Benha Journal of Applied Sciences (BJAS) print : ISSN 2356–9751
Vol.(5) Issue(7) Part (2) (2020), (175-181) online : ISSN 2356–976x
http://bjas.bu.edu.eg

On Existence and Uniqueness of Fractional Linear Integro Partial Differential Equation with Evolution Kernel Using Modified Bielecki Method and its Numerical Solution

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Abstract
The devote of this paper is discussing the existence and uniqueness of fractional linear integro partial differential equation with evolution kernel of heat type due to modified Bielecki method. In addition, Laplace homotopy perturbation method is used to obtain the numerical solutions in the space $C_E(E \times [0,T])$. Therefore, we estimate the error in different cases of α.

Keywords: Fractional linear integro partial differential equation (FLIPDE), Caputo derivative, Riemann integral, Modified Bielecki method, Laplace homotopy perturbation method (LHPM).

1. Introduction

Fractional partial differential equations have been interested in the recent literatures as it has many application in various fields of physics and engineering such as biophysics, bioengineering, quantum mechanics, finance, control theory, image and signal processing, viscoelasticity and material sciences [4-6, 14, 15]. Most fractional partial differential equations don't have analytic solutions so numerical techniques must be used as [7, 8] Laplace-Adomian decomposition method (LADM) is used to obtain the solution numerically. In [12] Kexue and Jigen discussed the Laplace transform (LT) method for solving fractional differential equations with constant coefficients. In [10, 18] the homotopy analysis method is applied to obtain the solution of a multi-order fractional differential equation in the Caputo sense. In [1] El-Borai *et al.* studied the Cauchy problem (CP) in Banach space E for linear fractional evolution equation.

In this work, we consider the following FLIPDE with evolution kernel of heat type has the following form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \int_0^t k(x,t-y)u(x,y)\,dy + h(x,t), \qquad 0<\alpha<1, \qquad (1.1)$$

with initial condition

$$u(x,0) = f(x), \qquad (1.2)$$

where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order α and $\frac{\partial^2 u(x,t)}{\partial x^2}$ is a linear closed bounded operator. Also, $\int_0^t k(x,t-y)\, u(x,y)dy; \quad t \in [0,T]$ is a linear closed bounded operator defined in the space $E \times [0,T]$, $h(x,t)$ is a free term of and $u(x,t) \in E \times [0,T]$.

The existence and uniqueness solution of (1.1) under condition (1.2) will be proved due to Modified Bielecki method. Moreover, the stability of the solution will be discussed. In addition, LHPM will be used to obtain the numerical solution of FLIPDE. Finally, numerical results will be discussed and the difference between exact solutions and approximate solutions will be calculated.

2. Preliminaries

Here, we summarize the definitions and lemmas in addition conditions of operators that we are used for discussion the existence, uniqueness, and the stability of the solution.

Definition 1. *The Caputo fractional derivative of order* $0<\alpha<1$, *is defined as [16]*

$$D_t^\alpha(f(t)) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial f(\theta)}{\partial\theta}(t-\theta)^{-\alpha}d\theta. \qquad (2.1)$$

Definition 2. *The Riemann-Liouville fractional integral operator of order* $0<\alpha<1$, *is defined as [17]*

$$I_t^\alpha(f(t)) = \frac{1}{\Gamma(\alpha)}\int_0^t f(\theta)(t-\theta)^{\alpha-1}d\theta. \qquad (2.2)$$

Definition 3. (E,d) *is said to be a complete metric spaces if metric space* d *is defined as[1]*

$$d(u(x,t),v(x,t)) = \max_{x,t}\begin{pmatrix} e^{-\lambda(t+x)} \parallel u(x,t) \\ -v(x,t)\parallel \end{pmatrix}, \ \lambda>1. \qquad (2.3)$$

Definition 4. *Laplace transform of a function* $u(x,t)$, $t>0$ *[2]*

$$u(x,s)\mathcal{L}[u(x,t)] = \int_0^\infty u(x,t)e^{-st}dt; \ \ s>0. \qquad (2.4)$$

Definition 5. *Laplace inverse of a function* $U(x,s)$ *defined as*

$$u(x,t) = \mathcal{L}^{-1}[U(x,s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} U(x,s)e^{st}ds. \qquad (2.5)$$

Proposition1 *LT for Caputo derivative of function* $u(x,t)$ *[2]*

$$L(D^\alpha u(x,t)) = L\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u(x,\tau)}{\partial\tau}d\tau\right] = \frac{1}{s^{1-\alpha}}[sL(u(x,t)) - u(x,0)]. \qquad (2.6)$$

Abdou et al. proved the following lemmas [1]

Lemma 1.

$$\int_0^t (t-\eta)^{(\delta-1)}d\eta \le \left(\frac{1}{\lambda}\right)^{(\delta-1)}t, \qquad (2.7)$$

where $0<\delta<1$, $\lambda>1$ *and* $t\in[0,T]$.

