Bayesian Inference for Truncated Modified Weibull Distribution

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Abstract

In this paper, Bayes estimators of the unknown parameters, reliability and hazard rate functions for the truncated modified Weibull distribution are obtained. The estimators are derived under squared error loss function as a symmetric loss function and linear exponential loss function as an asymmetric loss function. Also, the two-sample Bayesian prediction for some order statistics in a future random sample drawn from the same population independently is used. Finally, a Monte-Carlo simulation study is carried out to illustrate the theoretical results of the Bayesian estimation and prediction. A real life data are applied to illustrate the results derived.

Keywords:

Double truncated distribution; Bayesian estimation; Bayes predictive; two-sample prediction; Bayesian prediction bounds; Monte-Carlo simulation.

1. Introduction

Truncated distributions have many applications in different fields of science. Truncated distributions are used when the range of a random variable is limited from below and or above for different reasons; test conditions, cost and other restrictions, this situation commonly happens in lifetime and reliability analyses.

Many authors discussed the truncated distributions such as Okasha and Alqanoo (2014) proposed truncated Gamma distribution and Singh *et al*. (2014) introduced truncated Lindley distribution. Also, Isik *et al*. (2017) studied double truncated Dagum distribution and Najarzadegan, *et al*. (2017) presented truncated Weibull-G; which is more flexible and more reliable than Beta-G distribution. Aydin (2018) considered the five-parameter doubly-truncated exponentiated inverse Weibull distribution; where the truncation points are known, and studied its basic properties. Al-Omari (2018) considered an acceptance sampling plan problem based on truncated life tests when the lifetime have a Sushila distribution.

A main area of application for the Weibull distribution is lifetime research and reliability theory. Lai *et al*. (2003) derived the *modified Weibull* (MW) from the Weibull distribution and studied some of its properties and estimation using the *maximum likelihood* (ML) method. MW distribution is one of the most important distributions in lifetime modeling. Sarhan and Zaindin (2009) presented the MW distribution to provide a good fit to data sets. It can be used to describe several reliability models. It has three parameters; one scale and two shape parameters.

EL-Helbawy *et al*. (2018) derived the *truncated modified Weibull* (TMW) distribution and discussed its properties. Also, they used the ML method to estimate the parameters, the reliability and hazard rate functions. They obtained the ML prediction bounds for some order statistics of future observations based on two-sample prediction.

The *probability density function* (pdf) of the TMW distribution as derived by EL-Helbawy *et al*. (2018) has the following form

$$f(x;\underline{\vartheta}) = k\left[\left(a + b\theta x^{\theta-1}\right)exp\left(-ax - bx^{\theta}\right)\right], \quad c < x < d, \quad \underline{\vartheta} > \underline{0}, \qquad (1)$$

where the constant k is given by

$$k = \frac{1}{exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})},$$

where $\underline{\vartheta} = (a, b, \theta, c, d)'$, $\theta > 0$ and $a,\ b \geq 0$ such that $a + b > 0$, a is a scale parameter while θ, b are shape parameters, c is the lower point of truncation and d is the upper point of truncation. The corresponding *cumulative distribution function* (cdf) is

$$F(x;\underline{\vartheta}) = \frac{exp(-ac - bc^{\theta}) - exp(-ax - bx^{\theta})}{exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})},$$
$$c < x < d, \qquad \underline{\vartheta} > \underline{0}. \qquad (2)$$

The *reliability function* (rf), *hazard rate function* (hrf) and *reversed hazard rate function* (rhrf) have the following forms, respectively

$$R(x;\underline{\vartheta}) = \frac{exp(-ax - bx^{\theta}) - exp(-ad - bd^{\theta})}{exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})},$$
$$c < x < d, \qquad \underline{\vartheta} > \underline{0}, \qquad (3)$$

$$h(x;\underline{\vartheta}) = \frac{(a + b\theta x^{\theta-1})exp(-ax - bx^{\theta})}{exp(-ax - bx^{\theta}) - exp(-ad - bd^{\theta})},$$
$$c < x < d, \qquad \underline{\vartheta} > \underline{0}, \qquad (4)$$

and

$$rh(x;\underline{\vartheta}) = \frac{(a + b\theta x^{\theta-1})exp(-ax - bx^{\theta})}{exp(-ac - bc^{\theta}) - exp(-ax - bx^{\theta})},$$
$$c < x < d, \qquad \underline{\vartheta} > \underline{0}. \qquad (5)$$

The rest of this paper is organized as follows: in Section 2, Bayesian estimation, of the parameters, rf and hrf functions are obtained for the double TMW distribution based on a joint non-informative prior. Also, Bayesian prediction for a future observation based on two-sample prediction is discussed in Section 3. Finally, a simulation study and an application to real data are presented to illustrate the results derived in Section 4.

2. Bayesian Estimation of the Truncated Modified Weibull Distribution

In this section, the Bayesian approach is considered, under *squared error loss* (SEL) function; as a symmetric loss function, and *linear exponential* (LINEX) loss function; as an asymmetric loss function, to estimate the unknown parameters, rf, hrf and rhrf of the double TMW distribution based on complete samples. Also credible intervals for the parameters, rf and hrf are obtained.

