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ANALYTICAL SOLUTION OF SOME PROBLEMS OF APPLIED THEORY OF GYROSCOPES

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ABSTRACT

Motion of solid body with a fixed point is described by a system of nonlinear differential equations of motion of L. Euler. Gyroscopic instruments represent an axisymmetric solid body with a fixed point. The first partial solutions of the problem were obtained in works [1-4]. The subsequent development of Mechanics and Mathematics demonstrated that a nonlinear system of motion of L. Euler equations may describe a wide class of motions of celestial bodies, stability of motion of spacecrafts, Earth satellites, ships, aircrafts, monorail trains, etc.

The great interest to the problems with a fixed point is due to the gyroscopic effects, which became widespread in the modern technology, navigation and in many other areas.

More than 250 years have passed since the origination of nonlinear equations of motion of L. Euler, however, the interest in obtaining the solutions of these equations do not weaken. The world holds a huge amount of publications, for example, [5-14].

The analytical solutions of some problems characterizing the functioning of gyroscopes have been obtained based on the method of partial discretization of nonlinear differential equations, formed by the author of the report.

This paper demonstrates an analytical solution of the nonlinear problem on the motion of an axisymmetric solid body with a fixed point at a high velocity rate of self-rotation *n*. New analytical results have been obtained based on the method of partial discretization of nonlinear differential equations.

KEY WORDS

Gyroscope, Motion of solid body fixed at a point, Angular velocity of self-rotation, Nonlinear equations of motion, Unloading.

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NOTATION SYSTEM

$M_x, M_y, M_z,$	Moments of external forces relative to axes.
$x, y, z;\ \ p, g, r$	Projections of the angular velocity vector on the coordinates axis.
A, B, C	Inertia moments relative to axes $x, y, z;$
$\delta(z)$	Dirac delta function.
$f_1(t), f_2(t)$	Specified active forces.

MOTION OF SOLID BODY WITH A FIXED POINT AT A HIGH VELOCITY RATE OF SELF-ROTATION

First, let's consider the problem of solid body motion around a fixed point in resisting medium, which is described by nonlinear equations of motion of L. Euler

$$\begin{cases} A\dfrac{dp}{dt} + (C-B)qr = M_x, \\ B\dfrac{dq}{dt} + (A-C)rp = M_y, \\ C\dfrac{dr}{dt} + (B-A)pq = M_z, \end{cases} \tag{1}$$

where A, B, C – moments of body inertia relative to coordinate axes x, y, z connected with a body; p, q, r – projections of the body angular velocity vector onto these axes; M_x, M_y, M_z – moments of external resistance forces relative to axes x, y, z, and in general case depend on functions p, q, r and t.

Let's consider the system of differential equations (1) with the initial conditions

$$\begin{gathered} t=0: \quad p(0)=p_0, \quad q(0)=q_0, \quad r(0)=r_0, \\ \dot{p}(0)=\dot{p}_0, \quad \dot{q}(0)=\dot{q}_0, \quad \dot{r}(0)=\dot{r}_0. \end{gathered} \tag{2}$$

Without limiting the generality of the problem, then assume that

$$\dot{p}(0)=0, \quad \dot{q}(0)=0, \quad \dot{r}(0)=0\,. \tag{3}$$

We investigate the motion of an axisymmetric (A = B) solid body with a fixed point in a medium whose resistance, in general case, is proportional to arbitrariness of degree *n* of the body self-rotation velocity.

Assume, that

$$M_x = -\lambda_1 p, \quad M_y = -\lambda_2 q, \quad M_z = -\lambda_3 r^n . \tag{4}$$

Then, the system of differential equations (1) is written as

$$\begin{cases} \dot{p} + \mu r q + k_1 p = 0, \\ \dot{q} - \mu r p + k_2 q = 0, \\ r = \left[r_0^{1-n} - k_3 (1-n) t \right]^{\frac{1}{1-n}}. \end{cases} \tag{5}$$

where $M_1 = \frac{C-A}{A}, \quad k_1 = \frac{\lambda_1}{A}, \quad k_2 = \frac{\lambda_2}{A}, \quad k_3 = \frac{\lambda_3}{A}$

