Bayesian and Non- Bayesian Estimation for Parameters of Gompertz Distribution under Progressive Type-I Censoring Scheme

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Bayesian and Non- Bayesian Estimation for Parameters of Gompertz Distribution under Progressive Type-I Censoring Scheme

Dr. Berihan Elemary

Abstract

The challenging explored subject under non-Bayesian and Bayesian techniques is estimating parameters of Gompertz distribution based on scheme of progressive Type-I censoring. Therefore, Maximum likelihood estimators for the unknown parameters, as well as asymptotic confidence intervals, are determined. Bayes estimates with the estimates of the associated greatest posterior density credible interval are derived using squared error loss function. Using the Metropolis-Hasting algorithm and the method of Markov Chain Monte Carlo (MCMC), estimates of Bayes are summarized. To assess the proposed estimator's performance, a Monte Carlo simulation study is accomplished. Furthermore, the theoretical conclusions of Bayes estimates and maximum likelihood estimates at progressively Type-I censored samples specified schemes are illustrated using an examined analysis on real given data.

Key Words: Gompertz distribution; Progressive Type-I censoring scheme; Bayesian estimation; Maximum likelihood estimation; Markov Chain Monte Carlo.

1. Introduction

The Gompertz distribution was originally introduced by Gompertz (1825). This distribution is widely used to describe human mortality and establish actuarial tables. The Gompertz probability density function (pdf) and cumulative distribution function (cdf) are formulated respectively by

$$f(x;\lambda,\alpha) = \lambda \exp\left(\alpha x - \left(\frac{\lambda}{\alpha}\right)(\exp(\alpha x) - 1)\right); \quad x > 0, \lambda, \alpha > 0 \qquad (1)$$

and

$$F(x;\lambda,\alpha) = 1 - \exp\left(-\left(\frac{\lambda}{\alpha}\right)(\exp(\alpha x) - 1)\right); \quad x > 0, \lambda, \alpha > 0 \qquad (2)$$

where α represents the shape parameter and λ represents the scale parameter. Figure 1 indicated pdf and cdf behavior for the Gompertz distribution of several values of λ and α.

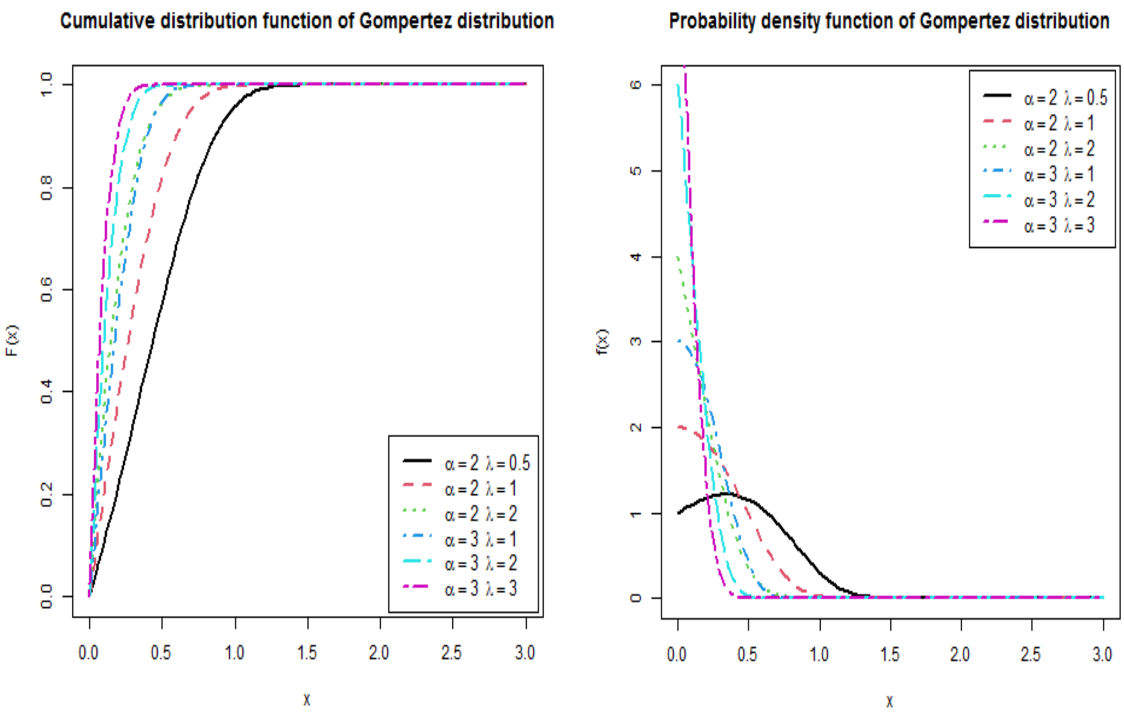

Figure 1: Pdf and cdf for Gompertz distribution at several α and λ values

The survival (reliability) function ($S(x)$) and hazard rate function ($h(x)$) of the Gompertz distribution function are provided respectively by

$$S(x) = \exp\left(-\left(\frac{\lambda}{\alpha}\right)(\exp(\alpha x) - 1)\right) \tag{3}$$

and

$$h(x) = \lambda \exp(\alpha x)\,; \quad x > 0 \tag{4}$$

The behavior of the hazard rate function and RF a are illustrated in Figure 2 with several values of α and λ.

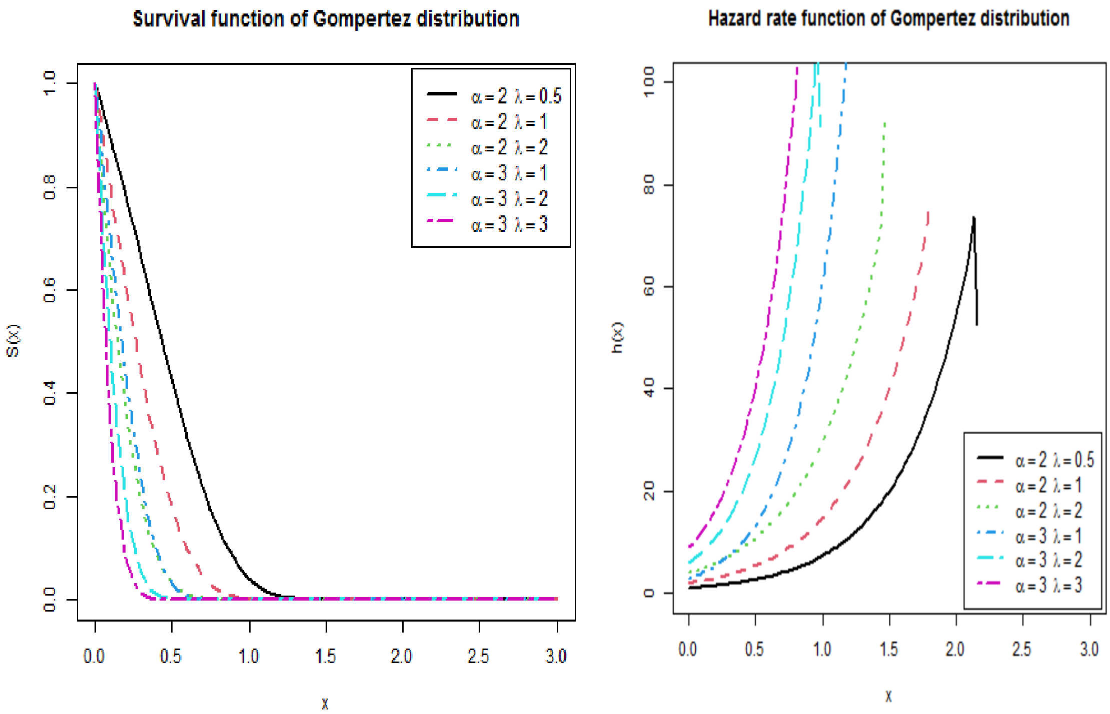

Figure 2: Function of Hazard rate and RF of Gompertz distribution of several α and λ values

Various authors studied the properties and characteristics of Gompertz distribution based on the pivotal quantity to estimate confidence intervals (CIs) of the parameters of interest more efficiently. Under the Type-II censoring scheme, Chen (1997) studied the exact CIs. For progressive Type-II censoring scheme, Wu *et al.* (2003) provided exact CIs. Studies of various estimation methods for this distribution have also been conducted by several researchers. Dey *et al.* (2018a) proposed different methods to estimate the PDF and CDF and compared the estimation methods based on the Monte Carlo simulations. Dey *et al.* (2018b) provided various mathematical and statistical properties and compared different estimation methods from both frequentist and Bayesian point of view. Moala and Dey (2018) provided Bayesian analysis methods under the objective and subjective priors including Jeffreys prior, maximal data information prior, Singpurwalla's prior, and elicited prior. Lee and Seo (2020) studied the weighted least-squared and pivot-based methods under progressive Type-II censoring. Chacko and Mohan (2018) studied the estimation of parameters for Gompertz distribution based on a progressively type-II censored sample with binomial removals. Gompertz distribution under progressive first failure censoring was studied by Soliman *et al.* (2012) and Soliman and Al Sobhi (2015). Abu-Moussa *et al.* (021) studied the estimation problem under adaptive progressive Type-II censoring scheme. Finally, Wang and Gui (2021) studied the estimation of the parameters for Gompertz distribution and prediction using general progressive Type-II censoring.