Lemma 2.

$$\int_0^t e^{\lambda\eta}(t-\eta)^{(\delta-1)}d\eta \le \left(\frac{1}{\lambda}\right)^{\delta}\left(1+\frac{1}{\delta}\right)e^{\lambda t}, \quad (2.8)$$

where $0<\delta<1,\ \lambda>1$ *and* $t\in[0,T]$.

For solution of (1.1) under initial condition (1.2) the following conditions must be satisfied:

- *The solution* $u(x,t)$ *and its fractional derivative* $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ *belong to the space* $C_E(E\times[0,T])$, *where* $C_E(E\times[0,T])$ *be the set of all continuous functions.*
- *Free term* $h(x,t)$ *is bounded and continuous in the space* $C_E(E\times[0,T])$.
- *The second derivative operator* $\frac{\partial^2}{\partial x^2}$ *generates an analytic semigroup* $Q(t,r(x))$ *and satisfies the following condition;*

$$\| Q(t,r(x)) \| \le k, \qquad \forall t\ge 0,$$

and $\| \frac{\partial^2}{\partial x^2}Q(t,r(x)) \| \le \frac{k}{t}$, *(2.9)*

where $\|\|$ *is the norm in* E *and* k *is a positive constant.*

- *The integral operator* $\int_0^t k(x,t-y)g(x,y)dy$ *satisfies uniformly Holder condition in* $t\in[0,T]$ *for every* $g(x,y)\in E\times[0,T]$ *as the following conditions;*

$$\| \int_0^{t_2} k(x,t-y)\,g(x,y)dy - \int_0^{t_1} k(x,t-y)\,g(x,y)dy \| \le \| k_1(x) \| (t_2-t_1)^\beta \le L_1\ (t_2-t_1)^\beta, \quad (2.10)$$

$\forall t_1,t_2\in[0,T],\ t_2>t_1$, *and* $0<\beta<1$.

And,

$$\| \int_0^t k(x,t-y)Q(t_1 r)dy \| \le \frac{\| k_2(x) \|}{(t_1 r)^\gamma} < \frac{L_2}{(t_1 r)^\gamma}, \quad \gamma\in(0,1),\ r\in E, \quad (2.11)$$

where, k_1, k_2 *and* r *are functions of* x *and* $Q(t_1 r(x))\in E\times[0,T]$. *Therefore,* L_1, l_2 *are constants where* $\| k_1(x) \| < L_1$ *and* $\| k_2(x) \| < L_2$.

3. Existence and uniqueness solution by using "Modified Bielecki method"

Now, we rewrite (1.1) under (1.2) by using (2.2) and the properties of fractional calculus as the following relation

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\partial^2 u(x,\theta)}{\partial x^2}(t-\theta)^{\alpha-1}d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\theta k(x,t-y)u(x,y)(t-\theta)^{\alpha-1}dyd\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}h(x,\theta)d\theta. \quad (3.1)$$

In this section, the technique of Modified Bielecki method is generalized as [14] to obtain the existence and uniqueness solution of (1.1) under condition (1.2) in Banach space E by searching the existence and uniqueness of equivalent equation (3.1) for $0<\alpha<1$.

Theorem 1. *Suppose that the integral operator* $\int_0^t k(x,t-y)\ u(x,y)dy$ *and the second derivative operator* $\frac{\partial^2\ u(x,t)}{\partial x^2}$ *are closed linear bounded operators for* $0<\alpha<1$ *then, (1.1) with initial condition (1.2) and its equivalent (3.1) have a unique solution in Banach spaceE .*

Proof. Let K be an operator defined by

$$Ku(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\partial^2 u(x,\theta)}{\partial x^2}(t-\theta)^{\alpha-1}d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\theta k(x,t-y)u(x,y)(t-\theta)^{\alpha-1}\ dyd\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}h(x,\theta)d\theta. \quad (3.2)$$