In accordance with r, the projection $p(t)$ of the angular velocity of a solid body satisfies the differential equation with variable coefficients

$$\ddot{p} + \left(k_1 + k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t} \right) \dot{p} + \left[\mu^2 \left[r_0^{1-n} - k_3(1-n)t \right]^{\frac{2}{1-n}} + k_1 \left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t} \right) \right] p = 0. \tag{6}$$

Usually, this kind of equation has no solution in elementary functions. By discretizing the last element of differential equation (6), we obtain:

$$\ddot{p} + \left(k_1 + k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t} \right) \dot{p} = -\frac{1}{2} \sum_{i=1}^{n} (t_i + t_{i+1}) \left\{ \left[\mu^2 \left[r_0^{1-n} - k_3(1-n) t_i \right]^{\frac{2}{1-n}} + \right.\right.$$

$$\left. + k_1 \left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n) t_i} \right) \right] p(t_i) \delta(t - t_i) - \left[\mu^2 \left[r_0^{1-n} - k_3(1-n) t_{i+1} \right]^{\frac{2}{1-n}} + \right. \tag{7}$$

$$\left.\left. + k_1 \left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n) t_{i+1}} \right) \right] p(t_{i+1}) \delta(t - t_{i+1}) \right\}.$$

The general solution of equation (7) has the form:

$$p(t)=C_2+\int\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\left\langle C_1-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\left\{\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{2}{1-n}}+\right.\right.\right.$$

$$\left.+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_i}\right)\right]\frac{e^{(k_1+k_2)t_i}}{\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{1}{1-n}}}p(t_i)H(t-t_i)- \tag{8}$$

$$-\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_{i+1}}\right)\right]\times$$

$$\left.\left.\times\frac{e^{(k_1+k_2)t_{i+1}}}{\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{1}{1-n}}}p(t_{i+1})H(t-t_{i+1})\right\}\right\rangle dt.$$

Considering initial conditions (2), (3), we obtain:

$$p(t)=p_0-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\left\{\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_i}\right)\right]\times\right.$$

$$\times\frac{e^{(k_1+k_2)t_i}}{\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{1}{1-n}}}p(t_i)H(t-t_i)\int_{t_i}^{t}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt-$$

$$-\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_{i+1}}\right)\right]\times \tag{9}$$

$$\left.\times\frac{e^{(k_1+k_2)t_{i+1}}}{\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{1}{1-n}}}p(t_{i+1})H(t-t_{i+1})\int_{t_{i+1}}^{t}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt\right\}.$$

In accordance with equation (9), the function $p(t)$ in points $t_i\left(i=\overline{1,n}\right)$ will be

$$p(t_i) = p_0 - \frac{1}{2}(t_1 + t_2)\left[\mu^2\left[r_0^{1-n} - k_3(1-n)t_1\right]^{\frac{2}{1-n}} + k_1\left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t_1}\right)\right]\times \tag{10}$$

$$\times\frac{e^{(k_1+k_2)t_1}}{\left[r_0^{1-n} - k_3(1-n)t_1\right]^{\frac{1}{1-n}}}\,p(t_1)\int_{t_1}^{t_i}\frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt +$$

$$+\frac{1}{2}\sum_{j=2}^{i-1}(t_{j+1} - t_{j-1})\left[\mu^2\left[r_0^{1-n} - k_3(1-n)t_j\right]^{\frac{2}{1-n}} + k_1\left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t_j}\right)\right]\times$$

$$\times\frac{e^{(k_1+k_2)t_j}}{\left[r_0^{1-n} - k_3(1-n)t_j\right]^{\frac{1}{1-n}}}\,p(t_j)\int_{t_j}^{t_i}\frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt.$$

Figure 1 shows the regularity of the projection change $p(t)$ of the body angular velocity for specific values of parameters $k_1 = k_2 = 0{,}1;\, k_3 = 0{,}5$ at $n = 3, 4, 5$. The graph shows that with the growth of parameter n, the body swings more and more.

Previously the solution of problem for n has been made equal to 1 and 2, which can be easily obtained from formula (10).