Censored data increases in actual data testing experiments when the experiment ended before the entire set of data is collected. For reasons, like cost minimization and time constraints, the censoring technique in practice is unavoidable and widespread, especially in reliability engineering. Multiple types of censoring are described in the literature, with Type-I censoring and Type-II censoring considered more common. In recent years, a generalization known as progressive censoring schemes got a lot of attention in the since it makes effective usage of available resources than the classic censoring schemes. Progressive censoring Type-I (PCTI) is included in these progressive censoring schemes. When a predefined number of life testing elements are terminated continuously from the experiment at the end of every predefined time durations, such schema is detected. It supplies the experiment with the benefit of practical realizing the termination time and with much greater flexibility to the phase of design by permitting the testing units to be terminated at points of non-terminal time ((Balakrishnan, et al. (2011)).

Let n represent units' number in a real testing experiment and $X_1, X_2, \dots, X_n$ represent the life-time for such n units selected from a population with pdf $f(x;\theta)$ and cdf $F(x;\theta)$, where θ is a vector of the unknown parameters from the distribution. The associated ordered lifetimes reported of the life testing are denoted by $x_{(1)} < x_{(2)} < \dots < x_{(n)}$. When R_i units are deleted of the survived items at the predefined time of censoring T_{q_i} corresponding to the q_ith quantiles, $i = 1,2,\dots,m$, where m is the stages number in the testing, $T_{q_i} > T_{q_{i-1}}$ and $n = r + \sum_{i=1}^{m} R_i$, PCTI is detected. The values T_{q_i} must be determined in advance as:

1. The choice of these times depends on expertise and prior knowledge on test the items of the experimenter (Balasooriya and Low (2004)), or
2. For lifetimes distribution, the q_ith quantiles
   $P(X \le T_{q_i}) = q_i \quad \Rightarrow \quad T_{q_i} = F^{-1}(q_i)$
   where $i = 1,2,\dots,m$ and $F^{-1}(.)$ is the inverse of the cdf.

In such cases R_i, T_{q_i} and n are pre-defined, l_i is the surviving items number at time T_{q_i} and $r = \sum_{i=1}^{m} l_i$ are random variables. The likelihood function in this case is discribed by

$$L(\theta) \propto \prod_{i=1}^{r} f(x_{(i)};\theta) \prod_{j=1}^{m} (1 - F\left(T_{(q_j)};\theta\right))^{R_j} \tag{5}$$

where $x_{(i)}$ is the observed lifetime of the i^{th} order statistic (Cohen (1963)). Figure 3 visualize this censoring scheme (Balakrishnan and Cramer (2010)).

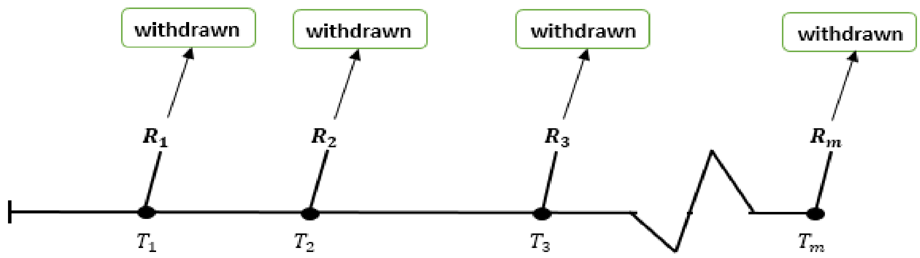

Figure 3: PCTI Scheme

Thus, Type-I censoring and complete samples can be considered as special cases of PCTI. For more details and implementation of this kind of censoring, one may mention Balakrishnan and Cramer (2010), Balakrishnan *et al.* (2011), and Balasooriya and Low (2004).

Mahmoud *et al.* (2018; 2021a) computed maximum likelihood estimates (MLE's) and Bayesian estimates (BEs) of the unknown parameters for the generalized inverted exponential (GIE) distribution model under PCTI. For the PCTI, there exist two publications that are closely associated. The first involved the unknown parameters estimates of MLEs and ACI, for the GIE model based on the assumption of two different kinds of failures (Mahmoud *et al.* (2020)). The second involved the unknown parameters estimates of MLEs and BEs, for the GIE (Mahmoud *et al.* (2021b)). For the Weibull, distribution, Balasooriya and Low (2004) proposed a competing risks model under PCTI.

This paper aims to study the estimation of Gompertz distribution parameters using maximum likelihood estimators (MLE) and Bayes estimators, concerning a life testing with only PCTI data have been available. The paper organized respectively as follow: assuming the lifetime of the units of test are Gompertz distributed independently and using PCTI, MLEs for the parameters of Gompertz(λ,α) distribution are obtained in Section 2. we assume prior distribution as gamma distribution for the two unknown parameters of the Gompertz distribution and suppose two separated loss functions; namely squared error and LINEX loss functions.

In Section 3, we got Bayesian estimates as well as the estimates of highest posterior density interval by applying Metropolis-Hasting algorithm and Markov Chain Monte Carlo (MCMC). Furthermore, in Section 4, for applicability purposes, a real data set is investigated, as well as simulations Monte Carlo to test the efficiency of the considered estimators, with providing comments derived based on the study.

2. Maximum Likelihood Estimation

For PCTI censored data, we generate MLEs of the unknown parameters of the Gompertz distribution in this section. The following steps can be adopted to implement a progressive Type-I censoring scheme:

- Assume a life-testing experiment is conducted on a random sample of units n with life time's follow the Gompertz (λ,α) distribution.
- Pre-fix m censoring time points $T_{q_1},\dots,T_{q_m}$, where a fixed number of survived items $R_1,\dots,\dots,R_{m-1}$ are eliminated of the test randomly. The censoring times T_{q_i} are specified to match $P(X \le T_{q_i}) = q_i$, where X is Gompertz (λ,α) distributed.
- The life test is terminating in a pre-determined time T_{q_m}.

Under such a censoring scheme, we may collect the PCTI samples $x = (x_{(1)}, x_{(2)}, \dots, x_{(r)})$, which reflect the observations for lifetime of the n units. Given the observations x and from equation 5, the related likelihood function of α and λ may be expressed as

$$L(\lambda,\alpha) \propto \lambda^r \exp\left(\sum_{i=1}^{r} \alpha x_{(r)} - \left(\frac{\lambda}{\alpha}\right)\left(\exp\left(\alpha x_{(r)}\right) - 1\right)\right) * \sum_{j=1}^{m}\left[\exp\left(-\left(\frac{\lambda}{\alpha}\right)\left(\exp\left(\alpha T_{q_j}\right) - 1\right)\right)\right]^{R_j} \quad (6)$$

Computing logarithm of $L(\lambda,\alpha)$ to get log-likelihood ln$L$ as

$$\ln L \propto r\log(\lambda) + \sum_{i=1}^{r}\left[\alpha x_{(r)} - \left(\frac{\lambda}{\alpha}\right)\left(\exp\left(\alpha x_{(r)}\right) - 1\right)\right] - \sum_{j=1}^{m}\left[R_j\left(\frac{\lambda}{\alpha}\right)\left(\exp\left(\alpha T_{q_j}\right) - 1\right)\right] \quad (7)$$

With respect to λ and α , the first partial derivatives of the Log-likelihood function ln$L$ are:

$$\frac{\partial lnL}{\partial\lambda} = \frac{r}{\lambda} + \sum_{i=1}^{r}\left[-\left(\frac{1}{\alpha}\right)\left(\exp\left(\alpha x_{(r)}\right) - 1\right)\right] - \sum_{j=1}^{m}\left[R_j\left(\frac{1}{\alpha}\right)\left(\exp\left(\alpha T_{q_j}\right) - 1\right)\right]$$

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{r}\left[x_{(r)} - \left(\frac{\lambda}{\alpha}\right)\left(x_{(r)} \exp\left(\alpha x_{(r)}\right)\right) + \left(\frac{\lambda}{\alpha^2}\right)\left(\exp\left(\alpha x_{(r)}\right) - 1\right)\right] - \sum_{j=1}^{m} R_j\left[\left(\frac{\lambda}{\alpha}\right)\left(T_{q_j} \exp\left(\alpha T_{q_j}\right)\right) - \left(\frac{\lambda}{\alpha^2}\right)\left(\exp\left(\alpha T_{q_j}\right) - 1\right)\right].$$

Equating $\frac{\partial \ln L}{\partial \lambda}|_{\lambda=\hat{\lambda}}$ and $\frac{\partial \ln L}{\partial \alpha}|_{\alpha=\hat{\alpha}}$ to zero as follows:

$$\frac{r}{\hat{\lambda}} + \sum_{i=1}^{r}\left[-\left(\frac{1}{\hat{\alpha}}\right)\left(\exp\left(\hat{\alpha} x_{(r)}\right) - 1\right)\right] - \sum_{j=1}^{m}\left[R_j\left(\frac{1}{\hat{\alpha}}\right)\left(\exp\left(\hat{\alpha} T_{q_j}\right) - 1\right)\right] = 0,$$

$$\sum_{i=1}^{r}\left[x_{(r)} - \left(\frac{\lambda}{\hat{\alpha}}\right)\left(x_{(r)} \exp\left(\hat{\alpha} x_{(r)}\right)\right) + \left(\frac{\hat{\lambda}}{\hat{\alpha}^2}\right)\left(\exp\left(\hat{\alpha} x_{(r)}\right) - 1\right)\right] - \sum_{j=1}^{m} R_j\left[\left(\frac{\hat{\lambda}}{\hat{\alpha}}\right)\left(T_{q_j} \exp\left(\hat{\alpha} T_{q_j}\right)\right) - \left(\frac{\hat{\lambda}}{\hat{\alpha}^2}\right)\left(\exp\left(\hat{\alpha} T_{q_j}\right) - 1\right)\right] = 0,$$

The MLEs of λ and α parameters, respectively, are the numerical solution of two equations above of $\hat{\lambda}$ and $\hat{\alpha}$.

