APPLICATION OF HOPOSCOTCH METHOD FOR SOLVING THE UNSTEADY TWO DIMENSIONAL, INVISCID, INTERNAL FLOWS

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ABSTRACT

In the present study, the hoposcotch method was tested for solving the axisymmetric, inviscid, compressible, shock-free flows. The physical space was transformed into a rectangular computational space. Hoposcotch method was applied to solve the unsteady, cylindrical, conserved governing equations of motion in the transformed computational space, with the steady state solution computed as the asymptote of the transient solution. The source flow and the transonic flow through a converging-diverging nozzle were considered as test problems. The solutions were compared with the available exact, experimental, and numerical results.

The method was found to be stable and in excellent agreement with the available results. An advantage of the method is that it is superior in speed (computing time); being 1.8 times faster than MacCormack's method and 2.9 to 7.3 times faster than the reference plane characteristics method.

NOMENCLATURE

a	: speed of sound.
E	: total internal energy per unit volume.
i,j	: indices denoting the location of a grid point in the ζ and η directions, respectively.
M	: Mach number.
n	: denotes the present time level
P	: Pressure.
u,v	: velocity components in the axial and radial directions, respectively.
x,y	: physical axial and radial coordinates
y_w	: wall radius
ρ	: density
ζ, η	: transformed coordinates
α, β	: partial derivatives of η with respect to x and y respectively.
γ	: specific heat ratio.
$\Delta\zeta, \Delta\eta, \Delta t$	: step sizes in the ζ, η, and t directions, respectively.

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INTRODUCTION

There are large variety of numerical methods that are used for solving the many flow problems to which the inviscid, compressible flow assumption are made. The success of any method depends on the applications of the method to real flow problems. An efficient numerical method must possess the following advantages: the finite-difference equations and the resulting computer programs tend to be simple and fast. The methods of Lax-Wendroff [1], MacCormack [2], and Rusanov [3] are ones of the mot popular methods because they have, to some extent, the above advantages. Among all of methods, is that due to MacCormack [2] which, in addition to its accuracy and simplicity, has proven to reach the steady state solution in minimum computing time (see [4] and [5]). However, researchers are continuing to provide new numerical methods in order to minimize the computing time. Of course computing time is of great importance for complicated problems such as flows through propulsive nozzles, Gas turbine and compressor blades.

In the present study, an interesting numerical method, the hoposcotch method, is tested with solutions of real fluid dynamic problems. This method was chosen because firstly, it has not recieved a great deal of applications; secondly, it is applicable to both viscid and inviscid flows; and thirdly, it seems to be simple and fast. Historically, hoposcotch method was first introduced to solve the Navier-Stokes equations for shear layer mixing problems [6]. It was later introduced to solve the inviscid flow problems [7] and it was successfully applied for the solution of the inviscid, one-dimensional flow problem [8]. The present work extends the applications of the method to solve the axisymmetric, two-dimensional flow problems. The accuracy and computing times are investigated and compared with the efficient MacCormak's method and also with the reference plane characteristics method.

The technique of hoposcotch method is based on numerically solving the unsteady governing equations of motion (in conservation forms). The steady state solution is computed as the asymptotic limit in time of the unsteady solution. An attractive solution procedure is accomplished with the technique which leads to considerable reduction in the computing time. This will be illustrated in the section of "Method of solution".

EQUATIONS OF MOTION

The appropriate conservation laws for the unsteady, axisymmetric, inviscid, compressible flow in vectorform are:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{1}{y}\frac{\partial (Gy)}{\partial y} + H = 0 \tag{1}$$

where

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ E \end{vmatrix} , \quad F = \begin{vmatrix} \rho u \\ \rho u^2 + P \\ \rho uv \\ u(E+P) \end{vmatrix} , \quad G = \begin{vmatrix} \rho v \\ \rho uv \\ \rho v^2 + P \\ v(E+P) \end{vmatrix} , \quad H = \begin{vmatrix} 0 \\ 0 \\ -P \\ 0 \end{vmatrix} \tag{2}$$

For purposes of applications of the hoposcotch method and the boundary conditions, the physical (x,y) plane is transformed into a rectangular computational (ζ,η) plane by the following transformations.

$$\zeta = x \quad , \quad \eta = y/y_w(x) \tag{3}$$

where y_w is function of the axial distance x, partial derivatives of the transformed variables with respect to the physical variables are defined as follow:

$$\beta = \frac{\partial \eta}{\partial y} = 1/y_w \tag{4}$$

$$\alpha = \frac{\partial \eta}{\partial x} = -\beta\eta \frac{\partial y_w}{\partial x} \tag{5}$$