ANALYTICAL SOLUTION OF PROBLEM ON MOTION OF SOLID BODY WITH A FIXED POINT UNDER OTHER LAWS FOR MEDIUM RESISTANCE

Suppose, that

$$M_x = f_1(t) - \lambda_1 p, \quad M_y = f_2(t) - \lambda_2 q, \quad M_z = -\lambda_3 r^n. \tag{11}$$

The system of differential equations (1) will be as

$$\begin{cases} \dot{p} + \mu r q + k_1 p = \dfrac{f_1(t)}{A}, \\ \dot{q} - \mu r p + k_2 q = \dfrac{f_2(t)}{A}, \\ r = \left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}. \end{cases} \tag{12}$$

And projection $p(t)$ of the body angular velocity is determined by inhomogeneous differential equation

$$\ddot{p}+\left(k_1+k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t}\right)\dot{p}+\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t}\right)\right]p=F(t), \tag{13}$$

where $$F(t)=\frac{1}{A}\left[\dot{f}_1(t)+\left(k_1+\frac{k_3}{r_0^{1-n}-k_3(1-n)t}\right)f_1(t)-\mu\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}f_2(t)\right].$$

By applying the method of partial discretization of differential equations, the equation (13) may be written as

$$\begin{aligned}
&\ddot{p}+\left(k_1+k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t}\right)\dot{p}=F(t)-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\times\\
&\times\left\{\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_i}\right)\right]p(t_i)\delta(t-t_i)-\right.\\
&\left.-\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_{i+1}}\right)\right]p(t_{i+1})\delta(t-t_{i+1})\right\}.
\end{aligned} \tag{14}$$

The general solution of equation (14) has a formula

$$\begin{aligned}
&p(t)=C_2+\int\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\left\langle C_1+\int\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}F(t)dt-\right.\\
&-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\left\{\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_i}\right)\right]\times\right.\\
&\times\frac{e^{(k_1+k_2)t_i}}{\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{1}{1-n}}}p(t_i)H(t-t_i)-\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{2}{1-n}}+\right.\\
&\left.\left.+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_{i+1}}\right)\right]\frac{e^{(k_1+k_2)t_{i+1}}}{\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{1}{1-n}}}p(t_{i+1})H(t-t_{i+1})\right\}\right\rangle dt.
\end{aligned} \tag{15}$$

After determining the integral constants, we have

$$
\begin{aligned}
p(t) = {} & p_0 - \left[\int \frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}} F(t)dt \right]_{t=0} \int_0^t \frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}} dt + \\
& + \int_0^t \frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}} \int \frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}} F(t)dtdt - \\
& - \frac{1}{2}\sum_{i=1}^{n}(t_i + t_{i+1})\left\{\left[\mu^2\left[r_0^{1-n} - k_3(1-n)t_i\right]^{\frac{2}{1-n}} + k_1\left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t_i}\right)\right]\times\right. \\
& \times \frac{e^{(k_1+k_2)t_i}}{\left[r_0^{1-n} - k_3(1-n)t_i\right]^{\frac{1}{1-n}}} p(t_i)H(t - t_i)\int_{t_i}^{t} \frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}} dt - \\
& - \left[\mu^2\left[r_0^{1-n} - k_3(1-n)t_{i+1}\right]^{\frac{2}{1-n}} + k_1\left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t_{i+1}}\right)\right]\times \\
& \left.\times \frac{e^{(k_1+k_2)t_{i+1}}}{\left[r_0^{1-n} - k_3(1-n)t_{i+1}\right]^{\frac{1}{1-n}}} p(t_{i+1})H(t - t_{i+1})\int_{t_{i+1}}^{t} \frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}} dt\right\}.
\end{aligned}
\tag{16}
$$

In accordance with equation (16) the solution integral curve $p(t)$ in points t_i $\left(i = \overline{1,3}\right)$ will have analytic formulas

$$
\begin{aligned}
p(t_1) = {} & p_0 - \left[\int \frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}} F(t)dt \right]_{t=0} \int_0^{t_1} \frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}} dt + \\
& + \int_0^{t_1} \frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}} \int \frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}} F(t)dtdt \; ,
\end{aligned}
$$

$$
\begin{aligned}
p(t_2) = {} & p_0 - \left[\int \frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}} F(t)dt \right]_{t=0} \int_0^{t_2} \frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}} dt + \\
& + \int_0^{t_2} \frac{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}} \int \frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}} F(t)dtdt - \frac{1}{2}(t_1 + t_2)\times
\end{aligned}
$$

$$\times\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_1\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_1}\right)\right]\times$$