Now, by inverting the observed information matrix of the elements that are minus the expected values for the second order derivatives of logarithms of the likelihood functions, the asymptotic variance-covariance matrix of the MLEs of λ and α can also be produced. This is

$$I(\lambda, \alpha) = \begin{bmatrix} -E\left(\frac{\partial^2 \ln \ell}{\partial \lambda^2}\right) & -E\left(\frac{\partial^2 \ln \ell}{\partial \lambda \partial \alpha}\right) \\ -E\left(\frac{\partial^2 \ln \ell}{\partial \alpha \partial \lambda}\right) & -E\left(\frac{\partial^2 \ln \ell}{\partial \alpha^2}\right) \end{bmatrix}$$

where

$$\frac{\partial^2 \ln \ell}{\partial \lambda^2} = -\frac{r}{\lambda^2}$$

$$\frac{\partial^2 \ln \ell}{\partial \alpha^2} = \sum_{i=1}^{r} \left[\left(\frac{\lambda}{\alpha}\right) \left(x_{(r)}^2 \exp\left(\alpha x_{(r)}\right)\right) - \left(\frac{\lambda}{\alpha^2}\right) \left(x_{(r)} \exp\left(\alpha x_{(r)}\right)\right) - \left(\frac{\lambda}{\alpha^2}\right) \left(x_{(r)} \exp\left(\alpha x_{(r)}\right)\right) - \left(\frac{2\lambda}{\alpha^3}\right) \left(\exp\left(\alpha x_{(r)}\right) - 1\right) \right] - \sum_{j=1}^{m} R_j \left[\left(\frac{\lambda}{\alpha}\right) \left(T_{q_j}^2 \exp\left(\alpha T_{q_j}\right)\right) - \left(\frac{\lambda}{\alpha^2}\right) \left(T_{q_j} \exp\left(\alpha T_{q_j}\right)\right) - \left(\frac{\lambda}{\alpha^2}\right) \left(T_{q_j} \exp\left(\alpha T_{q_j}\right)\right) - \left(\frac{2\lambda}{\alpha^3}\right) \left(\exp\left(\alpha T_{q_j}\right) - 1\right) \right]$$

$$\frac{\partial^2 \ln \ell}{\partial \lambda \partial \alpha} = \sum_{i=1}^{r} \left[-\left(\frac{1}{\alpha}\right) \left(x_{(r)} \exp\left(\alpha x_{(r)}\right)\right) + \left(\frac{1}{\alpha^2}\right) \left(\exp\left(\alpha x_{(r)}\right) - 1\right) \right] - \sum_{j=1}^{m} R_j \left[\left(\frac{1}{\alpha}\right) \left(T_{q_j} \exp\left(\alpha T_{q_j}\right)\right) - \left(\frac{1}{\alpha^2}\right) \left(\exp\left(\alpha T_{q_j}\right) - 1\right) \right]$$

Cohen (1965) confirmed that substituting expected values with their MLEs generated the approximate variance covariance matrix. The approximated sample information matrix will now be

$$I(\hat{\lambda}, \hat{\alpha}) = -\begin{bmatrix} \frac{\partial^2 \ln \ell}{\partial \alpha^2} & \frac{\partial^2 \ln \ell}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ln \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ln \ell}{\partial \lambda^2} \end{bmatrix}_{\alpha=\hat{\alpha}, \lambda=\hat{\lambda}}$$

As a consequence, the approximated variance covariance matrix of $\hat{\alpha}$ and $\hat{\lambda}$ will be

$$\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix} = -\begin{bmatrix}\frac{\partial^2 \ln\ell}{\partial\alpha^2} & \frac{\partial^2 \ln\ell}{\partial\alpha\partial\lambda}\\ \frac{\partial^2 \ln\ell}{\partial\lambda\partial\alpha} & \frac{\partial^2 \ln\ell}{\partial\lambda^2}\end{bmatrix}^{-1}_{\alpha=\hat{\alpha},\lambda=\hat{\lambda}} \quad (8)$$

Asymptotic Confidence Interval

The unknown parameters λ and α confidence intervals are derived under the asymptotic distribution of the estimates MLEs. For the MLE asymptotic distribution, it is proven that

$$(\hat{\alpha},\hat{\lambda}) - (\alpha,\lambda) \to N_2(0, I^{-1}(\hat{\alpha},\hat{\lambda}))$$

where $N_2(\cdot)$ is bivariate normal distribution and $I(\cdot)$ is the the Fisher information matrix expressed in equation 8. on specific regularity conditions, the two-sided $100(1-\gamma)\%, 0 < \gamma < 1,$ asymptotic confidence intervals of the vector of unknown parameters α and λ may be formulated as:

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\sigma_{11}}$$
$$\hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{\sigma_{22}}$$

where the MLEs asymptotic variances of λ and α, are σ_{11} and σ_{22} , $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$th percentile for the standard normal distribution.

3. Bayesian Estimation

The Bayesian estimates for the Gompertz distribution's unknown parameters is explored using a scheme of PCTI. The squared error loss function is considered for Bayesian parameter estimation. It can be noted that the shape parameter α seems to have a conjugate prior, that is an alpha prior, if the scale parameter λ is defined. for the parameters, there is no joint conjugate prior when both of the model's parameters are unknown. For both α and λ with pdfs, we suggest using independent alpha priors.

$$\pi_1(\alpha) \propto \alpha^{a_2-1}\exp(-b_2\alpha) \qquad \alpha > 0, a_2 > 0, b_2 > 0$$
$$\pi_2(\lambda) \propto \lambda^{a_1-1}\exp(-b_1\lambda) \qquad \lambda > 0, a_1 > 0, b_1 > 0$$

where a_1, b_1, a_2, b_2, the hyper-parameters, are chosen to represent the prior knowledge of the unknown parameters. The joint prior for α and λ is formulated by

$$\pi(\alpha,\lambda) = \pi_1(\alpha)\pi_2(\lambda)$$

$$\pi(\alpha,\lambda) \propto \alpha^{a_1-1}\lambda^{a_2-1}\exp(-b_1\alpha - b_2\lambda) \qquad (9)$$

Hyper-parameter elicitation

The informative priors are employed to select the hyper-parameters. Such informative priors are derived from the MLEs for (α,λ) via equating the mean and variance of $(\hat{\alpha}^j,\hat{\lambda}^j)$ along with the mean and variance of the considered priors (gamma priors), where $j = 1,2,\ldots,k$, k is the available samples number from the GIE distribution. On equating the mean and variance of $(\hat{\alpha}^j,\hat{\lambda}^j)$ and the mean and variance of alpha priors (Dey et al. (2016)), we get

$$\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j = \frac{a_1}{b_1} \quad \& \quad \frac{1}{k-1}\sum_{j=1}^{k}(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j)^2 = \frac{a_1}{b_1^2}$$

$$\frac{1}{k}\sum_{j=1}^{k}\hat{\lambda}^j = \frac{a_2}{b_2} \quad \& \quad \frac{1}{k-1}\sum_{j=1}^{k}(\hat{\lambda}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\lambda}^j)^2 = \frac{a_2}{b_2^2}.$$

The estimated hyper-parameters can therefore be written and illustrated by solving the two equations.

$$a_1 = \frac{(\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j)^2}{\frac{1}{k-1}\sum_{j=1}^{k}(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j)^2} \; \& \; b_1 = \frac{\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j}{\frac{1}{k-1}\sum_{j=1}^{k}(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j)^2}$$

$$a_2 = \frac{(\frac{1}{k}\sum_{j=1}^{k}\hat{\lambda}^j)^2}{\frac{1}{k-1}\sum_{j=1}^{k}(\hat{\lambda}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\lambda}^j)^2} \; \& \; b_2 = \frac{\frac{1}{k}\sum_{j=1}^{k}\hat{\lambda}^j}{\frac{1}{k-1}\sum_{j=1}^{k}(\hat{\lambda}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\lambda}^j)^2}.$$

The associating posterior density with the observations $x = (x_{(1)}, x_{(2)}, \ldots, x_{(r)})$ can therefore be formulated as

$$\pi(\lambda, \alpha | x) = \frac{\pi(\alpha, \lambda) L(\alpha, \lambda)}{\int_0^\infty \int_0^\infty \pi(\alpha, \lambda) L(\alpha, \lambda) d\lambda d\alpha}$$

The posterior density function is formulated as

$$\pi(\alpha, \lambda | x) = K^{-1} \left[\alpha^{a_1 - 1} \lambda^{r + a_2 - 1} \exp\left(-b_1 \alpha - b_2 \lambda + \sum_{i=1}^{r} \alpha x_{(r)} - \left(\frac{\lambda}{\alpha}\right) \left(\exp(\alpha x_{(r)}) - 1 \right) \right) * \sum_{j=1}^{m} \left[\exp\left(-\left(\frac{\lambda}{\alpha}\right) \left(\exp\left(\alpha T_{q_j}\right) - 1 \right) \right) \right]^{R_j} \right]$$

where k is a normalize constant. Thus, the posterior density may be reformulated as

$$\pi(\alpha, \lambda | x) \propto \alpha^{a_1 - 1} \lambda^{r + a_2 - 1} \exp\left(-b_1 \alpha - b_2 \lambda + \sum_{i=1}^{r} \alpha x_{(r)} - \left(\frac{\lambda}{\alpha}\right) \left(\exp(\alpha x_{(r)}) - 1 \right) \right) * \sum_{j=1}^{m} \left[\exp\left(-\left(\frac{\lambda}{\alpha}\right) \left(\exp\left(\alpha T_{q_j}\right) - 1 \right) \right) \right]^{R_j} \quad (10)$$

The Bayes Estimator of any function, say $g(\lambda, \alpha)$ with the squared error loss function, is as shown

$$\tilde{g}(\lambda, \alpha) = \int_0^\infty \int_0^\infty g(\lambda, \alpha) \pi(\lambda, \alpha | x) d\lambda d\alpha$$

Unfortunately, equation 10 can hardly be evaluated for general $g(\lambda, \alpha)$.The Markov Chain Monte Carlo (MCMC) and the most well-known approximated Bayes estimates of α and λ are recommended.