Taking the norm of (3.2), we get

$$\| Ku(x,t) \| \le \| u_0(x) \| + \frac{1}{\Gamma(\alpha)}\int_0^t \| \frac{\partial^2 u(x,\theta)}{\partial x^2}(t-\theta)^{\alpha-1} \| d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\theta \| k(x,t-y)u(x,y)(t-\theta)^{\alpha-1} \| dyd\theta + \frac{1}{\Gamma(\alpha)}\int_0^t \| (t-\theta)^{\alpha-1}h(x,\theta) \| d\theta. \quad (3.3)$$

Using properties of differential and integral operators, we have

$$\| \frac{\partial^2 u(x,t)}{\partial x^2} \| \le L \| u(x,t) \|,$$

$$\| \int_0^t k(x,t-y)u(x,y)dy \| \le M \| u(x,t) \|, \quad (3.4)$$

where L and M are positive constants.

Using (3.4) in (3.3), we get

$$\| Ku(x,t) \| \le \| u_0(x) \| + \frac{L+M}{\Gamma(\alpha)}\int_0^t \| u(x,\theta)(t-\theta)^{\alpha-1} \| d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t \| (t-\theta)^{\alpha-1}h(x,\theta) \| d\theta. \quad (3.5)$$

Using Cauchy Schwarz inequality and Lemma 1, we can obtain

$$\| Ku(x,t) \| \le \| u_0(x) \| + \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\lambda}\right)^{\alpha-1} T(\| (L+M)u(x,t) \| + \| h(x,t) \|),$$

$$T = \max_{0\le t\le T}. \quad (3.6)$$

It obvious that the operator K maps the ball $B_r\subset E$ into itself, this clear from inequality (3.6), since

$$r = \frac{\sigma}{1-\delta_1},\ \sigma = \| u_0(x) \| + \delta_1 \| h(x,t) \|$$

and $\delta_1 = \frac{L+M}{\Gamma(\alpha)}(\frac{1}{\lambda})^{\alpha-1}T$.

Therefore,

$$r>0\ and\ \sigma>0\ then,\ \delta_1<1.$$

Then, the inequality (3.6) involves the boundedness of the operator K.

Let the two functions $u(x,t)$ and $v(x,t)\in E\times[0,T]$ be two solutions of (1.1) then, formula (3.3) leads to

$$\| Ku(x,t)-Kv(x,t) \| \le \frac{1}{\Gamma(\alpha)}\int_0^t \| \left(\frac{\partial^2 u(x,\theta)}{\partial x^2}-\frac{\partial^2 v(x,\theta)}{\partial x^2}\right)(t-\theta)^{\alpha-1} \| d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t \| \int_0^\theta k(x,t-y)u(x,y)dy(t-\theta)^{\alpha-1} \| d\theta - \frac{1}{\Gamma(\alpha)}\int_0^t \| \int_0^\theta k(x,t-y)v(x,y)dy(t-\theta)^{\alpha-1} \| d\theta.$$

(3.7)

Using (3.4) and Cauchy-Schwartz inequality we can conclude that

$$\| Ku(x,t)-Kv(x,t) \| \le \frac{L+M}{\Gamma(\alpha)}e^{\lambda x}\int_0^t e^{-\lambda(x+\theta)} \| (u(x,\theta)-v(x,\theta)) \| d\theta \int_0^t e^{\lambda\theta}(t-\theta)^{\alpha-1}d\theta. \quad (3.8)$$

Equation (3.8) can be adapted in the following form by using Lemma 2,

$$\max_{x,t} e^{-\lambda(x+t)} \| Ku(x,t)-Kv(x,t) \| \le \frac{L+M}{\Gamma(\alpha)}(\frac{1}{\lambda})^\alpha(1+\frac{1}{\alpha})\max_{x,t} e^{-\lambda(x+t)} \| (u(x,t)-v(x,t) \|. \quad (3.9)$$

Using Definition 3, the formula (3.9) becomes have the following form

$$d(ku(x,t), kv(x,t)) \leq \sigma_1 d(u(x,t), v(x,t)), \quad (3.10)$$

where, $\sigma_1 = \frac{L+M}{\Gamma(\alpha)}\left(\frac{1}{\lambda}\right)^{\alpha}(1+\frac{1}{\alpha})$.