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a population having a double TMW$(\underline{\vartheta})$ at the truncation points $[c, d]$ given by (1). The *likelihood function* (LF) is given by

$$L(\underline{\vartheta}; \underline{x}) = \prod_{i=1}^{n} (a + b\theta x^{\theta-1})\, exp\left(-a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^{\theta}\right) \times \left[exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})\right]^{-n}. \quad (6)$$

2.1 Point Estimation

Suppose that the pdf of the double TMW $(\underline{\vartheta})$ distribution depends on the vector $\underline{\vartheta} = [a\ b\ \theta\ c\ d]'$ unknown parameters, where $\vartheta_1 = a,\ \vartheta_2 = b,\ \vartheta_3 = \theta$ are independent. In this case, the non-informative distribution are used as improper prior distribution, then the joint prior distribution of ϑ_1, ϑ_2 and ϑ_3 given by

$$\pi(a, b, \theta) \propto \frac{1}{a\, b\, \theta}, \quad a, b, \quad \theta > 0, \quad (7)$$

Let $\vartheta_4 = c,\ \vartheta_5 = d$ are dependent random variables $c < d$. Assuming that the conditional distribution of c and d is gamma distribution, then

$$\pi(c|d) = \frac{d^{\alpha}}{\Gamma(\alpha)} c^{\alpha-1} e^{-dc}, \qquad \alpha > 0, c < d, \tag{8}$$

Assuming that the distribution of d is gamma distribution, then

$$\pi(d) = \frac{{\alpha_2}^{\alpha_1}}{\Gamma(\alpha_1)} d^{\alpha_1-1} e^{-d\alpha_2}\ , \alpha_1, \alpha_2 > 0, c < d, \tag{9}$$

$$\pi(c,d) = \pi(c|d)\ \pi(d)$$

From (8) and (9)

$$\pi(c,d) \propto d^{(\alpha+\alpha_1)-1} c^{\alpha-1} e^{-d(c+\alpha_2)}, \quad \alpha_1, \alpha_2 > 0, c < d\ , \tag{10}$$

where $\alpha, \alpha_1, \alpha_2$ are hyper parameters of the gamma distribution.

Hence, from (7) and (10) the joint prior distribution of $\underline{\vartheta}$ is given by

$$\pi(\underline{\vartheta}) \propto \frac{1}{a\ b\ \theta}\ d^{(\alpha+\alpha_1)-1} c^{\alpha-1} e^{-d(c+\alpha_2)}, \quad a, b, \theta,$$
$$\alpha_1, \alpha_2 > 0, \quad c < d, \tag{11}$$

The joint posterior distribution can be obtained using (6) and (11) as follows:

$$\pi(\underline{\vartheta}|\underline{x}) \propto (a\ b\ \theta)^{-1}\ d^{(\alpha+\alpha_1)-1} c^{\alpha-1} \prod_{i=1}^{n} \left(a + b\theta x^{\theta-1}\right)$$

$$\times \left[exp\left(-ac - bc^{\theta}\right) - exp\left(-ad - bd^{\theta}\right)\right]^{-n}$$

$$exp\left(-d(c+\alpha_2)-a\sum_{i=1}^{n}x_i-b\sum_{i=1}^{n}x_i^{\theta}\right), \qquad (12)$$

$$\pi(\underline{\vartheta}|\underline{x}) = A(a\ b\ \theta)^{-1}\ d^{(\alpha+\alpha_1)-1}c^{\alpha-1}\prod_{i=1}^{n}(a+b\theta x^{\theta-1})$$

$$\times\left[exp(-ac-bc^{\theta})-exp(-ad-bd^{\theta})\right]^{-n}$$

$$exp\left(-d(c+\alpha_2)-a\sum_{i=1}^{n}x_i-b\sum_{i=1}^{n}x_i^{\theta}\right), \qquad (13)$$

where A is the normalizing constant and

$$A^{-1} = \int_{\underline{\vartheta}}(a\ b\ \theta)^{-1}\ d^{(\alpha+\alpha_1)-1}c^{\alpha-1}\prod_{i=1}^{n}(a+b\theta x^{\theta-1})$$

$$\times\left[exp(-ac-bc^{\theta})-exp(-ad-bd^{\theta})\right]^{-n}$$

$$exp\left(-d(c+\alpha_2)-a\textstyle\sum_{i=1}^{n}x_i-b\sum_{i=1}^{n}x_i^{\theta}\right)d\underline{\vartheta},$$

where

$$\int_{\underline{\vartheta}} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_c^{\infty} \quad \text{and} \quad d\underline{\vartheta} = da\ db\ d\theta\ dd\ dc\,. \qquad (14)$$

The marginal posteriors of $\underline{\vartheta}$ can be obtained by integrating the joint posterior distribution given by (13) with respect to the other parameters, hence the marginal posterior density is given by

$$\pi^*(\vartheta_j|\underline{x}) = \int_{\underline{\vartheta}_i} \pi(\underline{\vartheta}|\underline{x})\, d\underline{\vartheta}_i, \quad i,j = 1,2,3,4,5 \text{ and } j \neq i. \qquad (15)$$

I. Bayesian estimation under squared error loss function

Considering the SEL function, then the Bayes estimators of the parameters, is the mean of the posterior density and can be obtained as given below

$$\hat{\vartheta}_{jBs} = E(\vartheta_j|\underline{x}) = \int_{\underline{\vartheta}} \vartheta_j\; \pi(\underline{\vartheta}|\underline{x})\, d\underline{\vartheta}$$

$$= \int_{\underline{\vartheta}} \vartheta_j\; A\,(a\,b\,\theta)^{-1}\, d^{(\alpha+\alpha_1)-1}\, c^{\alpha-1} \prod_{i=1}^{n} (a + b\theta x^{\theta-1})$$

$$\times \left[exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})\right]^{-n} exp\left(-d(c+\alpha_2) - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^{\theta}\right) d\underline{\vartheta}, \qquad j = 1,2,\ldots,5 \qquad (16)$$