3.1 Markov Chain Monte Carlo (MCMC)

Markov Chain Monte Carlo (MCMC) is considered to be a technique of computer-driven sampling. It enables summarizing a distribution without recognizing all of its mathematical properties by randomly sampling data from it (Ravenzwaaij *et al.* (2018)). Since it focuses on posterior distributions, which are typically challenging to handle through analytic analysis, MCMC is very useful in Bayesian inference. In such cases or conditions, MCMC enables the user to define approximated characteristics of posterior distributions that are not be computed directly (e.g., random samples from the posterior, posterior means, etc.). To use MCMC to generated samples based on a distribution:

1. Start with an initial guess: a single value that could be selected reasonably from the distribution.
2. Using the initial guess, generate a series of new samples. Two phases are involved in generating each new sample:
    a. **Proposal:** Providing a small random perturbation to the recent sample generates a proposal of the new sample.
    b. **Acceptance:** The new proposal is either approved as a new sample or rejected (retain the old sample).

There are various methodologies for providing random noise to produce proposals, as well as various techniques to accept and reject proposals, like: Metropolis-Hastings and Gibbs-sampling algorithm.

3.1.1 Metropolis-Hasting algorithm

To implement the MH algorithm for the GIE distribution, an initial value and a proposal distribution for α and λ, the unknown parameters, need to be defined. For the proposed distribution, a bivariate normal distribution, $q((\alpha', \lambda')|(\alpha, \lambda)) \equiv N_2((\alpha, \lambda), S_{\alpha,\lambda})$, where $S_{\alpha,\lambda}$ is the matrix of the variance-covariance, is considered. we might get negative observations which cannot be accepted. The MLE for λ and α, $(\alpha^{(0)}, \lambda^{(0)}) = (\hat{\alpha}, \hat{\lambda})$, is considered for the initial values. $S_{\alpha,\lambda}$ is the asymptotic variance-covariance matrix $I^{-1}(\hat{\alpha}, \hat{\lambda})$, where $I(.)$ is the Fisher information matrix.the acceptance rate relies on selecting $S_{\alpha,\lambda}$, this is why it is an major matter in the MH algorithm. In this regard, the MH algorithm main steps for drawing samples from the posterior density given equation 10 are implemented as shown:

Step 1. Set an initial value of θ as $\theta^{(0)} = (\hat{\alpha}, \hat{\lambda})$.

Step 2. For $i = 1,2, \dots, M$ repeat the following: [label=2.:]

a. Set $\theta = \theta^{(i-1)}$.
b. Generate a new potential parameter value δ from $N_2(\ln\theta, S_\theta)$.
c. Set $\theta' = \exp(\delta)$.
d. Calculate $\beta = \frac{\pi(\theta'|x)}{\pi(\theta|x)}$, where $\pi(\cdot)$ is the posterior density in equation 9
e. Generate a sample u from the uniform, $U(0,1)$, distribution
f. Accept or reject the new candidate θ'

$$\begin{cases} If \quad u \le \beta \quad set \quad \theta^{(i)} = \theta' \\ otherwise \quad set \quad \theta^{(i)} = \theta. \end{cases}$$

Finally, some of the initial samples can be rejected (burn-in) for the random samples of size M selected from the posterior density, and the surviving samples may be used to construct Bayes estimates. The equation (10) could get estimated precisely as

$$\tilde{g}_{MH}(\alpha, \lambda) = \frac{1}{M - l_B} \sum_{i=l_B}^{M} g(\alpha_i, \lambda_i) \quad (11)$$

where l_B is the number of (burn-in) discarded samples.

3.1.2 Highest Posterior Density

In this sub-section, using samples chosen through the proposed MH algorithm, we generate HPD intervals for the unknown GIE distribution parameters α and λ using a PCTI censoring scheme. Consider that the γth quantiles of α and λ are $\alpha^{(\gamma)}$ and $\lambda^{(\gamma)}$.

$$(\alpha^{(\gamma)}, \lambda^{(\gamma)}) = \inf\{(\alpha, \lambda) : \Pi((\alpha, \lambda)|x) \geq \gamma\}$$

where $0 < \gamma < 1$ and $\Pi(\cdot)$ is the posterior distribution function of α and λ. For a given α^* and λ^*, a simulated consistent estimator of $\pi((\alpha, \lambda)|x)$ can be precisely estimated as

$$\Pi((\alpha^*, \lambda^*)|x) = \frac{1}{M - l_B} \sum_{i=l_B}^{M} I_{(\alpha,\lambda) \leq (\alpha^*, \lambda^*)}$$

Here $I_{(\lambda,\alpha) \leq (\lambda^*, \alpha^*)}$ is the indicator function. Therefore, the corresponding estimate is obtained as

$$\widehat{\Pi}((\alpha^*, \lambda^*)|x) = \begin{cases} 0 & if \quad (\alpha^*, \lambda^*) < (\alpha_{(l_B)}, \lambda_{(l_B)}) \\ \sum_{j=l_B}^{i} \omega_j & if \quad (\alpha_{(i)}, \lambda_{(i)}) < (\alpha^*, \lambda^*) < (\alpha_{(i+1)}, \lambda_{(i+1)}) \\ 1 & if \quad (\alpha^*, \alpha^*) > (\alpha_{(M)}, \lambda_{(M)}) \end{cases}$$

where $\omega_j = \frac{1}{M - l_B}$ and $(\alpha_{(j)}, \lambda_{(j)})$ are the ordered values of (α_j, λ_j). Now, for $i = l_B, \dots, M$, $(\alpha^{(\gamma)}, \lambda^{(\gamma)})$ can be approximatly estimated using

$$(\tilde{\alpha}^{(\gamma)}, \tilde{\lambda}^{(\gamma)}) = \begin{cases} (\alpha_{(l_B)}, \lambda_{(l-B)}) & if \quad \gamma = 0 \\ (\alpha_{(i)}, \lambda_{(i)}) & if \quad \sum_{j=l_B}^{i-1} \omega_j < \gamma < \sum_{j=l_B}^{i} \omega_j. \end{cases}$$

To have a $100(1-\gamma)\%$ HPD credible interval for α and λ, assume

$$HPD_j^{\alpha} = \left(\tilde{\alpha}^{\left(\frac{j}{M}\right)}, \tilde{\alpha}^{\left(\frac{j+(1-\gamma)M}{M}\right)}\right) \quad \& \quad HPD_j^{\lambda} = \left(\tilde{\lambda}^{\left(\frac{j}{M}\right)}, \tilde{\lambda}^{\left(\frac{j+(1-\gamma)M}{M}\right)}\right)$$

for $j = l_B, \dots, [\gamma M]$, here $[a]$ denotes the largest integer that is less than or equal to a. Then choose HPD_{j^*} among all the $HPD_j^{\prime}$s with the smallest width.

4. Simulation Study and Real Data Analysis

In this section, the purpose is to analyze the performance of the various estimation methods presented in sections above. For illustrative purposes, a real given dataset is used; additionally, a simulation study to examine the behavior of the suggested methods and to test the statistical performances of the estimators is utilized given a progressive Type-I censoring scheme. Calculations have been performed using the R-statistical programming language. Calculating MLEs and HPD intervals in R-language is done through utilizing the $bbmle$ and $HDInterval$ packages.

4.1 Simulation Study

In this sub-section, to analyze the performance of estimation methods, including MLE and Bayesian estimation, a Monte Carlo simulation study is employed, under PCTI scheme for Gompertz distribution. For the MLEs, 1000 observations are generated from NH distribution based on the assumptions:

1. $\alpha = 1.5$ and $\lambda = 2.5$, i.e., $Gompertz(0.5,1.5)$.
2. Sample sizes of $n = 25$, $n = 50$ and $n = 100$.
3. Number of progressive Type-I censoring stages are $m = 3,5$.
4. Time of censoringT_j (TC) are proposed as follows:

- $TC - I = (0.25, 0.55, 2)$
- $\mathrm{TC} - \mathrm{II} = (0.45, 1.25, 3.5)$
- $TC - III = (0.15, 0.45, 0.85, 1.5, 2.6)$
- $TC - IV = (0.30, 0.70, 1.25, 2.13, 4)$

where $j = 1, \ldots, m$. The patterns of TC can be classified according to m. In our study, $TC - I$ and $TC - II$ are used when $m = 3$ and $TC - III$ and $TC - IV$ are used when $m = 5$.

5. Removed items R_j are assumed at different sample size n as shown in Table 1

where $R_m = n - (\sum_{j=1}^{m-1} R_j + r)$ and r is the number of failure items.