If we choose λ sufficiently large then, $\sigma_1 < 1$ and d is a contraction mapping. Finally, by using Banach fixed point theorem K has a unique fixed point which is the unique solution of (3.1) and its equivalent (1.1) with initial condition (1.2). This completes the proof of Theorem 1.

4. The stability of the solution

Now, we will discuss the stability for the solution of (1.1) with initial conditions (1.2).

Theorem 2. *Let $u_n(x,t)$ be a sequence solution of (1.1) with initial condition $u_n(x,0) = g_n(x)$, where $g_n(x) \in E, (n = 1,2,3,\ldots.)$. If the sequences of the second derivative $\frac{d^2 g_n(x)}{dx^2}$ and the sequence of integral operator $\left(\int_0^t k(x,t-y)g_n(x)dy\right)$ converges uniformly on $E \times [0,T]$. Then, the sequence of solution $u_n(x,t)$ converges uniformly on $E \times [0,T]$ to a limit function $u(x,t)$ which is the solution of (1.1).*

Proof. If we suppose that $u_n(x,t) = u_n^\star(x,t) + g_n(x)$, then we obtain the following formula by substituting in (1.1)

$$\frac{\partial^\alpha u_n^\star(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_n^\star(x,t)}{\partial x^2} + z_n(x,t), \quad 0 < \alpha < 1, \quad (4.1)$$

where,

$$z_n(x,t) = \frac{d^2 g_n(x)}{dx^2} + \int_0^t k(x,t-y)g_n(x)dy$$
$$+ \int_0^t k(x,t-y)u_n^\star(x,y)dy + h(x,y). \quad (4.2)$$

By using semi-group method we can conclude that

$$u_n^\star(x,t) = \int_0^\infty \xi_\alpha(\theta)\varphi(t^\alpha\theta)d\theta$$
$$+\alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)$$
$$\varphi((t-\eta)^\alpha\theta)z_n(x,\eta)d\theta d\eta. \quad (4.3)$$

Here, $\xi_\alpha(\theta)$ is a probability density function defined on the interval $(0,\infty)$ as

$$\xi_\alpha(\mathrm{t}) = \frac{1}{\alpha}\mathrm{t}^{-1-\frac{1}{\alpha}}\rho_\alpha(\mathrm{t}^{-1/\alpha}), \quad (4.4)$$

where density function $\rho_\alpha(\mathrm{t})$ is defined as

$$\rho_\alpha(\mathrm{t}) = L^{-1}(e^{-p^\alpha}). \quad (4.5)$$

Using the properties of integral and derivative operators, we get

$$\| z_n(x,t) - z_m(x,t) \|$$
$$\leq \epsilon + \mu\epsilon + \alpha\int_0^t\int_0^\infty\int_0^\eta \theta(t-\eta)^{\alpha-1}$$
$$\xi_\alpha(\theta) \| k(x,t-y)g_n(x)\varphi((t-\eta)^\alpha\theta) \|$$
$$\| z_n(x,\eta) - z_m(x,\eta) \| dyd\theta d\eta. \quad (4.6)$$

Using the condition (2.11) and Lemma 2 we conclude that

$$\| z_n(x,t) - z_m(x,t) \|$$
$$\leq \mu^*\left(\frac{1}{\lambda}\right)^{\alpha v}\left(1+\frac{1}{\alpha v}\right) \| z_n(x,t) - z_m(x,t) \| + \mu(1+\epsilon), \quad (4.7)$$

where,

$$v = 1-\gamma,\ \epsilon > 0\ and\ \mu^* = \alpha\ L_2\ g_n(x)\int_0^\infty \theta^v\xi_\alpha(\theta)d\theta.$$

For all $n,m \geq N$, $u(x,t) \in E \times [0,T]$ and sufficiently large λ, we have

$$\max_{x,t} e^{-\lambda(x+t)} \| z_n(x,t) - z_m(x,t) \| < \mu(1+\epsilon). \quad (4.8)$$

Since E is a complete normed space, then the sequence $z_n(x,t)$ converges uniformly on $E \times [0,T]$ to a continuous function $z(x,t)$, therefore, the sequence $u_n^\star(x,t)$ converges uniformly on $E \times [0,T]$ to a continuous function. This completes the proof of Theorem 2.