By employing the chain-rule, the equations of motion in the transformed coordinates take the following form:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial \zeta} + \alpha \frac{\partial F}{\partial \eta} + \frac{\beta}{\eta}\frac{\partial (G\eta)}{\partial \eta} + H = 0 \tag{6}$$

where U, F, G and H are defined by Equation (2). To complete the system of equations, the equation of state for perfect gas is used:

$$P = (\gamma - 1)\,[E - \rho(u^2 + v^2)/2] \tag{7}$$

Equations (6) and (7) were employed for obtaining solution at the interior points of the flow field.

MOTHOD OF SOLUTION; HOPOSCOTCH METHOD

The numerical technique of the hoposcotch method is based on solving the unsteady governing equations of motion. The steady state solution is computed as the asymptotic limit in time of the unsteady solution. The equations are replaced by finite differences, where forward differences are used with time derivatives and central differences are used with space derivatives. At any time level n (where the solution is assumed to be known), the grid points are divided into two groups; the odd and even points [according to the summation of i+j+n; see Fig. 1]. The solution is explicity advanced from time level n to time level n+1 in two sweeps. The first sweep solves for flow properties at the even points by employing properties at time level n. The second sweep solves for flow properties at the odd points by employing properties at time level n+1. Thus applying the hoposcotch method to equation (6), yields the following solutions for the even and odd points: First sweep (i+j+n even, see Fig. 1):

$$\begin{aligned} U^{n+1}_{i,j} = U^{n}_{i,j} &- \frac{\Delta t}{2\Delta\zeta}\,(F^{n}_{i+1,j} - F^{n}_{i-1,j}) - \frac{\Delta t.\,\alpha_{i,j}}{2\,\Delta\eta}\,(F^{n}_{i,j+1} - F^{n}_{i,j-1}) \\ &- \frac{\Delta t}{2\Delta\eta}\left(\frac{\beta}{\eta}\right)_{i,j}\,[(G\eta)^{n}_{i,j+1} - (G\eta)^{n}_{i,j-1}] \\ &- \frac{1}{4}\,(H^{n}_{i+1,j} + H^{n}_{i-1,j} + H^{n}_{i,j+1} + H^{n}_{i,j-1}) \end{aligned} \tag{8}$$

second sweep (i+j+n odd; see Fig. 1)

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \frac{\Delta t}{2\Delta\zeta}(F_{i+1,j}^{n+1} - F_{i-1,j}^{n+1})$$
$$- \frac{\Delta t \alpha_{i,j}}{2\Delta\eta}(F_{i,j+1}^{n+1} - F_{i,j-1}^{n+1})$$
$$- \frac{\Delta t}{2\Delta\eta}\left(\frac{\beta}{\eta}\right)_{i,j}[(G\eta)_{i,j+1}^{n+1} - (G\eta)_{i,j-1}^{n+1}]$$
$$- \frac{1}{4}[H_{i+1,j}^{n+1} + H_{i-1,j}^{n+1} + H_{i,j+1}^{n+1} + H_{i,j-1}^{n+1}] \quad (9)$$

Gourlay [6] suggested a simplification to the standard two-sweep hoposcotch scheme which almost entirely removed the first sweep where equation (8) is replaced by

$$U_{i,j}^{n+1} = 2\,U_{i,j}^{n} - U_{i,j}^{n-1} \qquad (i+j+n \text{ even}) \qquad (10)$$

The use of equation (10) increases the speed of the hoposcotch method by a factor of two without requiring additional storage. Equations (9) and (10) were employed at all time levels, with one exception, where equation (8) replaces equation (10) at just one time level. This will be discussed in the next section.

BOUNDARY CONDITIONS AND TIME REGULATION

The boundaries consist of all points which lie on inlet, exit, wall and axis of symmetry. The solution at each boundary point was obtained by employing the method of reference plane characteristics. This method was applied in previous works to both interior and boundary points and was found to be very accurate for predicting solutions at the boundaries; see references [9] to [11]. The choice of the reference plane used for each boundary was such that, at least, one of the characteristic curves lies inside the computational domain. This method solves the unsteady equations in nonconservation forms, and the solution is advanced in consecutive time levels until the steady state solution is obtained(as hoposcotch method does). For more details of the applications of the method, the reader is referred to references [9] to [11].