$$\times\frac{e^{(k_1+k_2)t_1}}{\left[r_0^{1-n}-k_3(1-n)t_1\right]^{\frac{1}{1-n}}}p(t_1)\int_{t_1}^{t_2}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt,$$

$$p(t_3)=p_0-\left[\int\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}F(t)dt\right]_{t=0}\int_0^{t_3}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt+$$

$$+\int_0^{t_3}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}F(t)dtdt-\frac{1}{2}(t_1+t_2)\times$$

$$\times\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_1\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_1}\right)\right]\times$$

$$\times\frac{e^{(k_1+k_2)t_1}}{\left[r_0^{1-n}-k_3(1-n)t_1\right]^{\frac{1}{1-n}}}p(t_1)\int_{t_1}^{t_3}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt+$$

$$+\frac{1}{2}(t_3-t_1)\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_2\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_2}\right)\right]\times$$

$$\times\frac{e^{(k_1+k_2)t_2}}{\left[r_0^{1-n}-k_3(1-n)t_2\right]^{\frac{1}{1-n}}}p(t_2)\int_{t_2}^{t_3}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt.$$

Application of the method of mathematical induction gives:

$$p(t_i)=p_0-\left[\int\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}F(t)dt\right]_{t=0}\int_0^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt+$$

$$+\int_0^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}F(t)dtdt-\frac{1}{2}(t_1+t_2)\times$$

$$
\times\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_1\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_1}\right)\right]\times
$$

$$
\times\frac{e^{(k_1+k_2)t_1}}{\left[r_0^{1-n}-k_3(1-n)t_1\right]^{\frac{1}{1-n}}}p(t_1)\int_{t_1}^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt+ \tag{17}
$$

$$
+\frac{1}{2}\sum_{j=2}^{i-1}\left(t_{j+1}-t_{j-1}\right)\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_j\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_j}\right)\right]\times
$$

$$
\times\frac{e^{(k_1+k_2)t_j}}{\left[r_0^{1-n}-k_3(1-n)t_j\right]^{\frac{1}{1-n}}}p(t_j)\int_{t_j}^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt.
$$

Herein, the formula (16) together with (17) will represent the analytical solution of specified problem.

MOTION OF SOLID BODY IN A RESISTING MEDIUM WITH UNLOADING AT ARBITRARY INDEX OF VELOCITY RATE OF SELF-ROTATION

Let’s consider the motion of an axisymmetric body in a resisting medium at arbitrary index of the velocity rate of self-rotation. Furthermore, suppose that moments being time functions act on the body, which from time moment $t=\tau$ are exposed to unloading.

In this case, the nonlinear equations of motion of L. Euler are written in the following form

$$
\begin{cases}
A\dfrac{dp}{dt}+(C-B)qr=f_1(t)[H(t)-H(t-\tau)]-\lambda_1 p,\\[2ex]
B\dfrac{dq}{dt}+(A-C)rp=f_2(t)[H(t)-H(t-\tau)]-\lambda_2 q,\\[2ex]
C\dfrac{dr}{dt}+(B-A)pq=-\lambda_3 r^n.
\end{cases} \tag{18}
$$

Since we consider the motion of an axisymmetric body, the system of nonlinear differential equations (18) has the form:

$$\begin{cases} \dot{p} + \mu r q + k_1 p = \dfrac{f_1(t)}{A}\left[H(t) - H(t-\tau)\right], \\ \\ \dot{q} - \mu r p + k_2 q = \dfrac{f_2(t)}{A}\left[H(t) - H(t-\tau)\right], \\ \\ r = \left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}}. \end{cases} \tag{19}$$

The differential equation that determines $p(t)$, has the form:

$$\ddot{p} + \left(k_1 + k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t}\right)\dot{p} + \left[\mu^2\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{2}{1-n}} + k_1\left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t}\right)\right]p =$$

$$= \frac{1}{A}\left\{\left[\dot{f}_1(t) + \left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t}\right)f_1(t)\right]\left[H(t) - H(t-\tau)\right] - \mu\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}} \times\right.$$

$$\left. \times f_2(t)\left[H(t) - H(t-\tau)\right] + f_1(0)\delta(t) - f_1(\tau)\delta(t-\tau)\right\}. \tag{20}$$