The Bayes estimators of the rf and hrf under SEL function can be obtained using (3), (4) and (16) as follows:

$$\hat{R}_{Bs}(x) = \int_{\underline{\vartheta}} A\,(a\,b\,\theta)^{-1}\, d^{(\alpha+\alpha_1)-1} c^{\alpha-1} \frac{exp(-ax - bx^{\theta}) - exp(-ad - bd^{\theta})}{exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})}$$

$$\times \prod_{i=1}^{n} (a + b\theta x^{\theta-1}) \left[exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})\right]^{-n}$$

$$\times exp\left(-d(c+\alpha_2)-a\sum_{i=1}^{n}x_i-b\sum_{i=1}^{n}x_i^{\theta}\right)d\underline{\vartheta}\ ,$$

$$c<x<d, \qquad (17)$$

$$\hat{h}_{Bs}(x)=\int_{\underline{\vartheta}} A(a\, b\, \theta)^{-1}\, d^{(\alpha+\alpha_1)-1}\, c^{\alpha-1}\frac{(a+b\theta x^{\theta-1})exp(-ax-bx^{\theta})}{exp(-ax-bx^{\theta})-exp(-ad-bd^{\theta})}$$

$$\times\prod_{i=1}^{n}(a+b\theta x^{\theta-1})\left[exp(-ac-bc^{\theta})-exp(-ad-bd^{\theta})\right]^{-n}$$

$$\times exp\left(-d(c+\alpha_2)-a\sum_{i=1}^{n}x_i-b\sum_{i=1}^{n}x_i^{\theta}\right)d\underline{\vartheta}\ ,$$

$$c<x<d, \qquad (18)$$

II. Bayesian estimation under linear exponential loss function

Under LINEX loss function, the Bayes estimators of $\underline{\vartheta}$, rf and hrf are given, respectively, by

$$\hat{\vartheta}_{jBL}=\frac{-1}{v}ln\left[E\left(e^{-v\vartheta_j}\,|\underline{x}\right)\right]$$

$$\hat{\vartheta}_{jBL}=\frac{-1}{v}ln\int_{\underline{\vartheta}} e^{-v\vartheta_j}\ A(a\, b\, \theta)^{-1}\, d^{(\alpha+\alpha_1)-1}c^{\alpha-1}\prod_{i=1}^{n}(a+b\theta x^{\theta-1})$$

$$\times\left[exp(-ac-bc^{\theta})-exp(-ad-bd^{\theta})\right]^{-n}exp(-d(c+\alpha_2)-a\textstyle\sum_{i=1}^{n}x$$
$$b\textstyle\sum_{i=1}^{n}x_i^{\theta})\,d\underline{\vartheta}$$

$$,\qquad j=1,2,\dots,5 \qquad (19)$$

$$\hat{R}_{BL}(x) = \frac{-1}{v} ln\left[E\left(e^{-vR(x)} | \underline{x}\right)\right]$$

$$\hat{R}_{BL}(x) = \frac{-1}{v} ln \int_{\underline{\vartheta}} A(a\ b\ \theta)^{-1}\, d^{(\alpha+\alpha_1)-1} c^{\alpha-1} exp\left(-v\, \frac{exp(-ax-bx^{\theta}) - exp(-ad-bd^{\theta})}{exp(-ac-bc^{\theta}) - exp(-ad-bd^{\theta})}\right)$$

$$\times \prod_{i=1}^{n} (a + b\theta x^{\theta-1}) \left[exp(-ac-bc^{\theta}) - exp(-ad-bd^{\theta})\right]^{-n}$$

$$\times\, exp\left(-d(c+\alpha_2) - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^{\theta}\right) d\underline{\vartheta}\ , \qquad c < x < d, \quad (20)$$

and

$$\hat{h}_{BL}(x) = \frac{-1}{v} ln\left[E\left(e^{-vh(x)} | \underline{x}\right)\right]$$

$$= \frac{-1}{v} ln \int_{\underline{\vartheta}} A(a\ b\ \theta)^{-1}\, d^{(\alpha+\alpha_1)-1} c^{\alpha-1} exp\left(-v\, \frac{(a+b\theta x^{\theta-1}) exp(-ax-bx^{\theta})}{exp(-ax-bx^{\theta}) - exp(-ad-b\,d^{\theta})}\right)$$

$$\times \prod_{i=1}^{n} (a + b\theta x^{\theta-1}) \left[exp(-ac-bc^{\theta}) - exp(-ad-bd^{\theta})\right]^{-n}$$

$$\times\, exp\left(-d(c+\alpha_2) - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^{\theta}\right) d\underline{\vartheta}\,, \quad c < x < d, \qquad (21)$$

2.2 Credible intervals

In this subsection the credible intervals for $\underline{\vartheta}$ are given. In general, a two-sided $100(1-\tau)$ % credible intervals of $\underline{\vartheta}$ are

$$P[L(\underline{x}) < \vartheta_j < U(\underline{x})|\underline{x}] = \int_{L(\underline{x})}^{U(\underline{x})} \pi(\underline{\vartheta}|\underline{x})\, d\underline{\vartheta} = 1-\tau,$$
$$j = 1,2,3,4,5. \qquad (22)$$

where $L(\underline{x})$and $U(\underline{x})$ are the *lower limit* (LL) and *upper limit* (UL).

