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MILITARY TECHNICAL COLLEGE

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BICONVEX THIN AIRFOILS IN PERFECTLY CONDUCTING INCOMPRESSIBLE FLUIDS ORTHOGONAL TO A UNIFORM MAGNETIC FIELD

MINA B. ABD-EL-MALEK(+)

ABSTRACT

This paper treats an incompressible, inviscid, steady, and perfect electrically conducting fluid past a cylindrical body whose cross-section represents a biconvex thin non-conducting airfoil in such a position that its chord is assumed to be parallel to the flow direction. A uniform magnetic field is applied in the orthogonal direction to the direction of the flow. All the flow and magnetic field variables are assumed to differ by small amounts from their undisturbed values due to the presence of the thin airfoil.

The main purposes on this problem are to illustrate the effect of the biconvex thin airfoils on all the flow and magnetic field variables as well as the effect of the number "m", which measures the ratio of the undisturbed fluid to the speed of the Alfvén waves, on the same variables.

Flow and magnetic field quantities, at the body surface, such as speed, pressure coefficient, magnetic field intensity, lift force, and drag force are obtained in terms of the number "m". The effects of the thin airfoil and the number "m" on the flow and magnetic field quantities have been discussed.

(+) Associate Professor, Department of Engineering Mathematics and Physics. Faculty of Engineering, Alexandria University, Alexandria, EGYPT.

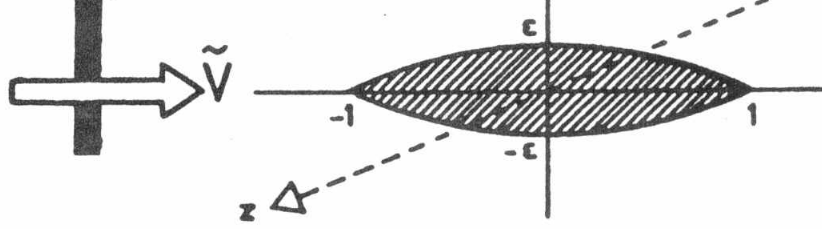
1-Introduction:-

The field of magneto-fluid-dynamics dates back to Faraday in 1836 and then developed by Hartman and Lazarus in 1937 where they used mercury as a conducting fluid in their experiments. This field deals with the study of the motion of electrically conducting fluids in the presence of magnetic field. In this conducting fluid,in the presence of a magnetic field,the Swedich scientist H.Alfvén,in 1940,discovered new waves which are unknown to both the fluid mechanics and electro-magnetism, propagate through it.
For more details about the electro-magnetic field equations see Jordan[1],Graffi[2],and for the topic consult Dragos[3], Shercliff[4],and Cabannes[5].
Recently,Walker[6] in 1986,considered the problem of the liquid-metal flow in a straight circular channel with a thin metal wall and a strong magnetic field is applied by a magnet with parallel poles that end abruptly.
This paper is concerned with the steady motion of a fluid which is incompressible,inviscid and electrically a perfect conductor,past a biconvex thin non-conducting airfoil in the presence of an orthogonal and uniform magnetic field. The assumption of perfect conducting fluid does not correspond exactly to the actual phenomena but simplifies to a great extent the equations of motion.
The interest in this problem because it throws the light on the effect of the obstacle and the number "m",which measures the ratio of the undisturbed fluid to the speed of the Alfvén waves,on the flow and magnetic field quantities.

2-Formulation of the Problem:-

Consider a perfectly conducting,steady,inviscid and incompressible fluid of density ρ. Relative to fixed axes Oxyz let the undisturbed velocity of the fluid be $\overline{V}=(V_o,0,0)$ and the undisturbed magnetic field be $\overline{H}=(0,H_o,0)$ where V_o and H_o are constants. A thin non-conducting cylinrical biconvex airfoil is fixed in the fluid near O,as shown in Figure(1),so that the equations of its upper and lower surfaces are:

$$y(x)=\varepsilon(1-x^2) \quad , \quad y>0 \quad ; \quad |x|\leq 1 \qquad (2.1)$$
$$y(x)=-\varepsilon(1-x^2) \quad , \quad y<0 \quad ; \quad |x|\leq 1$$



Fig(1) Biconvex Thin Airfoils in Orthogonal Fields.