Table 1: Numerous patterns for removing items from life test at different number of stages

m	Scheme	Patterns		
		$n=25$	$n=50$	$n=100$
3	PCTI-1	$\mathcal{R}=(0^{(2)},R_m)$	$\mathcal{R}=(0^{(2)},R_m)$	$\mathcal{R}=(0^{(2)},R_m)$
	PCTI-2	$\mathcal{R}=(3^{(2)},R_m)$	$\mathcal{R}=(5^{(2)},R_m)$	$\mathcal{R}=(9^{(2)},R_m)$
	PCTI-3	$\mathcal{R}=(5^{(2)},R_m)$	$\mathcal{R}=(9^{(2)},R_m)$	$\mathcal{R}=(18^{(2)},R_m)$
	PCTI-4	$\mathcal{R}=(6,0,R_m)$	$\mathcal{R}=(10,0,R_m)$	$\mathcal{R}=(18,0,R_m)$
	PCTI-5	$\mathcal{R}=(10,0,R_m)$	$\mathcal{R}=(18,0,R_m)$	$\mathcal{R}=(36,0,R_m)$
	PCTI-6	$\mathcal{R}=(0,6,R_m)$	$\mathcal{R}=(0,10,R_m)$	$\mathcal{R}=(0,18,R_m)$
	PCTI-7	$\mathcal{R}=(0,10,R_m)$	$\mathcal{R}=(0,18,R_m)$	$\mathcal{R}=(0,36,R_m)$
5	PCTI-8	$\mathcal{R}=(0^{(4)},R_m)$	$\mathcal{R}=(0^{(4)},R_m)$	$\mathcal{R}=(0^{(4)},R_m)$
	PCTI-9	$\mathcal{R}=(2^{(4)},R_m)$	$\mathcal{R}=(3^{(4)},R_m)$	$\mathcal{R}=(5^{(4)},R_m)$
	PCTI-10	$\mathcal{R}=(3^{(4)},R_m)$	$\mathcal{R}=(5^{(4)},R_m)$	$\mathcal{R}=(10^{(4)},R_m)$
	PCTI-11	$\mathcal{R}=(8,0^{(3)},R_m)$	$\mathcal{R}=(12,0^{(3)},R_m)$	$\mathcal{R}=(20,0^{(3)},R_m)$
	PCTI-12	$\mathcal{R}=(12,0^{(3)},R_m)$	$\mathcal{R}=(20,0^{(3)},R_m)$	$\mathcal{R}=(40,0^{(3)},R_m)$
	PCTI-13	$\mathcal{R}=(0^{(3)},8,R_m)$	$\mathcal{R}=(0^{(3)},12,R_m)$	$\mathcal{R}=(0^{(3)},20,R_m)$
	PCTI-14	$\mathcal{R}=(0^{(3)},12,R_m)$	$\mathcal{R}=(0^{(3)},20,R_m)$	$\mathcal{R}=(0^{(3)},40,R_m)$

Here, $(3^{(2)}, 0)$, for example, means that the censoring scheme employed is (3,3,0). It is indicated that scheme $\text{PCTI} - 1$ and $\text{PCTI} - 8$ are represent a special case of Type-I censoring scheme with number of failure items $R_m = n - r$ and CT is T_m. MLEs and related 95% asymptotic confidence intervals are produced based on the generated data. On deriving MLEs, be aware that the initial assume values are regarded as true parameter values.

We compute Bayesian estimates employing the Gompertz algorithm using informative priors for the Bayesian estimation method. As historical samples, we construct 1000 completed samples of size 60 each from the $Gompertz(0.5,1.5)$ distribution, then determine the values of the hyper parameter as $a_1 = 7.18, b_1 = 12.77, a_2 = 1.62, b_2 = 0.96$.

Such values of informative priors are plugged-in to evaluate the required estimates. Trough implementing the MH algorithm, the MLEs are used as initial guess values, as well as the corresponding variance-covariance matrix S_θ of $(\ln(\hat{\alpha}), \ln(\hat{\lambda}))$. In the end, 1600 burn-in samples were deleted from the total of 8,000 generated samples by the posterior density, and progressively produced Bayes estimates and HPD interval estimates adopting the technique of Chen and Shao (1999). All the average estimates for methods are reported in Table 2, Table 3, and Table 4 for samples size $n = 25$, $n = 50$, and $n = 100$, respectively. Further, the first row donates the average estimates (Avg.) and in the second row, related means square errors (MSEs). For confidence intervals, we have asymptotic confidence interval for MLEs and HPD for Bayesian estimates based on MCMC which reported in Table 5, Table 6, and Table 7 for samples size $n = 25$, $n = 50$, and $n = 100$, respectively. Further, the first row represents average interval lengths (AILs) and in the second row, related coverage probabilities (CPs).

According to the tabulated values, greater values of n inevitably lead to better estimates based on MSEs. It is also noted that MLEs compete directly well with informative Bayes estimates. Moreover, when the units are eliminated, the MSEs and AILs of related interval estimates are often smaller, when the units are terminated at early stages. The convergence of MCMC estimation of α and λ is illustrated in two figures. First; Figure 4 for $m = 3$ and pattern of censoring PTIC-3 and $TC - I$ for choosing sample size $n = 50$. Second; Figure 5 for $m = 5$ and pattern of censoring PTIC-10 and $TC - IV$ for choosing sample size $n = 50$.

Figure 4: Distribution and convergence of MCMC estimates for α and λ using MH algorithm under PCTI-3 and $TC - I$ where $m = 3$ and $n = 50$

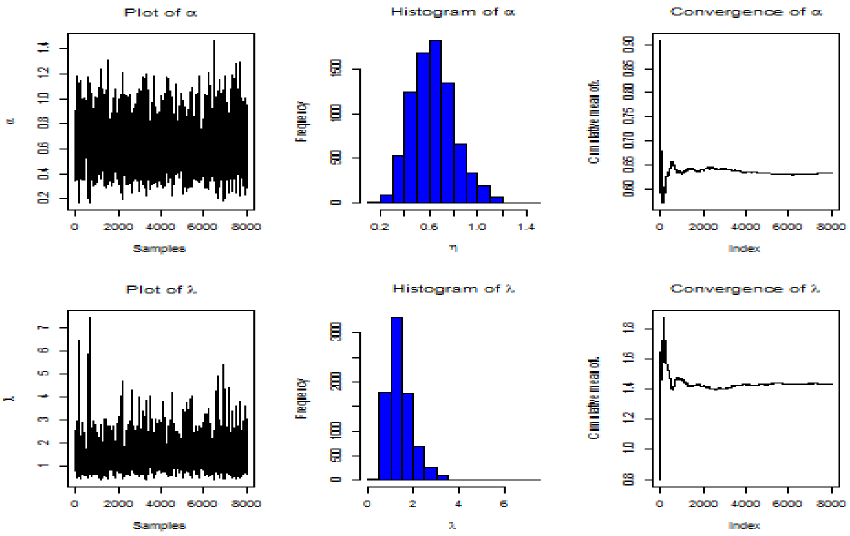

Figure 5: Distribution and convergence of MCMC estimates for α and λ using MH algorithm under PCTI-10 and $TC - IV$ where $m = 5$ and $n = 50$

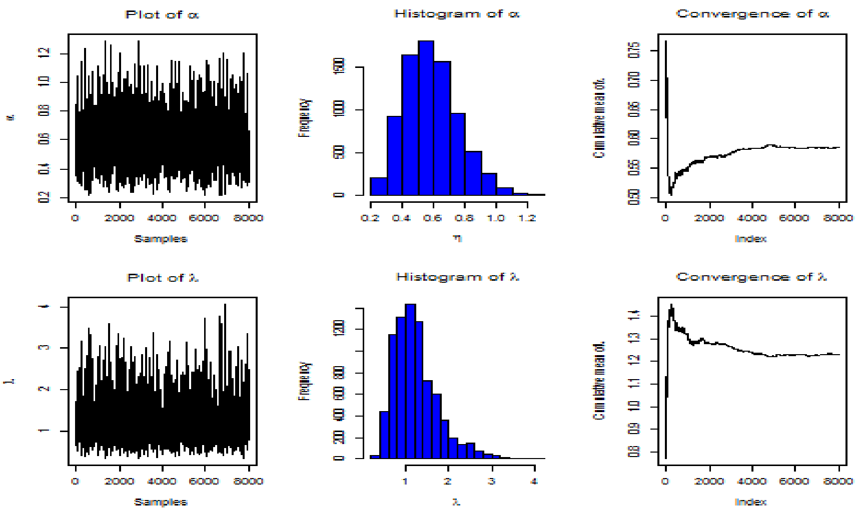

Table 2: Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of PCTI Gompertz data at sample size $n = 25$ with $\alpha = 0.5$ and $\lambda = 1.5$

m	Scheme		$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$
			TC-I = (0.25, 0.55, 2)				TC-II = (0.45, 1.25, 3.5)			
3	PCTI-1	Avg.	0.9677	2.5711	0.6088	1.3500	0.7589	2.8130	0.6045	1.3619
		MSE	0.7333	34.182	0.0234	0.8195	0.3200	35.632	0.0207	0.2352
	PCTI-2	Avg.	1.0269	2.3403	0.6014	1.3613	0.8306	2.4159	0.6091	1.3651
		MSE	0.8742	27.954	0.0172	0.2011	0.4246	27.475	0.0188	0.2094
	PCTI-3	Avg.	1.1328	2.5455	0.6067	1.3573	1.0323	2.3708	0.6105	1.3454
		MSE	1.0506	41.860	0.0476	0.9902	0.8440	34.427	0.0194	0.2514
	PCTI-4	Avg.	1.2426	2.5942	0.6021	1.3623	1.1909	1.9504	0.6125	1.3683
		MSE	1.4362	45.419	0.0146	0.2836	1.1793	22.839	0.0179	0.4712
	PCTI-5	Avg.	3.0034	1.7821	0.6009	1.2955	1.4002	2.4439	0.5928	1.3486
		MSE	13.235	30.722	0.1022	0.9647	1.8887	41.121	0.0150	0.4154
	PCTI-6	Avg.	0.9205	2.8128	0.6065	1.3878	0.7310	2.7088	0.6160	1.3833
		MSE	0.6404	38.119	0.0169	0.7542	0.3036	29.061	0.2117	0.2879
	PCTI-7	Avg.	1.0338	2.3847	0.6044	1.3880	0.7669	2.7028	0.6111	1.3665
		MSE	0.8263	31.541	0.0189	1.4207	0.3303	30.615	0.0459	0.2973
			TC-III = (0.15, 0.45, 0.85, 1.5, 2.6)				TC-IV = (0.30, 0.70, 1.25, 2.13, 4)			
5	PCTI-8	Avg.	0.7817	2.7769	0.5648	1.4344	0.7000	2.5971	0.5861	1.4083
		MSE	0.3706	11.788	0.0135	0.2107	0.2376	11.818	0.0155	0.2260
	PCTI-9	Avg.	0.9055	2.4300	0.5653	1.4502	0.8689	2.3697	0.5830	1.4113