5. Numerical schema for homotopy perturbation method (HPM)

The HPM has recently been reported to be useful for obtaining numerical solutions for fractional equations. HPM is more efficient method for solving fractional problems of limited scope in the period from [0,1]. The greater generality of the method often allows for strong convergence of the solution over larger spatial and parameter domains as HPM provides a simple way to ensure the convergence of the solution series. We describe the HPM as in [3, 9, 11] for a general type of the nonlinear differential equation with boundary conditions

$$A(u) - f(r) = 0, \quad r \in \Omega. \quad (5.1)$$

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \quad (5.2)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain. The operator A can be divided into two parts L and N where L is a linear operator and N is a nonlinear operator. Therefore, (5.1) have the following form

$$L(u) + N(u) - f(r) = 0. \quad (5.3)$$

We define a homotopy $H(r,p): \Omega \times [0,1] \to R$ as

$$H(u,p) = (1-p)[L(u) - L(u_0)] + p[A(u) - f(r)] = 0,$$
$$p \in [0,1], r \in \Omega. \quad (5.4)$$

where $p \in [0,1]$ and u_0 is an initial approximation for (5.1) with

$$H(u,0) = L(u) - L(u_0) = 0,$$
$$H(u,1) = A(u) - f(r) = 0. \quad (5.5)$$

Assume that the solution of (5.1) can be written as a power series in p, we get

$$v = \sum_{k=0}^\infty p^k u_k. \quad (5.6)$$

Substituting from (5.6) in (5.4) and comparing the coefficients of powers of p yields a successive procedure to determine u_k. Finally, by putting $p = 1$, we obtain the solution of (5.1).

6. Description of LHPM for FLIPDE with evaluation kernel of heat type

We obtain the following by using Definition 2.4 and proposition 2.6, LT can be applied on equations (1.1), (1.2)

$$\frac{1}{s^{1-\alpha}}[s\Phi(x,s) - \phi_0(x)] = \frac{\partial^2\Phi(x,s)}{\partial x^2} + \bar{k}(x,s)\Phi(x,s) + \bar{h}(x,s), \quad (6.1)$$

and

$$\Phi_0(x) = L(u_0(x)). \quad (6.2)$$

where $\Phi(x,s)$, $\bar{h}(x,s)$ and $\bar{k}(x,s)$ are the LT of $u(x,t)$, $h(x,t)$ and $k(x,t)$ respectively.

According to HPM we can obtain that

$$\sum_{j=0}^{\infty} p^j \Phi_j(x,s) = \frac{p}{s^\alpha}\left(\frac{\partial^2}{\partial x^2} + \bar{k}(x,s)\right)\sum_{j=0}^{\infty} p^j \Phi_j(x,s) + \frac{1}{s}[\phi_0(x)] + \frac{1}{s^\alpha}\bar{h}(x,s). \quad (6.3)$$

where $\Phi_j(x,s)$ are the unknown functions. By comparing the coefficients of powers of p we conclude that

$$p^0: \Phi_0(x,s) = \frac{1}{s}[\phi_0(x)] + \frac{1}{s^\alpha}\bar{h}(x,s)$$

$$p^1: \Phi_1(x,s) = \frac{1}{s^\alpha}\left(\frac{\partial^2}{\partial x^2} + \bar{k}(x,s)\right)\Phi_0(x,s)$$

$$\vdots$$

$$p^{n+1}: \Phi_n(x,s) = \frac{1}{s^\alpha}\left(\frac{\partial^2}{\partial x^2} + \bar{k}(x,s)\right)\Phi_n(x,s). \quad (6.4)$$

By taking the limit $p \to 1$, we conclude that

$$H_n(x,s) = \sum_{j=0}^{n} \Phi_j(x,s). \quad (6.5)$$

Taking Laplace inverse, then the formula (6.5) leads to

$$u(x,t) \simeq u_n(x,t) = L^{-1}(H_n(x,s)). \quad (6.6)$$

Equation (6.6) represents the approximate solution of (1.1).

7. Numerical examples

Here, the numerical results of two different examples are presented by using LHPM at different values of α.

Example 1 *Consider the following FLIPDE of heat type with evolution kernel* $k(x,t-y) = x(t-y)$.