The presence of the body disturbes both the velocity and magnetic fields,

$$\overline{V}=V_o(1+u_x\ ,\ u_y\ ,\ 0\ )\qquad , \tag{2.2}$$

$$\overline{H}=H_o(h_x\ ,\ 1+h_y\ ,\ 0\ )\qquad , \tag{2.3}$$

respectively.
It is assumed that at large distances the perturbations vanishes,i.e.

$$\lim_{x^2+y^2\to\infty} (u_x\ ,\ u_y\ ;\ h_x\ ,\ h_y\ ) = 0 \quad , \tag{2.4}$$

and are small enough that their squares and products may be neglected.

3-Equations of Motion:-

The fundamental equations governing the motion of steady, incompressible,and perfectly conducting fluid in an orthogonal direction of magnetic field,after neglecting squares and products of the small quantities,stated in section(2),are:

(i)Conservation of mass:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{3.1}$$

(ii)Consrvation of linear momentum:

$$V_o^2 \frac{\partial u_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{V_o^2}{m^2}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right) \quad , \tag{3.2}$$

$$V_o^2 \frac{\partial u_y}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{V_o^2}{m^2}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right) \quad , \tag{3.3}$$

(iii) Maxwell's Equations:

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} = 0 \quad , \tag{3.4}$$

$$\frac{\partial h_x}{\partial x} = \frac{\partial u_x}{\partial y} \quad , \tag{3.5}$$

$$\frac{\partial h_y}{\partial x} = \frac{\partial u_y}{\partial y} \quad , \tag{3.6}$$

where ρ denotes the fluid density,p the pressure function and $m=V_o/(\frac{H_o}{\sqrt{4\pi\rho}})$ is the ratio of the undisturbed fluid to the speed of Alfvén waves.

From (3.1) and (3.6) we get

$$h_y = -u_x \quad . \tag{3.7}$$

To reduce the system (3.1)-(3.6) to a simple equation, we do the following:
Eliminate p between (3.2) and (3.3), we get

$$\frac{\partial^2 u_x}{\partial x\,\partial y} - \frac{\partial^2 u_y}{\partial x^2} = -\frac{1}{m^2}\left(\frac{\partial^2 h_y}{\partial x\,\partial y} - \frac{\partial^2 h_x}{\partial y^2}\right) ,$$

then differentiate the obtained equation partially with respect to x and use (3.1) and (3.4) to eliminate u_x and h_x, we get

$$\frac{\partial}{\partial x}\left(\nabla^2 u_y\right) = \frac{1}{m^2}\frac{\partial}{\partial y}\left(\nabla^2 h_y\right) , \tag{3.8}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Differentiating (3.8) with respect to x and use (3.4) we get

$$\nabla^2\left(\frac{\partial^2 u_y}{\partial x^2}\right) = \frac{1}{m^2}\nabla^2\left(\frac{\partial^2 u_y}{\partial y^2}\right) \quad . \tag{3.9}$$

Similarly we get

$$\nabla^2\left(\frac{\partial^2 h_y}{\partial x^2}\right) = \frac{1}{m^2}\nabla^2\left(\frac{\partial^2 h_y}{\partial y^2}\right) \quad . \tag{3.10}$$

Equations (3.9) and (3.10) may be written in the form

$$T\,\nabla^2(u_y, h_y) = (0,0) \quad , \tag{3.11}$$

where

$$T \equiv \frac{\partial^2}{\partial x^2} - \frac{1}{m^2}\frac{\partial^2}{\partial y^2} \quad . \tag{3.12}$$

The operator T, defined in (3.12), is a hyperbolic operator, and the characteristic curves of T have slopes $\pm 1/m$.

4-Boundary Conditions:-

(i) On the body: The normal component of velocity is zero and hence

$$\left.\begin{aligned} u_y(x,0+) &= -2\varepsilon x \\ u_y(x,0-) &= 2\varepsilon x \end{aligned}\right\} \quad |x| \le 1 \quad , \tag{4.1}$$

because the parameters of the normal to the surface profile are $(\mp 2\varepsilon x\ ,\ -1)$ and we disregarded the products $\mp 2\varepsilon\, x u_x$ and imposed the boundary condition on the segment $y = 0\pm$ and not on the curves $y(x)=\pm\,\varepsilon(1-x^2)$.