Since, the joint posterior distribution is given by (13), then a two-sided 100 $(1-\tau)$ % credible intervals of $\underline{\vartheta}$ are given below

$$P(\vartheta_j > L_j(\underline{x})|\underline{x}) = \int_{L_j(\underline{x})}^{\infty} \pi(\vartheta_j|\underline{x})\ \ d\vartheta_j = 1-\frac{\tau}{2},\ \ j = 1,2,3,4,5, \qquad (23)$$

$$P(\vartheta_j > L_j(\underline{x})|\underline{x}) = \int_{L_j(\underline{x})}^{\infty} A(a\, b\, \theta)^{-1}\, d^{(\alpha+\alpha_1)-1} c^{\alpha-1} \prod_{i=1}^{n}(a+b\theta x^{\theta-1})$$
$$\times\, exp\left(-d(c+\alpha_2) - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^{\theta}\right)$$
$$\times \left[exp(-ac-bc^{\theta}) - exp(-ad-bd^{\theta})\right]^{-n} d\vartheta_j = 1-\frac{\tau}{2},$$
$$j = 1,2,3,4,5, \qquad (24)$$

and

$$P(\vartheta_j > U_j(\underline{x})|\underline{x}) = \int_{U_j(\underline{x})}^{\infty} \pi(\vartheta_j|\underline{x})\ \ d\vartheta_j = \frac{\tau}{2},\ j = 1,2,3,4,5, \qquad (25)$$

$$P\left(\vartheta_j > U_j(\underline{x})|\underline{x}\right) = \int_{U_j(\underline{x})}^{\infty} A(a\ b\ \theta)^{-1}\ d^{(\alpha+\alpha_1)-1} c^{\alpha-1} \prod_{i=1}^{n}\left(a + b\theta x^{\theta-1}\right)$$

$$\times exp\left(-d(c+\alpha_2) - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^{\theta}\right)$$

$$\times\left[exp\left(-ac - bc^{\theta}\right) - exp\left(-ad - bd^{\theta}\right)\right]^{-n} d\vartheta_j = \frac{\tau}{2},$$

$$j = 1,2,3,4,5. \qquad (26)$$

The Bayes estimators either point or interval can't be obtained in closed forms, so a numerical method is used to obtain the Bayes estimates.

3. Bayesian Prediction for a Future Observation of the Double Truncated Modified Weibull Distribution

In this section, the Bayesian two-sample prediction (point and interval) for a future observation $Y_{(s)}$, $1 \leq s \leq m$, from the double TMW$(\underline{\vartheta})$ distribution is considered.

Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the first *n* ordered life times in a random sample having the double TMW$(\underline{\vartheta})$ and $Y_{(1)} < Y_{(2)} < \cdots < Y_{(m)}$ is a future independent random sample of size *m* from the same distribution. Our aim is to predict a statistic in the future sample based on the informative sample.

For the future sample of size *m*, let $Y_{(s)}$ denotes the s^{th} order statistic, $1 \leq s \leq m$, the pdf for $Y_{(s)}$ is given by

$$h_s\left(y_{(s)}|\underline{\vartheta}\right) = D(s)\ f_x\left(y_{(s)}|\underline{\vartheta}\right)\left[F_x\left(y_{(s)}|\underline{\vartheta}\right)\right]^{s-1}\left[1 - F_s\left(y_{(s)}|\underline{\vartheta}\right)\right]^{m-s}, \quad y_{(s)} > 0, \qquad (27)$$

$$D(s) = s\binom{m}{s} = \frac{m!}{(s-1)!\,(m-s)!}, \quad s = 1,2,\dots,m. \qquad (28)$$

Using the binomial expansion, hence

$$h_s(y_{(s)}|\underline{\vartheta}) = D(s)\; Z\left(a + b\theta y_{(s)}{}^{\theta-1}\right)\left[exp\left(-ac - bc^{\theta}\right) - exp\left(-ad - bd^{\theta}\right)\right]^{-m}$$
$$\times\left(exp\left(-ac - bc^{\theta}\right)\right)^{s-j-1}\left(exp\left(-ad - bd^{\theta}\right)\right)^{i}\left(exp\left(-ay_{(s)} - by_{(s)}^{\theta}\right)\right)^{1+j+m-s-i} c \le y_{(s)} \le d, \qquad (29)$$

where

$$Z = \sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j}\sum_{i=0}^{m-s}(-1)^i\binom{m-s}{i}\;, \qquad (30)$$

and $D(s) = s\binom{m}{s} = \frac{m!}{(s-1)!(m-s)!}$, $s = 1,2,\dots,m$ is given in (28).

The *Bayesian predictive density* (BPD) of $Y_{(s)}$ given $\underline{x}$ is given by

$$h_{y_{(s)}}\left(y_{(s)}|\underline{x}\right) = \int_{\underline{\theta}} h_s\left(y_{(s)}|\underline{\vartheta}\right)\pi\left(\underline{\vartheta}|\underline{x}\right)d\underline{\vartheta}, \quad y_{(s)} > 0, \qquad (31)$$

where $\pi(\underline{\vartheta}|\underline{x})$ is the posterior pdf of $\underline{\vartheta}$ and $h_{y_{(s)}}\left(y_{(s)}|\underline{x}\right)$ is the pdf of $Y_{(s)}$.