Solving (20) by the method, frequently used above, we obtain:

$$\ddot{p} + \left(k_1 + k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t}\right)\dot{p} = \frac{1}{A}\left\{\left[\dot{f}_1(t) + \left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t}\right)f_1(t)\right]\times\right.$$

$$\times\left[H(t) - H(t-\tau)\right] - \mu\left[r_0^{1-n} - k_3(1-n)t\right]^{\frac{1}{1-n}} f_2(t)\left[H(t) - H(t-\tau)\right] +$$

$$+ f_1(0)\delta(t) - f_1(\tau)\delta(t-\tau) - \frac{1}{2}\sum_{i=1}^{n}(t_i + t_{i+1})\left\{\left[\mu^2\left[r_0^{1-n} - k_3(1-n)t_i\right]^{\frac{2}{1-n}} +\right.\right.$$

$$\left. + k_1\left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t_i}\right)\right]p(t_i)\delta(t-t_i) - \left[\mu^2\left[r_0^{1-n} - k_3(1-n)t_{i+1}\right]^{\frac{2}{1-n}} +\right. \tag{21}$$

$$\left.\left. + k_1\left(k_2 + \frac{k_3}{r_0^{1-n} - k_3(1-n)t_{i+1}}\right)\right]p(t_{i+1})\delta(t-t_{i+1})\right\}.$$

The general solution of equation (21) is represented as:

$$p(t)=C_2+C_1\int\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt+\frac{1}{A}\int f_1(t)[H(t)-H(t-\tau)]dt-$$

$$-\frac{k_1}{A}\int\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}f_1(t)[H(t)-H(t-\tau)]dtdt- \tag{22}$$

$$-\frac{\mu}{A}\int\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int e^{(k_1+k_2)t}f_2(t)[H(t)-H(t-\tau)]dtdt-$$

$$-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\left\{\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_i}\right)\right]\times\right.$$

$$\times\frac{e^{(k_1+k_2)t_i}}{\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{1}{1-n}}}p(t_i)H(t-t_i)\int_{t_i}^{t}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt-$$

$$-\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_{i+1}}\right)\right]\times$$

$$\left.\times\frac{e^{(k_1+k_2)t_{i+1}}}{\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{1}{1-n}}}p(t_{i+1})H(t-t_{i+1})\int_{t_{i+1}}^{t}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt\right\}.$$

Accounting for the initial conditions of motion (2) (3) gives:

$$p(t)=p_0-\frac{\dot f(0)}{Ar_0}\int_0^t\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt+\frac{1}{A}\left[H(t)\int_0^t f_1(t)dt-H(t-\tau)\int_\tau^t f_1(t)dt\right]-$$

$$-\frac{k_1}{A}\left[H(t)\int_0^t\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int_0^t\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}f_1(t)dtdt-\right.$$

$$\left.-H(t-\tau)\int_\tau^t\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int_\tau^t\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}f_1(t)dtdt\right]- \tag{23}$$

$$-\frac{\mu}{A}\left[H(t)\int_0^t \frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int_0^t e^{(k_1+k_2)t} f_2(t)dtdt-\right.$$

$$\left.-H(t-\tau)\int_\tau^t \frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int_\tau^t e^{(k_1+k_2)t} f_2(t)dtdt\right]-$$

$$-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\left\{\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_i}\right)\right]\times\right.$$

$$\times\frac{e^{(k_1+k_2)t_i}}{\left[r_0^{1-n}-k_3(1-n)t_i\right]^{\frac{1}{1-n}}}p(t_i)H(t-t_i)\int_{t_i}^t \frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt-$$

$$-\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_{i+1}}\right)\right]\times$$

$$\left.\times\frac{e^{(k_1+k_2)t_{i+1}}}{\left[r_0^{1-n}-k_3(1-n)t_{i+1}\right]^{\frac{1}{1-n}}}p(t_{i+1})H(t-t_{i+1})\int_{t_{i+1}}^t \frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt\right\}.$$

