$

$V_{2B}(t_0, x_0) \leq a(\|x(t_0)\|) \leq a(\lambda)$

From (4.4),(4.6) and(4.7), we get

$\omega_0 = V_1(t_0, x_0) + V_{2B}(t_0, x_0) \leq \lambda_0$

From (4.2) ,we get

$m(t,x) \leq r_2(t, t_0, \omega_0) < b(A)$

Then from the condition(II) , (4.5) and (4.8), we get at $t = t_1$

$b(A) = b(\|x(t_1)\|) \leq V_{2B}(t_1, x_1) < m(t_1, x(t_1)) \leq r_2(t_1, t_0, \omega_0) < b(A)$.

This is a contradiction ,then

$\| x(t, t_0, x_0)\| < A$ for $t \geq t_0$

provided that $\|x_0\| < \lambda$.

Therefore the zero solution of (1.1) is practically equistable.

5. practically $\Phi_0 -$ equistable

In this section we discuss the concept of practically $\phi_0 -$ equistable of the zero solution of (1.1) using cone valued perturbing Liapunov functions method and Comparison principle method.

The following definitions [6] will be needed in the sequal .

Definition 5.1

Let $0 < \lambda < A$ be given . The system (1.1) is said to be practically $\phi_0 -$ equistable, if for $t_0 \in J$ and $\phi_0 \in K_0^*$ such that the inequality

$$(\phi_0, x(t, t_0, x_0)) < A \quad \text{for } t \geq t_0 \tag{5.1}$$

holds ,provided that $(\phi_0, x_0) < \lambda$

where

$x(t, t_0, x_0)$ is the maximal solution of (1.1)

In case of uniformly practically $\phi_0 -$ equistable ,the inequality (5.1) holds for any t_0.

We define

$S^*(A) = \{x \in K, (\phi_0, x) < A, \phi_0 \in K_0^*\}$

Theorem 5.1

Suppose that there exist two functions $g_1, g_2 \in C[J \times R, R]$ with $g_1(t,0) = g_2(t,0) = 0$ and let there exist two cone valued Liapunov functions

$V_1 \in C[J \times S^*(A), K]$ and $V_{2B} \in C[J \times S^*(A) \cap S^*(B)^C, K]$ with

$V_1(t,0) = V_{2B}(t,0) = 0$ where

$S^*(B) = \{x \in K, (\phi_0, x_0) < B, 0 < B < A, \phi_0 \in K_0^*\}$ and $S^*(B)^C$ denotes the complement of $S^*(B)$ satisfying the following conditions:

(i) $V_1(t,x)$ is locally Lipschitzian in x relative to K .

$D^+(\phi_0, V_1(t,x)) \leq g_1(t, V_1(t,x)) \quad \forall (t,x) \in J \times S^*(A)$.

(ii) $V_{2B}(t,x)$ is locally Lipschitzian in x relative to K .

$b(\phi_0, x) \leq (\phi_0, V_{2B}(t,x)) \leq a(\phi_0, x) \quad \forall (t,x) \in J \times S^*(A) \cap S^*(B)^C$.

where $a, b \in \mathcal{K}$ are increasing functions.

(iii) (4.6)

$D^+(\phi_0, V_1(t,x)) + D^+(\phi_0, V_{2B}(t,x)) \leq g_2(t, V_1(t,x) + V_{2B}(t,x))$

(4.7)

$\forall (t,x) \in J \times S^*(A) \cap S^*(B)^C$.

(iv) If the zero solution of (1.3) is $\phi_0 -$ equistable, and the zero solution of (1.4) is uniformly practically ϕ_0 (4.8) equistable.

Then the zero solution of (1.1) is practically $\phi_0 -$ equistable.

Proof

Since the zero solution of the system (1.4) is uniformly practically $\phi_0 -$ equistable, given given $0 < \lambda_0 < a(B)$ for $a(B) > 0$ such that the inequality

$$(\phi_0, r_2(t, t_0, \omega_0)) < a(B) \tag{5.2}$$

holds provided $[(\phi_0, \omega]_0) \leq \lambda_0$ .where $r_2(t, t_0, \omega_0)$ is the maximal solution of (1.4).