The time step, Δt, between successive solution surfaces was determined by applying the Courant-Friedrichs-Lewy (CFL) stability criterion. In the transformed coordinates the time step is given by:

$$\Delta t = A/\left[\frac{|u|}{\Delta\zeta} + \frac{|v|\beta}{\Delta\eta} + a\sqrt{\frac{1}{\Delta\zeta^2} + \frac{\beta^2}{\Delta\eta^2}}\right] \qquad (11)$$

where A is a constant depending on the problem at hand (in the present analysis, A = 1). Equation (11) was applied at each grid point and the minimum of Δt was chosen. It is to be noticed that Δt must be recalculated for each new time level.

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SOLUTION PROCEDURE

To begin the solution procedure, an initial value surface (n = 0) was generated. This was done by assigning the one-dimensional flow to all grid points of the flowfield. The time step size Δt was then calculated from equation (11). The solution was advanced by one time level by first obtaining solution at the boundary points, then obtaining solution at the interior points. The following procedure was performed for the solution of the interior points. At time level n = 1, equation (8) was employed to obtain solution at the even points [This equation was only used at n = 1, since equation (11) can not be used at this time level because it requires two predetermined time levels, which are not available at n = 1] and equation (9) was employed to obtain solutions at the odd points. At time levels $n \geq 2$, equation (10) [which replaces equation (8)] was employed first for the even points, then equation (9) was employed for the odd points [where the required flow properties at the even points at time level n+1 are now available, see Fig. 1]. The current solution surface was treated as the initial-value surface and the procedure was repeated until convergence was achieved.

RESULTS AND DISCUSSIONS

In order to examin the accuracy and the computing time of the hoposcotch method, two flow cases were considered, where comparison was made with the available results. The first case is the source flow problem and the second case is the flow through transonic convirging -diverging nozzle.

In the source flow problem, a conical nozzle of 15° half-angle was used. The values of x at the inlet and exit planes are 1.7 in and 4.9 in respectively. The inlet Mach number at the centerline is 1.1, the inlet stagnation conditions are 70 psia and 540 R. The results are presented in figures (2) to (4). The results of the exact solution and of Ref. [10] are alos presented for comparison purposes. Ref.[10] employed reference plane characteristics method for the solution of all grid points (boundary as well as interior). As shown from the figures, the agreement is very good. In purticular the maximum disagreement in static pressure is about 0.3 psi and that of the axial velocity is 30 ft/s (In this problem, the exact solution leads to minimum and maximum velocities of 1124 ft/s and 2184 ft/s, respectively). Calculation of the stagnation pressure indicated that the disagreement is within 0.7%. The computing time steps are 74 while that of Ref.[10] are 63. The computing times for hoposcotch method and that of reference plane characteristics method (Ref.[10]) were recorded and it was found that, for the problem at hand, hoposcotch method is 2.9 times faster than the reference plane characteristics method. Since the computing time of the boundary were included, it is expected that the above ratio is of higher value and as will be shown in the next paragraph, this ratio is a function of the total number of time steps and the number of grid points.

The second flow case considered in the present study is flow through an axisymmetric converging-diverging nozzle. The nozzle geometry is shown in Fig. 5 . At the inlet plane the stagnation properties are 70 psia and 540 R, and the flow angle is zero degree. The results are presented in figures (6) and (7). The experimental and numerical results of Ref.[11] are included in the same figures for comparison purposes. The numerical results of Ref.[10] are also included. As mentioned in the previous paragraph, Ref.[10] employed reference plane characteristics method for the solution of all points. Ref.[11] employed the same method of Ref.[10] for the solution of boundary points and employed MacCormack's method for the solution of the interior points. Thus the comparison of the computing time of Ref.[11] and the present work, is a comparison between MacCormack method and hoposcotch method, since both works, employed the same boundary solutions. Again the results as indicated by figures (6) and (7) are in excellent agreement for all methods. The computing time steps required for reaching the steady state solution are 256 for the reference plane characteristics method (Ref.[10]), 299 for MacCormack method (Ref.[11]), and 312 for the hoposcotch method (present work). The computing time for each method was recorded and it was found that hoposcotch method is 1.8 times faster than MacCormak's method, and 7.3 times faster than the reference plane characteristics method, for this problem.

CONCLUDING REMARKS

The hoposcotch method was found to be comparable in accuracy with the other efficient numerical methods. This method is superior in speed (computing time); being 1.8 faster than MacCormak's method and 2.9 to 7.3 faster than the reference plane characteristic method, for the two problems considered in this study. For the above reasons, the hoposcotch method is recommended for solutions of shock-free problems. However, the method does not yet applied to real flow problems which include shock wave discontinuities. More applications of the method are still in need to cover this kind of flow.