		MSE	0.5834	9.2526	0.0119	0.1968	0.5121	9.3213	0.0163	0.2135
	PCTI-10	Avg.	1.0701	1.9935	0.5680	1.4721	0.9858	2.4346	0.5854	1.3696
		MSE	0.8933	9.4475	0.0083	0.3912	0.7927	9.5905	0.0447	0.2088
	PCTI-11	Avg.	0.9396	2.2192	0.5775	1.4362	0.7754	2.8923	0.6000	1.4119
		MSE	0.6276	8.5497	0.0115	0.2235	0.3384	8.1076	0.1939	0.2250
	PCTI-12	Avg.	0.9526	2.3661	0.5665	1.4503	0.8747	2.2995	0.5844	1.4012
		MSE	0.6925	9.0145	0.0105	0.2317	0.5308	9.1926	0.0142	0.2442
	PCTI-13	Avg.	0.9861	2.7320	0.5671	1.4674	0.9010	2.2049	0.5867	1.4199
		MSE	0.7864	11.963	0.0117	0.4533	0.5538	9.7758	0.0149	0.1977
	PCTI-14	Avg.	1.1152	2.3162	0.5659	1.4685	0.8690	2.8618	0.5770	1.4398
		MSE	1.0137	11.837	0.0109	0.5452	0.5383	12.009	0.0147	0.2702

Table 3: Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of PCTI Gompertz data at sample size $n = 50$ with $\alpha = 0.5$ and $\lambda = 1.5$

m	Scheme		$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$
			TC-I = (0.25, 0.55, 2)				TC-II = (0.45, 1.25, 3.5)			
3	PCTI-1	Avg.	0.7390	3.0519	0.5915	1.4403	0.6291	2.6418	0.5921	1.4592
		MSE	0.3189	9.4902	0.0191	0.3766	0.1515	6.0803	0.0211	0.3927
	PCTI-2	Avg.	0.8011	2.5679	0.5967	1.4029	0.6910	2.6517	0.5986	1.4192
		MSE	0.3819	10.186	0.0179	0.2453	0.2129	9.3341	0.0229	0.3498
	PCTI-3	Avg.	0.8748	2.5555	0.5989	1.4188	0.7514	2.4376	0.5995	1.4221
		MSE	0.5119	8.2528	0.0186	0.2351	0.3019	6.9196	0.0233	0.3843
	PCTI-4	Avg.	0.8847	2.4505	0.6039	1.3750	0.7478	2.4788	0.6017	1.3817

		MSE	0.5184	6.2125	0.0170	0.2104	0.3016	9.2935	0.0202	0.2124
	PCTI-5	Avg.	1.5613	2.0127	0.5969	1.3882	1.0700	2.7945	0.5912	1.4193
		MSE	2.6891	10.379	0.0129	0.2520	0.9750	12.411	0.0179	0.2548
	PCTI-6	Avg.	0.7677	2.5999	0.5999	1.4383	0.6435	2.9360	0.5927	1.4778
		MSE	0.3513	11.256	0.0260	0.3241	0.1720	11.907	0.0236	0.5429
	PCTI-7	Avg.	0.7734	2.7190	0.6021	1.4087	0.6199	2.8267	0.5869	1.4717
		MSE	0.3526	8.3449	0.0190	0.2738	0.1620	8.9953	0.0239	0.5349
			TC-III = (0.15, 0.45, 0.85, 1.5, 2.6)				TC-IV = (0.30, 0.70, 1.25, 2.13, 4)			
5	PCTI-8	Avg.	0.6434	3.2483	0.5725	1.5699	0.5873	2.7209	0.5673	1.5385
		MSE	0.1978	7.1641	0.0183	0.5322	0.1198	4.3949	0.0196	0.4664
	PCTI-9	Avg.	0.7140	2.7001	0.5710	1.5417	0.6416	2.7907	0.5741	1.4724
		MSE	0.2690	6.2278	0.0234	0.4999	0.1686	4.7119	0.0168	0.2712
	PCTI-10	Avg.	0.7642	2.4687	0.5773	1.5185	0.7284	2.3271	0.5831	1.4470
		MSE	0.3385	4.8471	0.0175	0.4061	0.2763	5.2684	0.0244	0.2292
	PCTI-11	Avg.	0.7226	2.5457	0.5800	1.4920	0.6412	2.6695	0.5754	1.4718
		MSE	0.2580	7.1739	0.0181	0.4182	0.1580	7.6975	0.0176	0.3041
	PCTI-12	Avg.	0.7376	2.7943	0.5815	1.4680	0.6839	2.6584	0.5861	1.4290
		MSE	0.3051	6.7629	0.0164	0.3396	0.2051	3.5665	0.0176	0.2436
	PCTI-13	Avg.	0.7377	2.7426	0.5719	1.5541	0.6869	2.5124	0.5759	1.4500
		MSE	0.2757	4.1830	0.0190	0.6194	0.2110	5.4127	0.0183	0.2452
	PCTI-14	Avg.	0.8091	2.6045	0.5708	1.5191	0.6991	3.0343	0.5812	1.4651
		MSE	0.4184	2.7520	0.0162	0.2822	0.2509	4.3184	0.0164	0.2932

Table 4: Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of PCTI Gompertz data at sample size $n = 100$ with $\alpha = 0.5$ and $\lambda = 1.5$

m	Scheme		$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$
			TC-I = (0.25, 0.55, 2)				TC-II = (0.45, 1.25, 3.5)			
3	PCTI-1	Avg.	0.6189	3.0645	0.5672	1.5930	0.5374	2.5959	0.5564	1.6506
		MSE	0.1620	4.0252	0.0233	0.8349	0.0782	4.8447	0.0215	0.9735
	PCTI-2	Avg.	0.6574	2.5812	0.5787	1.5273	0.5539	2.7196	0.5712	1.5800
		MSE	0.1899	2.6238	0.0220	0.5667	0.0929	4.5614	0.0864	0.7077
	PCTI-3	Avg.	0.6746	2.8032	0.5831	1.4912	0.6034	2.2969	0.5731	1.5555
		MSE	0.2122	3.6927	0.0205	0.4459	0.1157	3.4254	0.0215	0.6735
	PCTI-4	Avg.	0.6966	2.5780	0.5905	1.4483	0.5968	2.6056	0.5745	1.5474
		MSE	0.2340	2.7398	0.0208	0.3532	0.1232	5.3136	0.0215	0.6288
	PCTI-5	Avg.	0.9371	2.8187	0.5957	1.4186	0.7725	3.1949	0.5769	1.4964
		MSE	0.6709	3.5062	0.0165	0.1914	0.3820	2.3891	0.0210	0.3405
	PCTI-6	Avg.	0.6171	2.9533	0.5710	1.5559	0.5339	2.8224	0.5476	1.6856
		MSE	0.1646	2.5731	0.0229	0.6724	0.0835	4.2485	0.0241	1.0595
	PCTI-7	Avg.	0.6313	2.8073	0.5748	1.5591	0.5488	2.5607	0.5559	1.6815
		MSE	0.1646	3.5396	0.0216	0.5936	0.0798	3.2904	0.0225	1.1047
			TC-III = (0.15, 0.45, 0.85, 1.5, 2.6)				TC-IV = (0.30, 0.70, 1.25, 2.13, 4)			
5	PCTI-8	Avg.	0.5674	2.6574	0.5414	1.6309	0.5353	2.6575	0.5314	1.6675
		MSE	0.1035	2.8039	0.0180	0.6796	0.0711	2.2160	0.0190	0.7130
	PCTI-9	Avg.	0.5905	2.6067	0.5465	1.6010	0.5576	2.5566	0.5340	1.6341
		MSE	0.1113	2.1867	0.0158	0.4730	0.0834	1.5693	0.0142	0.5404
	PCTI-10	Avg.	0.6395	2.6969	0.5580	1.5617	0.5949	2.8591	0.5340	1.6210

		MSE	0.1755	2.8176	0.0156	0.4596	0.1343	2.6221	0.0151	0.5106
	PCTI-11	Avg.	0.5744	2.5835	0.5482	1.5517	0.5435	2.6863	0.5278	1.6843
		MSE	0.1032	1.3209	0.0146	0.3749	0.0797	2.0733	0.0148	0.6958
	PCTI-12	Avg.	0.6418	2.5706	0.5616	1.5183	0.5639	2.6105	0.5354	1.6468
		MSE	0.1564	2.2589	0.0169	0.3603	0.0965	1.7152	0.0208	0.6054
	PCTI-13	Avg.	0.6039	2.7270	0.5480	1.6150	0.5710	2.5270	0.5317	1.6607
		MSE	0.1354	2.9669	0.0177	0.5920	0.0988	2.2898	0.0153	0.6869
	PCTI-14	Avg.	0.6651	2.8963	0.5537	1.5810	0.5926	2.7871	0.5326	1.6773
		MSE	0.2057	2.9896	0.0171	0.4387	0.1293	2.3861	0.0160	0.7674