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \int_0^t x(t-y)\,u(x,y)\,dy + h(x,t), \quad (x,t) \in [0,1]\times[0,T], \quad (7.1)$$

with initial-boundary conditions

$$u(x,0) = u(0,t) = 0, \quad (7.2)$$

and the exact solution is

$$u(x,t) = t^2\sin(x). \quad (7.3)$$

Results of exact solutions (Exact) and approximate solutions (Appro) moreover, the difference between them (error) of Example 1 are obtained numerically by using LHPM at different values of α in Table (1) and Table (2).

x=t		$\alpha = 0.98$		$\alpha = 0.8$	
	Exact	Appro	Error	Appro	error
0	0	0	0.00E+00	0	0.00E+00
0.04	6.39829E-05	6.39832E-05	2.57E-10	6.39844E-05	1.46E-09
0.08	0.000511454	0.000511471	1.65E-08	0.000511526	7.21E-08
0.12	0.001723856	0.001724046	1.90E-07	0.001724566	7.10E-07
0.16	0.004078546	0.004079631	1.09E-06	0.004082165	3.62E-06
0.2	0.007946773	0.007950991	4.22E-06	0.007959626	1.29E-05
0.24	0.013691671	0.013704509	1.28E-05	0.013728008	3.63E-05
0.28	0.021666283	0.021699302	3.30E-05	0.021754034	8.78E-05
0.32	0.032211616	0.032286671	7.51E-05	0.032400447	1.89E-04
0.36	0.045654741	0.045809961	1.55E-04	0.046026805	3.72E-04
0.4	0.062306935	0.0626048441	2.98E-04	0.062990779	6.84E-04
0.44	0.08246188	0.083000073	5.38E-04	0.083649951	1.19E-03
0.48	0.106393922	0.107318739	9.25E-04	0.108364173	1.97E-03
0.52	0.134356389	0.135880062	1.52E-03	0.137498494	3.14E-03
0.56	0.166579992	0.169001753	2.42E-03	0.171426706	4.85E-03
0.6	0.20327129	0.207002991	3.73E-03	0.210535536	7.26E-03
0.64	0.244611253	0.250208047	5.60E-03	0.255229528	1.06E-02
0.68	0.290753894	0.298950564	8.20E-03	0.305936648	1.52E-02

Table (1) Results of Example 1 at α=0.98 and α=0.8.

x=t		$\alpha = 0.48$		$\alpha = 0.38$	
	Exact	Appro	Error	Appro	error
0	0	0	0.00E+00	0	0.00E+00
0.04	6.39829E-05	6.40139E-05	3.10E-08	6.40621E-05	7.92E-08
0.08	0.000511454	0.000512423	9.69E-07	0.000513607	2.15E-06
0.12	0.001723856	0.001731142	7.29E-06	0.001738759	1.49E-05
0.16	0.004078546	0.004109117	3.06E-05	0.004137503	5.90E-05
0.2	0.007946773	0.008039996	9.32E-05	0.008118478	1.72E-04
0.24	0.013691671	0.013924033	2.32E-04	0.014103789	4.12E-04
0.28	0.021666283	0.022170308	5.04E-04	0.022532005	8.66E-04
0.32	0.032211616	0.033199312	9.88E-04	0.033861461	1.65E-03
0.36	0.045654741	0.047445954	1.79E-03	0.048573961	2.92E-03
0.4	0.062306935	0.065363044	3.06E-03	0.067178967	4.87E-03
0.44	0.08246188	0.087425305	4.96E-03	0.090218364	7.76E-03
0.48	0.106393922	0.114133991	7.74E-03	0.118271876	1.19E-02
0.52	0.134356389	0.146022135	1.17E-02	0.151963235	1.76E-02
0.56	0.166579992	0.183660558	1.71E-02	0.191967193	2.54E-02
0.6	0.20327129	0.227664649	2.44E-02	0.239017485	3.57E-02
0.64	0.244611253	0.278702048	3.41E-02	0.293915863	4.93E-02
0.68	0.290753894	0.337501312	4.67E-02	0.357542315	6.68E-02

Table (2) ***Results of Example 1 at*** $\alpha = 0.48$ ***and*** $\alpha = 0.38$.

Example 2 *Consider the following FLIPDE of heat type with evolution kernel* $k(x, t-y) = x(t^2 - y)$.