(ii) Inside the body: Since the body is non-conducting material, the variation of the internal field, determined by the equations

$$\bar{\nabla}\times\bar{H}=\bar{0}\ ,\quad \bar{\nabla}\cdot\bar{H}=0\ ,$$

is small.
Following Stewartson [7] and Dragos [3], we get

$$\left.\begin{aligned} h_x(x,0+) &= h_x(x,0-) \\ h_y(x,0+) &= h_y(x,0-) \end{aligned}\right\}\quad |x|\le 1 \tag{4.2}$$

5-Solution of the Problem:-

The hyperbolic operator T, defined by (3.12), has solutions of the form $F(x\mp my)$, where F is an arbitrary function.
Thus writing

$$u_y^{\pm} = F_{\pm}(x\mp my) + \frac{\partial S}{\partial y}\ ,\quad \nabla^2 S = 0 \tag{5.1}$$

and we find successively that:

$$h_y^{\pm} = G_{\pm}(x\mp my) + \frac{\partial W}{\partial y}\ ,\quad \nabla^2 W = 0\ , \tag{5.2}$$

$$u_x^{\pm} = \pm m\ F_{\pm}(x\mp my) + \frac{\partial S}{\partial x}\ , \tag{5.3}$$

$$h_x^{\pm} = \pm m\ G_{\pm}(x\mp my) + \frac{\partial W}{\partial x}\ . \tag{5.4}$$

From (3.7) we get

$$h_y^{\pm} = -\,u_x^{\pm}\ . \tag{5.5}$$

From (5.2), (5.3) and (5.5) we have

$$G_{\pm} = \mp mF_{\pm}\ , \tag{5.6}$$

and

$$\frac{\partial W}{\partial y} = -\frac{\partial S}{\partial x}\ . \tag{5.7}$$

Similarly, we can show that

$$\frac{\partial W}{\partial x} = \frac{\partial S}{\partial y}\ . \tag{5.8}$$

Using (3.2) and (3.3), the pressure may be obtained as

$$\frac{p}{\rho V_o^2} = \pm\frac{1}{m}\,F_{\pm}(x\mp my) - \frac{\partial S}{\partial x}\ . \tag{5.9}$$

The two arbitrary functions appeared in the solutions (5.1)-(5.9) may be determined from the boundary conditions given in the previous section.
From (5.1) and (4.1) we have

$$F_{\pm}(x) + \frac{\partial S}{\partial y}(x,0\pm) = \mp 2\varepsilon x \quad , \quad |x|\leq 1 \quad .$$

Thus

$$[F] \mp \left[\frac{\partial S}{\partial y}\right] = -2\varepsilon x\, U(1-x^2) \quad , \tag{5.10}$$

where

$$[F] = F(x,0+) - F(x,0-) \quad ,$$

and U(x) is the unit step function.
From (5.4) and (4.2),using (5.6) and (5.8),we get:

$$m\left(F_{+}(x) - F_{-}(x)\right) - \frac{1}{m}\left[\frac{\partial S}{\partial y}\right] = 0 \quad , \quad |x|\leq 1 \quad . \tag{5.11}$$

Likewise,using (5.5),(4.2),and (5.7) we get

$$F_{+}(x) + F_{-}(x) + \frac{1}{m}\left[\frac{\partial S}{\partial y}\right] = 0 \quad . \tag{5.12}$$

Since the derivatives of S and W are continuous outside the airfoil,from (5.10) and (5.12) we conclude that

$$F_{+}(x) \equiv F_{-}(x) \equiv 0 \quad , \quad |x|>1 \quad . \tag{5.13}$$

Writing

$$f(x) = \left[\frac{\partial S}{\partial x}\right] \quad . \tag{5.14}$$

Equations (5.10),(5.11),and (5.12) may be written as a linear system of F_{+}, F_{-},and $\left[\frac{\partial S}{\partial y}\right]$ as follows

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -(\frac{1}{m}+m) \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} F_{+} \\ F_{-} \\ \left[\frac{\partial S}{\partial y}\right] \end{pmatrix} = \begin{pmatrix} -4\varepsilon xU(1-x^2) \\ 4\varepsilon mxU(1-x^2) \\ \frac{f}{m} + 4\varepsilon xU(1-x^2) \end{pmatrix}$$