Table 5: AILs and CP (%) values of the ML and BE using MCMC for different schemes of PCTI Gompertz data at sample size $n = 25$ with $\alpha = 0.5$ and $\lambda = 1.5$

m	Scheme		$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$
			TC-I = (0.25, 0.55, 2)				TC-II = (0.45, 1.25, 3.5)			
3	PCTI-1	AIL	2.0769	9.7252	0.2683	1.5319	1.5844	8.0636	0.3498	1.6219
		CP	92.0	95.5	97.9	95.7	93.2	94.6	96.4	96.4
	PCTI-2	AIL	2.2048	9.0339	0.2458	1.6641	1.7872	8.8720	0.2966	1.5850
		CP	91.1	95.1	97.6	95.9	94.8	94.6	97.7	95.6
	PCTI-3	AIL	2.3278	7.1622	0.2210	1.7685	2.0699	5.7636	0.2778	1.6563
		CP	91.5	94.8	98.5	95.9	90.3	94.6	98.1	96.2
	PCTI-4	AIL	2.5119	6.7320	0.1978	1.9051	2.3641	5.0209	0.2394	1.6137
		CP	90.2	95.7	98.5	96.1	90.7	94.6	97.4	96.7
	PCTI-5	AIL	5.6018	7.6781	0.1798	2.5968	2.9459	8.4617	0.1955	1.8762

		CP	91.6	95.4	97.6	95.1	93.1	94.6	97.8	96.5
	PCTI-6	AIL	2.0501	11.5786	0.2506	1.5753	1.5929	9.7017	0.4006	1.5942
		CP	92.5	92.9	97.6	96.6	94.1	94.6	98.9	95.6
	PCTI-7	AIL	2.1249	10.2432	0.2580	1.6047	1.6087	8.4279	0.4083	1.5983
		CP	91.2	93.3	99.0	96.4	93.7	94.6	98.9	96.2
			TC-III = (0.15, 0.45, 0.85, 1.5, 2.6)				TC-IV = (0.30, 0.70, 1.25, 2.13, 4)			
5	PCTI-8	AIL	1.6711	10.7931	0.3053	1.5183	1.4899	10.7152	0.3331	1.6238
		CP	93.6	94.8	98.5	96.5	94.8	94.4	98.1	95.4
	PCTI-9	AIL	1.9388	9.7287	0.2411	1.6045	1.8174	9.0732	0.3284	1.6317
		CP	92.7	95.2	97.8	96.8	92.3	95.4	98.6	96.3
	PCTI-10	AIL	2.2640	9.9578	0.2113	1.6642	2.0989	8.9071	0.2992	1.6897
		CP	93.2	95.5	97.9	96.8	92.2	95.6	98.8	95.7
	PCTI-11	AIL	1.9215	7.5084	0.2570	1.6446	1.6383	9.5342	0.2775	1.7119
		CP	91.1	95.2	98.6	97.0	94.1	95.0	97.5	96.8
	PCTI-12	AIL	1.9851	9.8999	0.2154	1.7204	1.8416	10.0424	0.2351	1.8600
		CP	91.5	95.6	97.3	95.6	92.7	96.1	98.2	96.3
	PCTI-13	AIL	2.0411	8.7487	0.2706	1.6310	1.8843	6.7860	0.2732	1.6100
		CP	90.5	95.6	97.8	95.7	92.6	95.4	98.8	95.5
	PCTI-14	AIL	2.3353	6.3256	0.2584	1.6327	1.8334	8.6165	0.3049	1.6779
		CP	91.5	95.2	98.6	95.7	92.6	95.2	98.4	96.1

Table 6: AILs and CP (%) values of the ML and BE using MCMC for different schemes of PCTI Gompertz data at sample size $n = 50$ with $\alpha = 0.5$ and $\lambda = 1.5$

m	Scheme		$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$
			TC-I = (0.25, 0.55, 2)				**TC-II = (0.45, 1.25, 3.5)**			
3	PCTI-1	AIL	1.5660	6.0747	0.3839	1.5053	1.2894	14.9474	0.4782	1.8238
		CP	92.8	79.5	99.4	95.4	95.8	99.0	98.3	95.6
	PCTI-2	AIL	1.7122	6.3099	0.3309	1.4569	1.3977	15.8915	0.4867	1.6225
		CP	93.9	69.5	99.2	96.1	93.4	99.1	98.6	95.6
	PCTI-3	AIL	1.8509	7.7779	0.3108	1.5310	1.5138	14.1922	0.4903	1.7146
		CP	92.6	96.1	98.7	96.3	93.0	99.1	98.4	95.9
	PCTI-4	AIL	1.8740	6.4181	0.2802	1.3324	1.5593	16.3550	0.3766	1.3803
		CP	93.3	95.9	98.9	95.3	92.8	99.6	98.6	95.2
	PCTI-5	AIL	3.1771	6.2094	0.1822	1.6200	2.1454	19.4794	0.3239	1.5017
		CP	88.0	96.1	97.3	97.6	89.8	100.0	99.4	96.4
	PCTI-6	AIL	1.6175	7.7169	0.4106	1.5621	1.3141	16.2431	0.5098	2.0338
		CP	92.8	95.2	98.8	95.6	95.1	98.4	98.4	95.1
	PCTI-7	AIL	1.6038	5.2287	0.3825	1.5110	1.2694	14.6012	0.5457	1.9924
		CP	92.5	93.4	98.7	96.7	93.8	98.6	99.1	95.6
			TC-III = (0.15, 0.45, 0.85, 1.5, 2.6)				**TC-IV = (0.30, 0.70, 1.25, 2.13, 4)**			
5	PCTI-8	AIL	1.3427	4.9492	0.4895	2.1345	1.1925	3.4769	0.5105	2.0054
		CP	93.5	94.5	98.8	96.1	95.6	98.7	98.3	96.0
	PCTI-9	AIL	1.4730	3.4065	0.4888	2.0162	1.3375	4.5505	0.4254	1.6078
		CP	92.8	96.6	99.6	96.2	96.2	98.5	98.7	96.2

	PCTI-10	AIL	1.5993	2.5140	0.4291	1.7461	1.5100	5.1118	0.3566	1.4566
		CP	93.8	95.5	98.8	96.1	93.8	95.3	98.8	95.6
	PCTI-11	AIL	1.4967	5.5498	0.4063	1.7180	1.2920	5.1912	0.4724	1.6427
		CP	93.3	96.5	99.1	95.5	94.0	95.9	98.2	95.5
	PCTI-12	AIL	1.5751	6.1973	0.3490	1.6305	1.3799	5.4425	0.3504	1.3788
		CP	94.2	96.8	98.9	95.8	95.0	95.3	98.6	96.2
	PCTI-13	AIL	1.5239	5.3348	0.4837	1.7761	1.4305	7.1870	0.4193	1.4846
		CP	94.4	96.9	99.5	95.5	94.8	95.4	99.6	95.5
	PCTI-14	AIL	1.7237	5.9576	0.4213	1.7118	1.4474	5.8059	0.3501	1.4859
		CP	92.1	95.4	99.1	95.5	93.4	97.4	98.8	95.7

Table 7: AILs and CP (%) values of the ML and BE using MCMC for different schemes of PCTI Gompertz data at sample size $n = 100$ with $\alpha = 0.5$ and $\lambda = 1.5$

m	Scheme		$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}$	$\tilde{\lambda}$
			TC-I = (0.25, 0.55, 2)				TC-II = (0.45, 1.25, 3.5)			
3	PCTI-1	AIL	1.3093	4.9428	0.5654	2.3447	1.0808	2.2393	0.5725	2.9260
		CP	95.1	97.7	99.4	95.2	97.6	97.8	98.7	95.0
	PCTI-2	AIL	1.3734	4.9199	0.5447	1.8886	1.1073	2.5922	0.5732	2.2361
		CP	94.9	98.6	98.3	95.3	95.6	98.0	97.0	95.3
	PCTI-3	AIL	1.4720	6.4409	0.4520	1.7238	1.2411	3.3637	0.5481	2.2377
		CP	95.1	98.5	99.8	95.7	97.2	98.8	98.1	95.3
	PCTI-4	AIL	1.4552	5.2888	0.4266	1.3794	1.2236	2.3496	0.5628	2.1016
		CP	95.0	99.3	98.6	95.6	95.9	97.8	98.2	95.1