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \int_0^t x(t^2 - y)\, u(x,y)\, dy + h(x,t),$$

$$(x,t) \in [0,1] \times [0,T]. \qquad (7.4)$$

with initial- boundary conditions

$$u(x,0) = 0. \; u(0,t) = 0. \qquad (7.5)$$

and the exact solution is

$$u(x,t) = xt. \qquad (7.6)$$

Results of exact solution and approximate solution moreover, the difference between exact solution and approximate solution of Example 2 are obtained numerically by using LHPM at different values of α in Table (3) and Table (4).

x=t		$\alpha = 0.98$		$\alpha = 0.8$	
	Exact	Appro	Error	Appro	error
0	0	0	0.00E+00	0	0.00E+00
0.08	0.0064	0.006400005	4.75E-09	0.00640001	1.01E-08
0.16	0.0256	0.025600321	3.00E-07	0.025600565	5.65E-07
0.24	0.0576	0.057603391	3.39E-06	0.057605938	5.94E-06
0.32	0.1024	0.102418943	1.89E-05	0.102431501	3.15E-05
0.4	0.16	0.160071941	7.19E-05	0.160114924	1.15E-04
0.48	0.2304	0.230614038	2.14E-04	0.230730884	3.31E-04
0.56	0.3136	0.314138081	5.38E-04	0.314409081	8.09E-04
0.64	0.4096	0.410795803	1.20E-03	0.411355422	1.76E-03
0.72	0.5184	0.520818703	2.42E-03	0.521876306	3.48E-03
0.8	0.64	0.644542085	4.54E-03	0.646405928	6.41E-03
0.88	0.7744	0.782432322	8.03E-03	0.785536612	1.11E-02
0.96	0.9216	0.935117372	1.35E-02	0.940052162	1.85E-02

Table (3) ***Results of Example 2 at*** $\alpha = 0.98$ ***and*** $\alpha = 0.8$.

x=t		$\alpha = 0.48$		$\alpha = 0.2$	
	Exact	Appro	Error	Appro	error
0	0	0	0.00E+00	0	0.00E+00
0.08	0.0064	0.006400038	3.85E-08	0.006400121	1.21E-07
0.16	0.0256	0.025601717	1.72E-06	0.025604462	4.46E-06
0.24	0.0576	0.057615839	1.58E-05	0.057636748	3.67E-05
0.32	0.1024	0.102476629	7.66E-05	0.10256404	1.64E-04
0.4	0.16	0.160260308	2.60E-04	0.160523538	5.24E-04
0.48	0.2304	0.231107041	7.07E-04	0.231751436	1.35E-03
0.56	0.3136	0.315245804	1.65E-03	0.316613491	3.01E-03
0.64	0.4096	0.413021902	3.42E-03	0.415637189	6.04E-03
0.72	0.5184	0.524926958	6.53E-03	0.529545511	1.11E-02
0.8	0.64	0.651631339	1.16E-02	0.659292398	1.93E-02
0.88	0.7744	0.794018991	1.96E-02	0.806100106	3.17E-02
0.96	0.9216	0.953224759	3.16E-02	0.971498701	4.99E-02

Table (4) ***Results of Example 2 at*** $\alpha = 0.48$ ***and*** $\alpha = 0.2$**.**

8. Conclusions

The fundamental goal of this paper is to propose an efficient algorithm for the solution of FLIPDE with evolution kernel of heat type moreover, we discussed the existence and uniqueness using modified Bielecki method in addition, stability of these equations is discussed using the semi-group method. Finally, LHPM is introduced for solving with evaluation kernel FLIPDE of heat type to show the applicability and efficiency of the proposed method. we conclude that the LHPM is very powerful and efficient in founding numerical solutions for FLIPDE of heat type with evaluation kernel as the results of exact solutions and approximate solutions moreover, the difference between exact solutions and approximate solutions (errors) are obtained numerically by using LHPM at different cases of α in Table (1), Table (2), Table (3) and Table (4).

In Table (1) and Table (2) for Example 1 we conclude that

- Exact solution and approximate solution are identical at $x = t = 0$.
- Since x and t increase then, error also increases for different $\alpha = 0.95,\ \alpha = 0.8, \alpha = 0.48\ and\ \alpha = 0.35$.
- The error increases when α decreases as maximum error for Example 1 is $6.68E - 02$ at $\alpha = 0.38$.

In Table 3 and Table 4 for Example 2, we conclude that

- Exact solutions and approximate solutions are the same at $x = t = 0$.
- When x and t increase then, error also increases for different $\alpha = 0.95$, $\alpha = 0.8$, $\alpha = 0.48$ and $\alpha = 0.2$.
- The error increases when α is decrease as maximum error for Example 2 is $4.99E - 02$ at $\alpha = 0.2$, $x = 0.96$, and $t = 0.96$.

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