Solving this linear system by Gauss-elimination method,we get

$$2(m^2+1)F_{+} = -(m+\frac{1}{m})f - 4\varepsilon x \quad , \tag{5.15}$$

$$2(m^2+1)F_{-} = -(m+\frac{1}{m})f + 4\varepsilon x \quad , \tag{5.16}$$

$$(m^2+1)\left[\frac{\partial S}{\partial y}\right] = -4\varepsilon m^2 x \quad . \tag{5.17}$$

Upon substituting (5.15)-(5.17) in (5.9) we find

$$\left[\frac{p}{\rho V_0^2}\right] = -\left(1+\frac{1}{m^2}\right)f \tag{5.18}$$

Now,to determine the function f(x) consider the following complex variable function

$$K(z) = \frac{\partial S}{\partial x} - i\frac{\partial S}{\partial y} \quad , \tag{5.19}$$

which is holomorphic in $|x|>1$ and vanishes at infinity.
From (5.14),(5.17) and (5.19) we have

$$[K(z)] = \left(f + i4\varepsilon\left(\frac{m^2}{1+m^2}\right)x\right) U(1-x^2) \quad . \tag{5.20}$$

Using the Hilbert transformation, the function K(z) has the following expression

$$K(z)=\frac{1}{2\pi i}\int_{-1}^{1}\left(f(t)+i\frac{4\varepsilon m^2}{1+m^2}t\right)\frac{dt}{t-z} , \qquad (5.21)$$

for more details about the Hilbert transformation see Muskhelishvili [8] and Ditkin and Prudnikov [9] .
Define $K_{\pm}(x) = \lim_{z\to x+i(0\pm)} K(z)$.

Thus we can write

$$K_{+}(x)=\frac{1}{2\pi i}\int_{-1}^{1}\left(f(t)+i\frac{4\varepsilon m^2}{1+m^2}t\right)\frac{dt}{t-x} , \qquad (5.22)$$

so that by Cauchy's residue theorem, and using (5.19), separating the imaginary part, we have

$$\frac{\partial S}{\partial y}(x,0+)=-\frac{2\varepsilon m^2}{1+m^2}x + \frac{1}{2\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt , \qquad (5.23)$$

where $\int$ here (and below), denote a singular integral in the sence of Cauchy which we define as

$$\lim_{\varepsilon\to 0}\left\{\int_{-1}^{-x-\varepsilon} + \int_{x+\varepsilon}^{1}\right\}\frac{f(t)}{t-x}\,dt , \qquad (5.24)$$

provided this limit exists.
(5.20) leads to the Plemelj formulae, see Pennline [10] ,

$$K_{+}(x) + K_{-}(x) = \frac{1}{i\pi}\int_{-1}^{1}\left(f(t)+i\frac{4\varepsilon m^2}{1+m^2}t\right)\frac{dt}{t-x} , \qquad (5.25)$$

from which, and using (5.19), we have

$$\frac{\partial S}{\partial y}(x,0+) + \frac{\partial S}{\partial y}(x,0-) = \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt . \qquad (5.26)$$

From (5.15) and (5.16), we have

$$F_{+} + F_{-} = -\frac{f(x)}{m} , \qquad (5.27)$$

and from (5.10) we have

$$F_{+} + F_{-} + \frac{\partial S}{\partial y}(x,0+) + \frac{\partial S}{\partial y}(x,0-) = -4\varepsilon x . \qquad (5.28)$$

Hence, (5.26) and (5.27) in (5.28) we find

$$f(x) = \frac{m}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt + 4\varepsilon m x , \qquad (5.29)$$

which is a singular integral equation of the Carleman type, see Manwell [11] , Chapter 9.
Solution of (5.29), given by Manwell [11] , has the form

$$f(x)=\frac{4\varepsilon m(2r+x)}{\sqrt{1+m^2}}\left(\frac{1-x}{1+x}\right)^r , \quad 0\le r<1 , \qquad (5.30)$$

where $\tan r\pi = m$.
Therefore, the general solution of the problem may be determined from the following equations:

$$u_y(x,0+)=-4\varepsilon x \tag{5.31}$$

$$u_y(x,0-)=0 \tag{5.32}$$

$$u_x(x,0+)=-\frac{2\varepsilon m}{1+m^2}\left\{x+\sqrt{1+m^2}(2r+x)\left(\frac{1-x}{1+x}\right)^r-\frac{m}{\pi}\left(2+x\ln\frac{1-x}{1+x}\right)\right\} \tag{5.33}$$