Since the zero solution of the system (1.3) is $\phi_0 -$ equistable , given $\frac{\lambda_0}{2}$ and $t_0 \in R_+$ there exist $\delta = \delta(t_0, \lambda_0)$ such that the inequality

$$(\phi_0, r_1(t, t_0, u_0)) < \frac{\lambda_0}{2} \tag{5.3}$$

From the condition (ii), assume that

$a(B) \le b(A)$ (5.4)

also we can choose $\lambda_1 > 0$ such that

$a(\lambda) + \frac{\lambda_0}{2} \le \lambda_0$ (5.5)

To show that the zero solution of (1.1) is practically ϕ_0- equistable. It must be show that for $0 < \lambda < A$, $t_0 \in J$ and $\phi_0 \in K_0^*$ such that the inequality

$(\phi_0, x(t,t_0,x_0)) < A$ for $t \ge t_0$

holds ,provided that $(\phi_0, x_0) < \lambda$ where $x(t,t_0,x_0)$ is the maximal solution of (1.1).

Suppose that is false, then there exists a solution $x(t,t_0,x_0)$ of (1.1) with $t_2 > t_1 > t_0$ such that for $(\phi_0, x_0) < \lambda$ where $\lambda = \min(\lambda_0, \lambda_1)$

$(\phi_0, x(t_1,t_0,x_0)) = \lambda_1$, $(\phi_0, x(t_2,t_0,x_0)) = A$ (5.6)

$\lambda_1 \le (\phi_0, x(t,t_0,x_0)) \le A$ for $t \in [t_1, t_2]$

Let $\lambda_1 = B$ and setting

$m(t,x) = V_1(t,x) + V_{2B}(t,x)$

From the condition (iii) we obtain the differential inequality

$D^+ (\phi_0, m(t,x)) \le (\phi_0, g_2(t, m(t,x)))$ for $t \in [t_1, t_2]$

$\omega_0 = m(t_1, x(t_1)) = V_1(t_1, x(t_1)) + V_{2B}(t_1, x(t_1))$

Applying the comparison Theorem of [7] , yields

$(\phi_0, m(t,x)) \le (\phi_0, r_2(t,t_0,\omega_0))$

To prove that

$(\phi_0, r_2(t,t_0,\omega_0)) < a(B)$

It must be show that

$(\phi_0, \omega_0) \le \lambda_0$

Choose $u_0 = V_1(t_0, x_0)$ From the condition (i) and applying the comparison Theorem [7] it yields

$(\phi_0, V_1(t,x)) \le (\phi_0, r_1(t,t_0,u_0))$

From (5.3) at $t = t_1$

$(\phi_0, V_1(t,x)) \le (\phi_0, r_1(t,t_0,u_0)) < \frac{\lambda_0}{2}$ (5.8)

From the condition (ii) and (5.6) , at $t = t_1$

$(\phi_0, V_{2B}(t_1, x(t_1))) \le (\phi_0, x(t_1)) \le a(\lambda_1)$ (5.9)

From (5.5),(5.8) and(5.9), we get

$\phi_0, V_{2B}(t_1, x(t_1)) \le \lambda_0$

From (5.2) ,we get

$(\phi_0, m(t,x)) \le (\phi_0, r_2(t,t_0,\omega_0)) < a(B)$ (5.10)

Then from the condition (ii) ,(5.4) , (5.6) and (5.10), we get at $t = t_2$

$b(A) = b(\phi_0, x(t_2))$

$\le (\phi_0, m(t_2, x(t_2)))$

$< (\phi_0, r_2(t_2, t_0, \omega_0))$

$< a(B)$

$\le a(A).$

which leads to a contradiction ,then it must be

$(\phi_0, x(t,t_0,x_0)) < A$ for $t \ge t_0$

holds ,provided that $(\phi_0, x_0) < \lambda$ Therefore the zero solution of (1.1) is practically ϕ_0- equistable.

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