Assuming that the parameters $\underline{\vartheta}$ are unknown, then the BPD of $Y_{(s)}$ given $\underline{x}$ can be obtained by substituting (13) and (29) into (31) as follows:

$$h_{y_{(s)}}\left(y_{(s)}|\underline{x}\right) = \int_{\underline{\theta}} A\,D(s)\,Z(a\;b\;\theta)^{-1}\,d^{(\alpha+\alpha_1)-1}c^{\alpha-1}\prod_{i=1}^{n}\left(a + b\theta x^{\theta-1}\right)$$

$$\times\left[exp(-ac-bc^{\theta})-exp(-ad-bd^{\theta})\right]^{-n-m}exp\left(-d(c+\alpha_2)-a\sum_{i=1}^{n}x_i-b\sum_{i=1}^{n}x_i^{\theta}\right)$$

$$\times\left(a+b\theta y_{(s)}{}^{\theta-1}\right)\left(exp(-ad-bd^{\theta})\right)^{i}\left(exp(-ay_{(s)}-by_{(s)}^{\theta})\right)^{1+j+m-s-i}$$

$$\times\left(exp(-ac-bc^{\theta})\right)^{s-j-1}d\underline{\vartheta},\quad c\le y_{(s)}\le d;\ \underline{\vartheta}>\underline{0}\,, \qquad (32)$$

where $D(s)$ is given by (28) and Z is given by (30).

3.1 Point prediction

Bayesian prediction is considered under two types of loss functions, SEL function; as a symmetric loss function, and LINEX loss function; as an asymmetric loss function.

I. Squared error loss function

The *Bayes predictive* (BP) for the future observation, $Y_{(s)}, 1\le s\le m$, under SEL function can be derived using (32) as given below

$$\hat{y}_{(s)(SEL)}=E\left(y_{(s)}|\underline{x}\right)$$

$$=\int_{y_{(s)}}y_{(s)}\,h_{y_{(s)}}\left(y_{(s)}|\underline{x}\right)\,dy_{(s)} \qquad (33)$$

$$\hat{y}_{(s)(SEL)}=\int_{\underline{\vartheta}^*}y_{(s)}\,AD(s)Z(a\,b\,\theta)^{-1}\,d^{(\alpha+\alpha_1)-1}c^{\alpha-1}\prod_{i=1}^{n}\left(a+b\theta x^{\theta-1}\right)$$

$$\times\left[exp(-ac-bc^{\theta})-exp(-ad-bd^{\theta})\right]^{-n-m}exp\left(-d(c+\alpha_2)-a\sum_{i=1}^{n}x_i-b\sum_{i=1}^{n}x_i^{\theta}\right)$$

$$\times\left(a+b\theta y_{(s)}{}^{\theta-1}\right)\left(exp(-ad-bd^{\theta})\right)^{i}\left(exp(-ay_{(s)}-by_{(s)}^{\theta})\right)^{1+j+m-s-i}$$

$$\times\left(exp(-ac-bc^{\theta})\right)^{s-j-1}d\underline{\vartheta}^{*},\quad c\le y_{(s)}\le d,\underline{\vartheta}>\underline{0}, \qquad (34)$$

where

$$\int_{\underline{\vartheta}^{*}}=\int_{y_{(s)}}\int_{\underline{\vartheta}} \qquad \text{and} \qquad d\underline{\vartheta}^{*}=d\,y_{(s)}\,d\underline{\vartheta}\ .$$

II. Linear exponential loss function

The BP for the future observation, $Y_{(s)}, 1\le s\le m$, under LINEX loss function can be obtained using the following equation

$$\hat{y}_{(s)(LINX)}=\frac{-1}{v}\,ln\ E\left(exp(-vy_{(s)})|\underline{x}\right)$$

$$=\frac{-1}{v}\ ln\int_{y_{(s)}}exp(-vy_{(s)})\,h_{y_{(s)}}(y_{(s)}|\underline{x})\ dy_{(s)}\,, \qquad (35)$$

where v is constant and $v \neq 0$.Substituting (32) in (35), then

$$\hat{y}_{(s)(LINX)} = \frac{-1}{v} \ln \int_{\underline{\vartheta}^*} exp(-v y_{(s)})\ A\ D(s) Z(a\ b\ \theta)^{-1}\ d^{(\alpha+\alpha_1)-1} c^{\alpha-1}$$

$$\prod_{i=1}^{n} (a + b\theta x^{\theta-1})$$

$$\times \left[exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})\right]^{-n-m} exp\left(-d(c + \alpha_2) - a \sum_{i=1}^{n} x_i - b \sum_{i=1}^{n} x_i^{\theta}\right)$$

$$\times \left(a + b\theta y_{(s)}{}^{\theta-1}\right) \left(exp(-ad - bd^{\theta})\right)^{i} \left(exp(-ay_{(s)} - by_{(s)}^{\theta})\right)^{1+j+m-s-i}$$

$$\times \left(exp(-ac - bc^{\theta})\right)^{s-j-1} d\underline{\vartheta}^*, c \leq\ y_{(s)} \leq d, \underline{\vartheta} > \underline{0},$$

$$s =\ 1, 2, 3, \dots, m. \qquad (36)$$

Special cases:

I. If $s = 1$, in (34) and (36), one can predict the minimum observable, $Y_{(1)}$, which represents the first failure time in a future sample of size m, under SEL and LINEX loss functions.