$$u_x(x,0-)=-\frac{2\varepsilon m}{1+m^2}\left\{x-3\sqrt{1+m^2}(2r+x)\left(\frac{1-x}{1+x}\right)^r+\frac{m}{\pi}\left(2+x\ln\frac{1-x}{1+x}\right)\right\} \tag{5.34}$$

$$h_y(x,0+)=\frac{2\varepsilon m}{1+m^2}\left\{x+\sqrt{1+m^2}(2r+x)\left(\frac{1-x}{1+x}\right)^r+\frac{m}{\pi}\left(2+x\ln\frac{1-x}{1+x}\right)\right\} \tag{5.35}$$

$$h_y(x,0-)=\frac{2\varepsilon m}{1+m^2}\left\{x-\sqrt{1+m^2}(2r+x)\left(\frac{1-x}{1+x}\right)^r+\frac{m}{\pi}\left(2+x\ln\frac{1-x}{1+x}\right)\right\} \tag{5.36}$$

$$h_x(x,0+)=-2\varepsilon\left\{x-\sqrt{1+m^2}(2r+x)\left(\frac{1-x}{1+x}\right)^r\right\} \tag{5.37}$$

$$h_x(x,0-)=\frac{2\varepsilon}{1+m^2}\left\{(m^2-1)x-(m^2-1)\sqrt{m^2+1}(2r+x)\left(\frac{1-x}{1+x}\right)^r\right\} \tag{5.38}$$

$$p(x,0+)=-\frac{2\varepsilon\rho V_o^2}{m(m^2+1)}\left\{x+\sqrt{m^2+1}(2r+x)\left(\frac{1-x}{1+x}\right)^r-\frac{m^3}{\pi}\left(2+x\ln\frac{1-x}{1+x}\right)\right\} \tag{5.39}$$

$$p(x,0-)=-\frac{2\varepsilon\rho V_o^2}{m(m^2+1)}\left\{x-(2m^2+1)\sqrt{m^2+1}(2r+x)\left(\frac{1-x}{1+x}\right)^r+\frac{m^3}{\pi}\left(2+x\ln\frac{1-x}{1+x}\right)\right\} \tag{5.40}$$

$$C_{p_s}(x,0+)=-\frac{4\varepsilon}{m(m^2+1)}\left\{x+\sqrt{m^2+1}(2r+x)\left(\frac{1-x}{1+x}\right)^r-\frac{m^3}{\pi}\left(2+x\ln\frac{1-x}{1+x}\right)\right\} \tag{5.41}$$

$$C_{p_s}(x,0-)=-\frac{4\varepsilon}{m(m^2+1)}\left\{x-(2m^2+1)\sqrt{m^2+1}(2r+x)\left(\frac{1-x}{1+x}\right)^r+\frac{m^3}{\pi}\left(2+x\ln\frac{1-x}{1+x}\right)\right\} \tag{5.42}$$

We notice that the solution is determined only if $m^2+1\neq 0$. Homentcovschi [12] has shown that this singularity is due to the linearization of the boundary conditions.

6-Determination of the Lift and Drag Forces:-

The flow field is now known, in section (5), and the lift and drag form on the airfoil may now be worked out. Thus on a thin body, the lift is given by

$$L=-\int_{-1}^{1}[p]\,dx \quad , \tag{6.1}$$

using (5.39), (5.40) and carrying out the integrations, we get

$$\frac{L}{2\pi\varepsilon\rho V_o^2}=-\frac{4(m^4+m^2+1)}{m(m^2+1)\sqrt{m^2+1}}\left\{\frac{r(2r+1)}{\sin r\pi}-\frac{2}{\pi}\left(\frac{1}{r+2}+\frac{r}{r+3}+\frac{r(r+1)}{2!(r+4)}+\dots\right)\right\} \tag{6.2}$$

The drag force is given by

$$D=-2\varepsilon\int_{-1}^{1}\left[p(x,0+)+p(x,0-)\right]x\,dx \quad , \tag{6.3}$$

Using (5.39),(5.40) and carrying out the integration we get

$$\frac{D}{8\varepsilon\rho v_o^2} = -\frac{m}{\sqrt{m^2+1}}\left\{(2r+1)\frac{2\pi r}{\sin r\pi} + 8\left(\frac{1-r}{2+r} - \frac{1-r+r^2}{3+r} - \frac{r(1+r^2)}{2!(4+r)} - \frac{r(1+r)(1+r+r^2)}{3!(5+r)} - \dots\right)\right\} + \frac{2}{3m(m^2+1)} \qquad (6.4)$$

7-Numerical Results and Discusions:

From the foregoing analysis we find that:

(1) $u_x(x,0)$ decreases with the increase of x and over the upper body surface, the magnitude increases with the increase of m, as shown in Figure 2, while along the lower body surface, the magnitude decreases with the increase of m, as shown in Figure 3.