In accordance with equation (23) the function $p(t)$ at the points t_i $\left(i=\overline{1,n}\right)$ will have the formulas

$$p(t_i)=p_0-\frac{\dot{f}(0)}{Ar_0}\int_0^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt+\frac{1}{A}\left[\int_0^{t_i} f_1(t)dt-H(t_i-\tau)\int_\tau^{t_i} f_1(t)dt\right]-$$

$$-\frac{k_1}{A}\left[\int_0^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int_0^t\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}f_1(t)dtdt-\right.$$

$$\left.-H(t_i-\tau)\int_\tau^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int_\tau^t\frac{e^{(k_1+k_2)t}}{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}f_1(t)dtdt\right]- \qquad (24)$$

$$-\frac{\mu}{A}\left[\int_0^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int_0^t e^{(k_1+k_2)t}f_2(t)dtdt-\right.$$

$$\left.-H(t_i-\tau)\int_\tau^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}\int_\tau^t e^{(k_1+k_2)t}f_2(t)dtdt\right]-$$

$$-\frac{1}{2}(t_1+t_2)\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_1\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_1}\right)\right]\times$$

$$\times\frac{e^{(k_1+k_2)t_1}}{\left[r_0^{1-n}-k_3(1-n)t_1\right]^{\frac{1}{1-n}}}p(t_1)\int_{t_1}^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt-$$

$$-\frac{1}{2}\sum_{j=2}^{i-1}(t_{j+1}-t_{j-1})\left[\mu^2\left[r_0^{1-n}-k_3(1-n)t_j\right]^{\frac{2}{1-n}}+k_1\left(k_2+\frac{k_3}{r_0^{1-n}-k_3(1-n)t_j}\right)\right]\times$$

$$\times\frac{e^{(k_1+k_2)t_j}}{\left[r_0^{1-n}-k_3(1-n)t_j\right]^{\frac{1}{1-n}}}p(t_j)\int_{t_j}^{t_i}\frac{\left[r_0^{1-n}-k_3(1-n)t\right]^{\frac{1}{1-n}}}{e^{(k_1+k_2)t}}dt.$$

The method of partial discretization of nonlinear differential equations allows to obtain a solution of problem practically for any laws of function change $f_1(t)$ и $f_2(t)$.

Figure 2 gives projection curve $p(t)$ of the body angular velocity at $\mathbf{n}=\mathbf{2}$.

In case of the simultaneous unloading of external effects $f_1(t)$ and $f_2(t)$ the change $p(t)$ is subject to two different laws that are performed in one case - in the time range of these perturbations action, and in the other - after the instantaneous removal of these moments.

The figure shows that at the time of unloading, the motion amplitude receives a second burst, after which the rotational vibration is attenuated with time.

CONCLUSION

Finally, we would like to note that analysis of solutions (10) of the first problem, as well as diagrams in Figure (1), show that growth of the parameter n in general has a little effect on change in the law of fluctuation the projection $p(t)$ of the angular velocity of the body, though with increasing time this change is much greater than initially. Meanwhile, presence of a resisting medium causes attenuation in a finite time range of fluctuation.

We solved the important problem of the motion of a rigid body in a resisting medium relieving existing forces. It is shown that when instantaneous relief the maximum amplitude of velocity has a degree equal to the maximum velocity when loading.

It is easy to obtain regularity of changes in the function $\mathrm{q(t)}$ in accordance with the above procedure of definition of angular velocity of a spherical motion $p(t)$.

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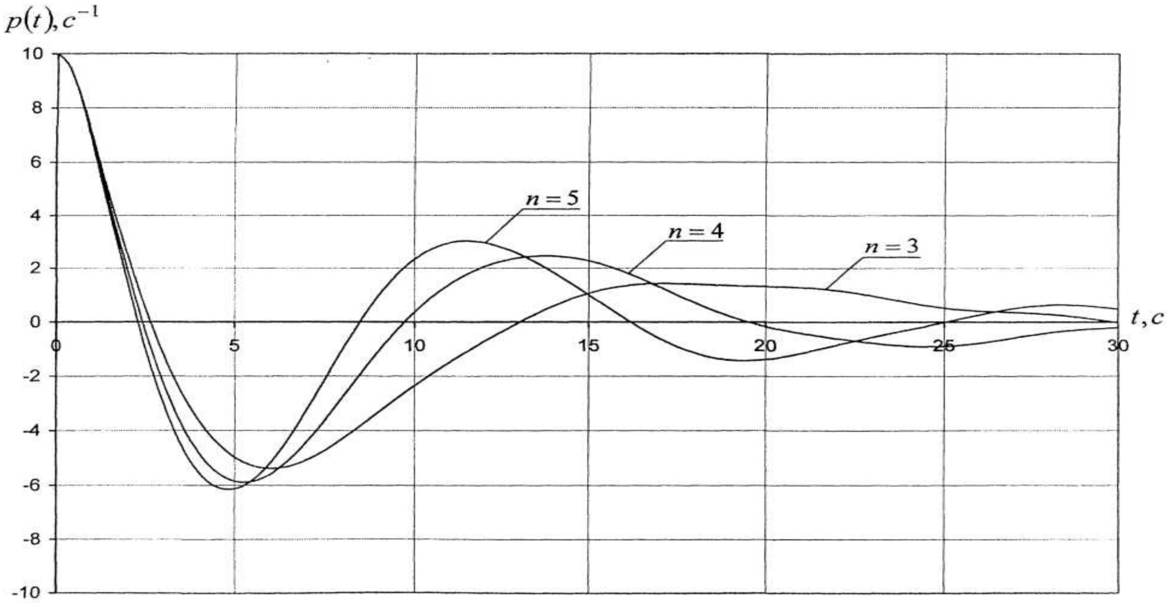