II. If $s = m$, in (34) and (36), one can predict the maximum observable, $Y_{(m)}$, which represents the largest failure time in a future sample of size *m* under SEL and LINEX loss functions.

III. If $s = \frac{m+1}{2}$, in (34) and (36), one can predict the median observable when m is odd, $Y_{\left(\frac{m+1}{2}\right)}$, which represents the median failure time in a future sample of size m under SEL and LINEX loss functions.

3.2 Bayesian prediction bounds

A 100 $(1-\tau)\%$ *Bayesian predictive bounds* (BPB) for the future observation, such that $P\left(L_{(s)}(\underline{x}) < Y_{(s)} < U_{(s)}(\underline{x})|\underline{x}\right)$ can be obtained as given below

$$P\left(Y_{(s)} > L_{(s)}(\underline{x})|\underline{x}\right) = \int_{L_{(s)}(\underline{x})}^{\infty} h_{y_{(s)}}\left(y_{(s)}|\underline{x}\right) dy_{(s)} = 1 - \frac{\tau}{2}, \qquad (37)$$

and

$$P\left(Y_{(s)} > U_{(s)}(\underline{x})|\underline{x}\right) = \int_{U_{(s)}(\underline{x})}^{\infty} h_{y_{(s)}}\left(y_{(s)}|\underline{x}\right) dy_{(s)} = \frac{\tau}{2}. \qquad (38)$$

Substituting (32) in (37) and (38) lower and upper bounds are obtained as follows:

$$P\left(Y_{(s)} > L_{(s)}(\underline{x})|\underline{x}\right) = \int_{L_{(s)}(\underline{x})}^{\infty} A\,D(s) Z(a\,b\,\theta)^{-1}\, d^{(\alpha+\alpha_1)-1}\, c^{\alpha-1} \prod_{i=1}^{n} (a + b\theta x^{\theta-1})$$

$$\times \left[exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})\right]^{-n-m} exp\left(-d(c+\alpha_2) - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^{\theta}\right)$$

$$\times \left(a + b\theta y_{(s)}^{\theta-1}\right)\left(exp(-ad - bd^{\theta})\right)^{i}\left(exp(-ay_{(s)} - by_{(s)}^{\theta})\right)^{1+j+m-s-i}$$

$$\times \left(exp(-ac - bc^{\theta})\right)^{s-j-1}\, dy_{(s)} = 1 - \frac{\tau}{2}, \qquad (39)$$

$$P\left(Y_{(s)} > U_{(s)}(\underline{x})|\underline{x}\right) = \int_{U_{(s)}(\underline{x})}^{\infty} A\,D(s) Z(a\,b\,\theta)^{-1}\, d^{(\alpha+\alpha_1)-1}\, c^{\alpha-1} \prod_{i=1}^{n} (a + b\theta x^{\theta-1})$$

$$\times \left[exp(-ac - bc^{\theta}) - exp(-ad - bd^{\theta})\right]^{-n-m} exp\left(-d(c+\alpha_2) - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^{\theta}\right)$$

$$\times \left(a + b\theta y_{(s)}^{\theta-1}\right)\left(exp(-ad - bd^{\theta})\right)^{i}\left(exp(-ay_{(s)} - by_{(s)}^{\theta})\right)^{1+j+m-s-i}$$

$$\times \left(exp(-ac - bc^{\theta})\right)^{s-j-1} \, dy_{(s)} = \frac{\tau}{2}, \qquad (40)$$

where $s = 1,2, \ldots, m$.

Special cases:

- Note that all results obtained in this paper for the TMW distribution give corresponding results for the left TIW distribution when $d = \infty$ in (1).
- Results can also be obtained for right TMW distribution when $c = 0$ in (1).

4. Numerical Results

This section aims to illustrate the theoretical results of both estimation and prediction problems on the basis of simulated data and real data.

4.1 Simulation study

- Several data sets are generated from double TMW distribution for a combination of the initial parameter values of a, b, θ, c and d and for samples of size 30, 50 and 100 using $M =$ 10000 replications for each sample size.
- The transformation between uniform distribution and double TMW distribution is given as follows:

$$x_i = \left[\frac{1}{(-a-b)} ln\left[(1-u_i)e^{(-ac-bc^{\theta})} + u_i e^{(-ad-bd^{\theta})}\right]\right]^{\frac{1}{\theta+1}}. \qquad (41)$$

- The Bayes estimates are obtained under SEL and LINEX loss functions using non- informative prior.
- Evaluating the performance of the estimates has been considered through some measurements of accuracy; *estimated risk* (ER) of the Bayes estimates of the parameters, rf and hrf are computed as follows:

$$ER(\hat{\theta}) = \frac{1}{M}\sum_{i=1}^{M}(\hat{\theta}_i - \theta)^2.$$

- The simulation results of the Bayes estimates are displayed in Tables 1-3. The Bayes averages, ERs and credible intervals of the unknown parameters; under SEL and LINEX loss functions, are presented in Table 1. While Table 2 and 3 present the Bayes averages and credible intervals of rf and hrf for different values of the time x_0; under SEL and LINEX loss functions, respectively.
- The Bayes predictive for a future observation is obtained under SEL and LINEX loss functions using non- informative prior; considering two-sample prediction.
- Table 4 presents the Bayes predictive and bounds for a future observation.