(2) $u_y(x,0+)$ decreases linearly with x and does not depend on m, while there is no variation in $u_y(x,0-)$.

(3) $h_x(x,0+)$ increases with the increase of m, in the negative x-direction, and decreases in the positive x-direction.

(4) $h_y(x,0-)$ decreases with the increase of m, and reaches its minimum faster as the parameter m increases, as shown in Fig.4.

(5) Along the upper body surface, the negative pressure coefficient increases parabolically for $m<1$ and increases linearly for $m=1$, while for $m>1$, it decreases parabolically in the negative x-direction, while it increases parabolically in the positive x-direction, as shown in Figure 5.

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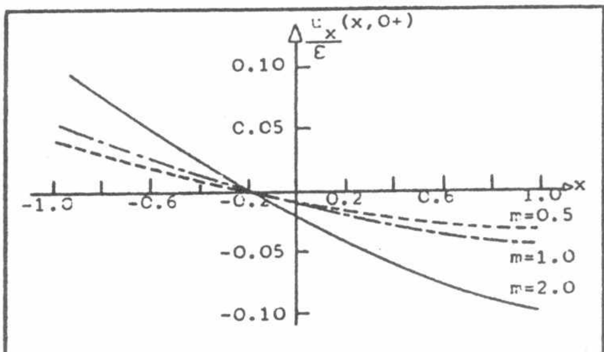
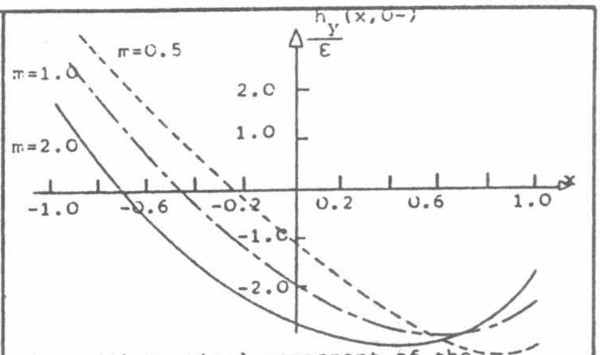
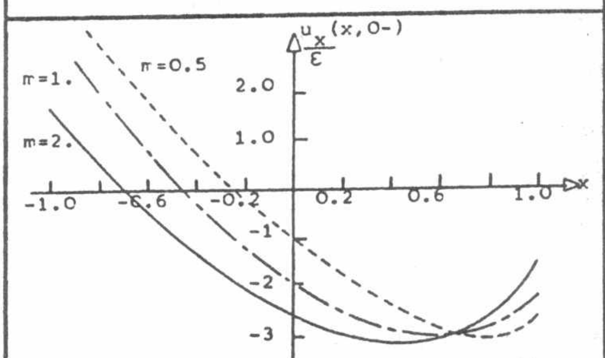


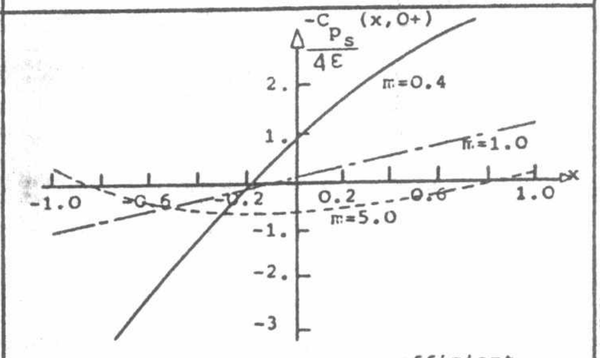
Figure(2):Horizontal component of the disturbed velocity versus x for various m



Figure(4):Vertical component of the disturbed magnetic field versus x .



Figure(3):Horizontal component of the disturbed velocity versus x for various values of m



Figure(5):Surface pressure coefficient versus x for various values of m .