Figure 1

Projection vibration curves $p(t)$ of angular velocity for parameters values $A = 10\ kg \cdot m^2$, $C = 20\ kg \cdot m^2$, $r_0 = 1c^{-1}$, $p_0 = 10c^{-1}$.

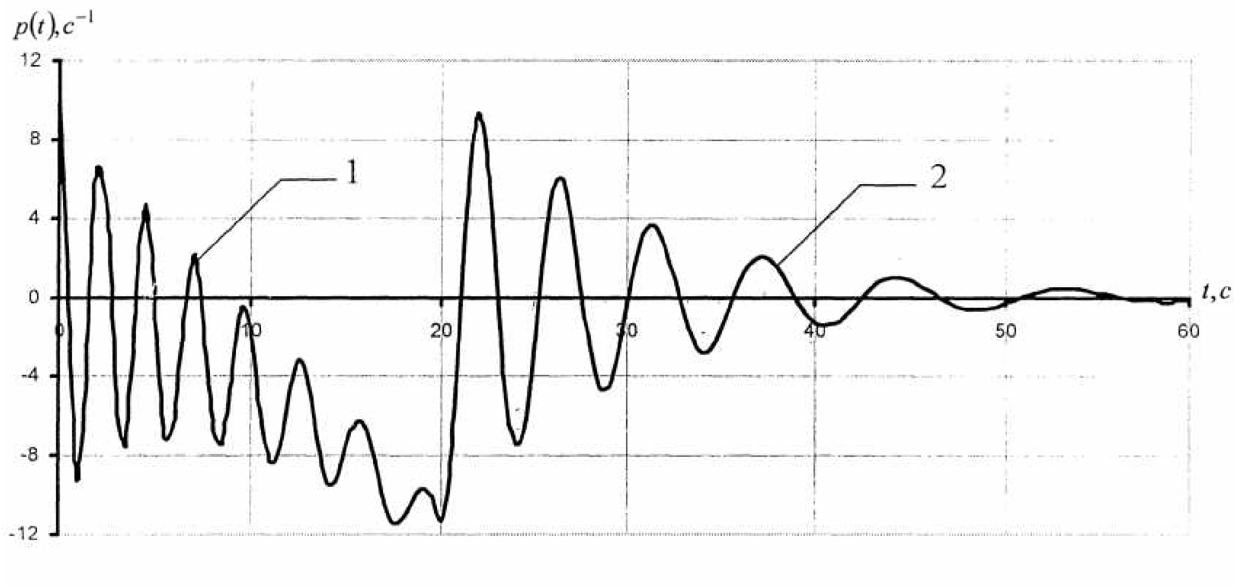

Figure 2

Schedule of unloading of disturbing components $f_1(t)$, $f_2(t)$

1 – Projection curve $p(t)$ of angular velocity before unloading;

2 – Projection curve $p(t)$ of angular velocity after unloading, at

$f_1(t) = 0{,}5t,\ f_2(t) = 10t,\ \tau = 20c,\ \lambda_1 = \lambda_2 = 1,\ \lambda_3 = 0{,}6,\ A = 10kg \cdot m^2,\ C = 20kg \cdot m^2$.