	PCTI-5	AIL	1.9096	4.8576	0.2787	1.2333	1.6731	4.4877	0.5336	1.8188
		CP	90.5	99.4	98.8	95.5	93.4	98.6	99.2	95.3
	PCTI-6	AIL	1.3285	3.6700	0.5487	2.2829	1.0673	3.0921	0.6196	3.1686
		CP	94.5	98.3	98.7	95.6	96.0	97.4	96.6	95.1
	PCTI-7	AIL	1.3241	4.5526	0.5491	2.2755	1.0802	4.9253	0.6103	3.2220
		CP	95.6	98.0	98.2	95.2	96.2	97.3	96.2	95.2
			TC-III = (0.15, 0.45, 0.85, 1.5, 2.6)				TC-IV = (0.30, 0.70, 1.25, 2.13, 4)			
5	PCTI-8	AIL	1.1567	3.7013	0.5482	2.4948	1.0225	1.2223	0.5041	2.0363
		CP	96.4	98.0	96.6	95.4	95.9	96.3	98.8	95.7
	PCTI-9	AIL	1.2138	2.9957	0.4792	2.0098	1.0960	2.5120	0.4845	2.2491
		CP	96.2	98.0	98.5	95.7	96.7	97.0	97.7	95.0
	PCTI-10	AIL	1.3380	4.3689	0.4783	2.0010	1.2260	1.7541	0.4928	2.2318
		CP	94.6	97.9	98.8	95.2	95.3	97.5	97.9	95.1
	PCTI-11	AIL	1.2029	3.6790	0.4619	1.6351	1.0678	2.5227	0.5067	2.5152
		CP	97.0	98.2	98.5	95.1	97.5	97.3	97.4	95.1
	PCTI-12	AIL	1.3219	4.9162	0.4628	1.5810	1.1067	1.3594	0.4830	2.0814
		CP	96.5	98.2	98.3	95.6	95.4	98.6	99.2	95.3
	PCTI-13	AIL	1.2435	3.4322	0.5128	2.4050	1.1157	1.5757	0.5260	2.2962
		CP	95.5	97.7	98.8	95.1	95.8	97.7	98.3	95.6
	PCTI-14	AIL	1.4013	5.1401	0.4882	1.8429	1.2098	1.1004	0.4999	2.6056
		CP	95.3	97.8	98.5	95.2	94.8	98.2	98.0	95.2

4.2 Real data analysis

In this section, a real given data set is analyzed. The original dataset contains 34 observations of the vinyl chloride data obtained from Bhaumik *et. al* (2009) which represents clean up gradient ground–water monitoring wells in mg/L. The data are as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2

We first check whether the Gompertz distribution is suitable for analyzing this data set. The calculated Kolmogorov-Smirnov (K-S) distance between the empirical and the fitted extended for the Gompertz distribution was 0.0904 and its p-value is 0.9438 where $\hat{\alpha} = 0.0028$ and $\hat{\lambda} = 205.8005$ which indicate that this distribution can be considered as an adequate model for the given dataset. From the original data, seven PCTI schemes are generated with different m stages and terminated items R_j at CT T_j, where $j = 1,2, \dots, m$. Such several schemes is explained in Table 8. Note that: $R_m = n - (\sum_{j=1}^{m-1} R_j + r)$ and r is the number of failure units. Also, it can be summarized that Type-I censoring scheme, Scheme PCTI-6, is considered as a special case of PCTI and complete sampling is considered as a special case of PCTI when $T_m = \max(x) = 8$ and $R_1 = R_2 = \cdots = R_m = 0$.

Table 8: Different schemes for progressively Type-I censored samples

Scheme	m	CT (T_j)	Removed items (R_i)
PCTI-1	3	(0.5, 3, 6)	(10, 0 ,R_m)
PCTI-2	3	(0.5, 3, 6)	(5, 5, R_m)
PCTI-3	3	(0.5, 3, 6)	(0, 10, R_m)
PCTI-4	5	(0.5, 1.5, 3, 4.5, 6)	(10, 0, 0, 0, R_m)
PCTI-5	5	(0.5, 1.5, 3, 4.5, 6)	(0, 0, 0, 10, R_m)
PCTI-6	5	(0.5, 1.5, 3, 4.5, 6)	(0, 0, 0, 0, $n - r$)
PCTI-7	5	(0.5, 1.5, 3, 4.5, 8)	(0,0,0,0,0)

We compute the parameters of MLEs of α and λ and the associated 95 % asymptotic confidence interval estimates. Bayes estimates is computed utilizing the MH algorithm with the informative prior. Note that the non-informative prior is assumed where $a_1 = b_1 = a_2 = b_2 = 0$. While generating samples from the posterior distribution utilizing the MH algorithm, initial values of (α, λ) are considered as $(\alpha^{(0)}, \lambda^{(0)}) = (\hat{\alpha}, \hat{\lambda})$, where $\hat{\alpha}$ and $\hat{\lambda}$ are the MLEs of the parameters α and λ respectively. Thus, we considered the varianceâ€“covariance matrix S_θ of $(\ln(\hat{\alpha}), \ln(\hat{\lambda}))$, Using the delta approach, this can be easily accomplished. Eventually, 1600 burn-in samples are terminated from the entire 8000 samples generated by the posterior density, and adopted technique to produce Bayes estimates and HPD interval estimates of Chen and Shao (1999).

All of the estimated MLE values, as well as the related interval estimates (Asymptotic CI) and standard errors (St.Er), are illustrated in Table 9. Bayesian estimates utilising MCMC employing the MH algorithm, as well as associated HPD intervals and St.Er, are also calculated.

Table 9: MLE's and BEs with associated St.Er (in practices) and CIs based on different PCTI schemes for the given real data set

Scheme	Parm.	MLE		Bayesian: MCMC	
		Estimate (St.Er)	Asy CI	Estimate (St.Er)	HPD
PCTI-1	α	0.00270 (٥.٣٥× 10^{-3})	(0.0026, 0.0028)	0.0039 (0.0004)	(0.0032, 0.0041)
	λ	٢٢٠.007 (6.50× 10^{-3})	(220.006, 220.008)	225.045 (0.0129)	(218.891, 232.194)
PCTI-2	α	0.00211 (4.23× 10^{-3})	(0.0020, 0.0023)	0.0024 (0.0002)	(0.0032, 0.0041)
	λ	220.132 (5.59× 10^{-3})	(220.006, 220.008)	222.155 (0.0082)	(217.312, 227.304)

PCTI-3	α	0.00722 (1.89× 10^{-2})	(0.0051, 0.0093)	0.0045 (0.0008)	(0.0032, 0.0049)
	λ	220.137 (2.53× 10^{-2})	(218.012, 222.326)	218.320 (0.0015)	(215.125, 230.897)
PCTI-4	α	0.002369 (4.54× 10^{-4})	(0.00236, 0.00237)	0.0024 (0.0044)	(0.0021, 0.0027)
	λ	220.0069 (4.87× 10^{-3})	(220.0068, 220.0071)	220.632 (0.0031)	(218.311, 222.021)
PCTI-5	α	0.00321 (1.07× 10^{-3})	(0.00291, 0.00331)	0.0026 (0.0051)	(0.0024, 0.0028)
	λ	220.0757 (4.87× 10^{-2})	(219.1133, 221.2350)	220.632 (0.0031)	(218.311, 222.021)
PCTI-6	α	0.00252 (2.10× 10^{-3})	(0.00221, 0.00253)	0.00275 (0.0021)	(0.00256, 0.00294)
	λ	220.132 (0.51× 10^{-2})	(219.1133, 221.2350)	220.12037 (0.0067)	(220.1083, 220.1285)
PCTI-7	α	0.00281 (6.01× 10^{-4})	(0.00280, 0.00281)	0.00280 (0.0001)	(0.00269, 0.00291)
	λ	205.8005 (3.49× 10^{-3})	(204.1352, 208.0211)	205.1522 (0.0053)	(205.1343, 205.1553)

Conclusion

In this paper, we investigated estimation problem of the parameters for the Gompertz distribution under PCTI from both a classical perspective and a Bayesian perspective We calculated MLEs and related asymptotic confidence intervals for the Gompertz distribution's unknown parameters. Then, with informative priors, Bayes estimates is generated with MCMC and the corresponding HPD interval estimates squared error loss function. Furthermore, if an informative prior is used, how to select hyper-parameter values in Bayesian estimates is investigated using historical samples. The results of the simulation show that MLEs informative Bayes estimates using MCMC better than both MLEs. For future, work, we used Bayesian estimation by utilizing MCMC, other methods such as Lindely's approximation or importance sampling can be employed under PCTI. Also, maximum product spacing can be used as alternative to classical estimation (MLEs). Thus, the current approach can be applied to establishing an optimal progressive censoring system, in addition to other censoring techniques.

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تقدير معالم توزيع جومبرتز باستخدام الطرق البييزية وغير البييزية في وجود مراقبة معجلة من النوع الأول

د. بريهان رشاد العمري

الملخص:

في هذا البحث تم تقدير معالم توزيع جومبرتز باستخدام الطرق البييزية وغير البييزية في حالة وجود مراقبة معجلة من النوع الأول. تم التقدير الغير البييزي للمعالم غير المعلومة للتوزيع المفترض باستخدام طريقة الأمكان الأعظم وتقدير فترة الثقة التقاربية. كما تم التقدير باستخدام أسلوب بايز بطريقة سلاسل ماركوف مونت كارلو باستخدام دالة الخسارة المربعة مع توظيف خوارزمية Metropolis-Hasting. أيضا تم الحصول على فترة الثقة للتقديرات البييزية باستخدام طريقة أكبر كثافة بعدية لدراسة خصائص وسلوك المقدرات عملياً لتوزيع جومبرتز في حالة المراقبة المعجلة من النوع الأول. في هذا البحث تم عمل محاكاة وتقدير المعالم بالطرق المقترحة لبيان اهمية النتائج النظرية للمقدرات ودراسة خصائصها من خلال مجموعة من البيانات الحقيقية. تم الاعتماد على المحاكاة العددية باستخدام أسلوب مونت كارلو للمحاكاة كما تم الاعتماد على بيانات واقعية للتحقق من مدي إمكانية تطبيق أنماط المراقبة في حال ان البيانات تتبع توزيع جومبرتز.