4.2 Concluding Remarks

- It is clear from Tables 1- 3 that the ERs of the Bayes averages of the parameters, rf and hrf perform better when the sample size increases, also the lengths of the credible intervals become narrower.
- It can be observed that the ERs of the estimates and lengths of the parameters, rf and hrf under LINEX loss function have the less values than the corresponding ERs of the estimates under SEL loss function.
- One can notice from Table 4 that the Bayes predictive for the future observation is located between LL and UL.

4.3 Application

The main aim of this subsection is to demonstrate how the TMW$(\underline{\vartheta})$ distribution can be used in practice. This data set were taken from Lee and Wang (2003) which are the calm times of a random sample of 128 bladder cancer patients.

The data are given below:

0.08,2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

- The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p values is 0.559, it showed that the TMW $(\underline{\vartheta})$ distribution fits the data very well.
- For the real data set, the Bayes averages of the parameters, their ERs and the credible intervals; using non-informative prior under SEL and LINEX loss functions, are shown in Table 5.
- Bayes predictive for the future observation and the bounds; considering two-sample prediction are given in Table 6.
- From Tables 5 and 6, one can notice that the Bayes predictive for the future observation is located between LL and UL.
- The results based on the real data ensure the simulation results.

Table 1

Bayes averages, estimated risks and 95% credible interval of the parameters, using non-informative prior under SEL and LINEX loss functions

($M = 10000, a = 1, b = 1.7, \theta = 1.5, c = 0.3$ and $d = 4$)

n	Loss functions	$\hat{\underline{\theta}}$	Average	ER	Credible interval		
					UL	**LL**	**Length**
30	**SEL**	$\hat{a}$	1.0001	2.605e-06	1.0011	0.9994	0.0017
		$\hat{b}$	1.6991	9.198e-06	1.6999	1.6984	0.0015
		$\hat{\theta}$	1.4978	7.718e-06	1.5004	1.4952	0.0052
		$\hat{c}$	0.3009	1.140e-06	0.3018	0.3003	0.0015
		$\hat{d}$	3.9988	4.319e-06	4.0003	3.9966	0.0037
	LINEX	$\hat{a}$	0.9978	7.959e-06	1.0001	0.9948	0.0059
		$\hat{b}$	1.7008	1.032e-05	1.7018	1.6999	0.0019
		$\hat{\theta}$	1.4993	8.531e-05	1.5004	1.4982	0.0021
		$\hat{c}$	0.3021	5.140e-06	0.3033	0.3005	0.0028
		$\hat{d}$	4.0001	5.056e-06	4.0003	3.9966	0.0037
50	**SEL**	$\hat{a}$	0.9987	2.598e-06	0.9995	0.9982	0.0013
		$\hat{b}$	1.6988	5.673e-06	1.6999	1.6984	0.0014
		$\hat{\theta}$	1.4969	2.818e-06	1.5002	1.4965	0.0037
		$\hat{c}$	0.2991	0.965e-06	0.2999	0.2984	0.0014
		$\hat{d}$	3.9978	2.171e-06	4.0002	0.9967	0.0035
	LINEX	$\hat{a}$	0.9957	6.577e-06	0.9972	0.9943	0.0027
		$\hat{b}$	1.6999	8.194e-06	1.7006	1.6988	0.0017
		$\hat{\theta}$	1.4871	5.688e-05	1.4885	1.4866	0.0020
		$\hat{c}$	0.3007	4.753e-06	0.3017	0.2998	0.0019
		$\hat{d}$	3.9820	1.118e-05	3.9828	3.9797	0.0030

n	Loss functions	$\hat{\theta}$	Average	ER	Credible interval		
					UL	**LL**	**Length**
100	**SEL**	$\hat{a}$ $\hat{b}$ $\hat{\theta}$ $\hat{c}$ $\hat{d}$	0.9866 1.6975 1.4942 0.2977 3.9971	1.108e-06 2.667e-06 8.749e-06 0.796e-06 1.036e-06	0.9875 1.6985 1.5001 0.3004 4.0007	0.9862 1.6969 1.4931 0.2951 3.9964	0.0011 0.0012 0.0019 0.0011 0.00031
	LINEX	$\hat{a}$ $\hat{b}$ $\hat{\theta}$ $\hat{c}$ $\hat{d}$	0.9948 1.6117 1.4611 0.2994 3.9781	4.823e-06 5.784e-06 1.477e-06 2.379e-06 7.452e-06	0.9967 1.6125 1.4618 0.3006 3.9803	0.9942 1.6105 1.4599 0.2989 3.9775	0.0023 0.0015 0.0018 0.0017 0.0028

Table 2

Bayes averages, estimated risks and credible intervals of the rf and hrf at $x_0 = (0.5, 1)$, from TMW distribution using SEL function for different samples size and replications M = 10000

N	x_0	Estimators	Average	ER	UL	LL	Length
30	**0.5**	$\hat{R}(x_0)$	**0.5945**	**7.161e-06**	**0.5953**	**0.5935**	**0.0018**
		$\hat{h}(x_0)$	**0.9023**	**0.3584**	**0.9048**	**0.9020**	**0.0026**
	1	$\hat{R}(x_0)$	**0.1209**	**5.170e-06**	**0.1218**	**0.1199**	**0.0017**
		$\hat{h}(x_0)$	**2.5507**	**1.1019**	**2.5595**	**2.5523**	**0.0028**
50	**0.5**	$\hat{R}(x_0)$	**0.5934**	**5.052e-06**	**0.5941**	**0.5923**	**0.0016**
		$\hat{h}(x_0)$	**0.9102**	**0.3558**	**0.9105**	**0.9087**	**0.0023**

	1	$\hat{R}(x_0)$	0.1199	3.052e-06	0.1209	0.1192	0.0015
		$\hat{h}(x_0)$	2.5509	0.4125	2.5512	2.5488	0.0023
100	0.5	$\hat{R}(x_0)$	0.5913	2.259e-06	0.5925	0.5909	0.0013
		$\hat{h}(x_0)$	0.9352	0.3543	0.9365	0.9346	0.0019
	1	$\hat{R}(x_0)$	0.1187	1.298e-06	0.1203	0.1184	0.0014
		$\hat{h}(x_0)$	2.5518	0.1035	2.5526	2.5495	0.0019

Table 3

Bayes estimates, estimated risks and credible intervals of the rf and hrf at $x_0 = (0.5, 1)$, from TMW distribution using LINEX loss function for different samples size, and replications M = 10000

n	x_0	Estimators	Estimates	ER	UL	LL	Length
30	0.5	$\hat{R}(x_0)$	0.5941	8.717e-06	0.5958	0.5931	0.0027
		$\hat{h}(x_0)$	0.9013	9.171e-06	0.9031	0.8996	0.0035
	1	$\hat{R}(x_0)$	0.1205	8.537e-06	0.1227	0.1196	0.0032
		$\hat{h}(x_0)$	2.5494	9.171e-06	2.5515	2.5480	0.0030
	0.5	$\hat{R}(x_0)$	0.5936	5.912e-06	0.5948	0.5923	0.0025

50		$\hat{h}(x_0)$	0.9016	7.126e-06	0.9120	0.8997	0.0027
	1	$\hat{R}(x_0)$	0.1201	5.913e-06	0.1213	0.1188	0.0025
		$\hat{h}(x_0)$	2.5497	2.126e-06	2.5515	2.5488	0.0028
100	0.5	$\hat{R}(x_0)$	0.5913	5.715e-06	0.5926	0.5899	0.0022
		$\hat{h}(x_0)$	0.9022	2.588e-06	0.9038	0.9014	0.0023
	1	$\hat{R}(x_0)$	0.1178	5.77e-06	0.1195	0.1172	0.0021
		$\hat{h}(x_0)$	2.5499	2.015e-06	2.5514	2.5489	0.0024

Table 4

Bayes predictive and bounds (non-informative prior) for a future observation

($n_1 = 40,\ n_2 = 20,\ a = 1,\ b = 1.7,\ \theta = 1.5,\ c = 0.3$ and $d = 4$)

s	Loss function	$\hat{y}_{(s)B}$	Credible interval		
			UL	LL	Length
3	SEL	1.7998	1.8007	1.7989	0.0018
	LINEX	1.7989	1.8001	1.7978	0.0023
9	SEL	1.8004	1.8021	1.7993	0.0027
	LINEX	1.7996	1.8006	1.7977	0.0028
18	SEL	1.8011	1.8021	1.7987	0.0033
	LINEX	1.8017	1.8041	1.8006	0.0035

Table 5

Bayes averages and estimated risks of the parameters, using non-informative prior under SEL and LINEX loss functions for the real data

n	Loss function	$\hat{\underline{\theta}}$	Estimates	ER	UL	LL	Length
128	SEL	$\hat{a}$	1.5003	6.579e-06	**1.5030**	**1.4990**	**0.0039**
		$\hat{b}$	1.9998	2.175e-06	**2.0001**	**1.9981**	**0.0019**
		$\hat{\theta}$	0.6999	4.887e-06	**0.7010**	**0.6980**	**0.0030**
		$\hat{c}$	0.6**005**	4.864e-06	**0.6020**	**0.6000**	**0.0029**
		$\hat{d}$	2.0017	3.262e-06	**2.0028**	**2.0010**	**0.0018**
	LINEX	$\hat{a}$	1.4999	2.154e-06	**2.0005**	**1.9987**	**0.0022**
		$\hat{b}$	1.9990	1.599e-06	**1.9997**	**1.9980**	**0.0017**
		$\hat{\theta}$	0.6990	3.753e-06	**0.7010**	**0.6985**	**0.0025**
		$\hat{c}$	0.5995	1.060e-06	**0.6000**	**0.5981**	**0.0021**
		$\hat{d}$	2.0001	1.517e-06	**2.0007**	**1.9995**	**0.0015**

Table 6

Bayes predictive estimates and bounds using non-informative prior of the future observation for real data under two-sample prediction

S	SEL				LINEX			
	$\hat{y}_{(s)B}$	Credible interval			$\hat{y}_{(s)B}$	Credible interval		
		UL	LL	Length		UL	LL	Length
7	**0.1310**	**0.1324**	**0.1299**	**0.0025**	**0.1304**	**0.1309**	**0.1295**	**0.0014**
15	**0.1879**	**1.1903**	**0.1832**	**0.0070**	**1.1885**	**1.1901**	**1.1859**	**0.0040**
37	**2.2015**	**2.5050**	**2.4967**	**0.0083**	**2.4996**	**2.5027**	**2.4948**	**0.0077**

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