REFERENCES

1- Lax, P.D., and Wendroff, B. "Difference Schemes with High Order of Accuracy for Solving Hyperbolic Equations", Communications on Pure and Applied Mathematics, V. 17, P. 381, 1964.

2- MacCormack, R.W. "The Effect of Viscosity in Hypervelocity Impact Cratering", AIAA Hypervelocity Impact Conf., AIAA paper No. 69-354, 1969.

3- Rusanov, V. "Calculation of Interaction of Non-Steady Shock Waves with Obstacle", National Research Council of Canada, Translation No. 1027, 1962.

4- Sod, G.A. "Review. A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws", J. Comp. Physics, V. 27, 1978.

5- Kemry, M.M.; and El-Kady, M.A. "Solution of the Shock Tube Problem be Severeal Finite Difference Methods", Proc. of the Fifth Int. Conf. for Mech. Power Engineering, Cairo, 1984.

6- Gourlay, A.R. "Hoposcotch: A Fast Second-Order Partial Differential Equation Solver", J. Inst. Maths. Applics, V. 6, 1970.

7- Gourlay, A.R., and Morris, J.L. "Hoposcotch Method for Non-Linear Hyperbolc Systems", IBM (United Kingdom) Ltd., Peterlee, County Durham England, 1971.

8- Holst, T.L. "The Relative Merits of Several Numerical Techniques for Solving the Compressible Navier-Stokes Equations", NASA CP-2001, Advances in Engineering Science, V. 4, 1974.

9- Shin, Y.W., and Cot, C.A. "Two-Dimensional Fluid Transient Analysis by the Method of Near Characteristics", J. Comp. Phys., V. 28, No. 2, 1978

10- Kemry, M.M. "Computation of the Axisymmetric Compressible Internal Flow by the Reference Plane Characteristic Method", Proc. Int. AMSE Conf. "Modelling & Simulation", V. 3, Nice, 1983.

11- Cline, M.C. "VNAP: A Computer Program for Computation of Two-Dimensional, Time-Dependent, Compressible, Viscous, Internal Flow", Los Alamos Science Laboratory, Report LA-7326, 1978.

12- Roach, P.J. "Computational Fluid Dynamics", Hemosa Publishers, Albuquerque, N.M., 1976.

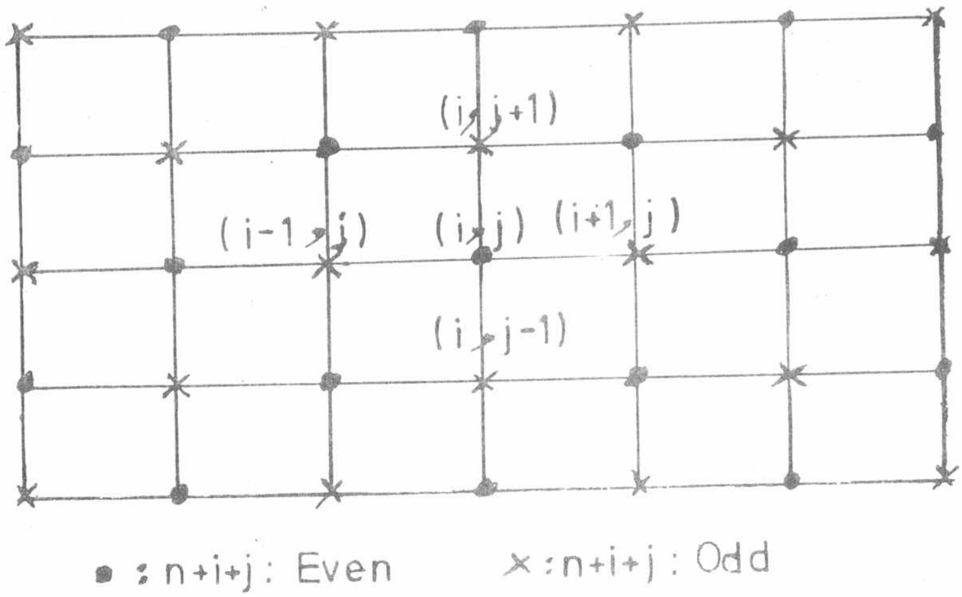

Fig.1. Interior, even and odd, points

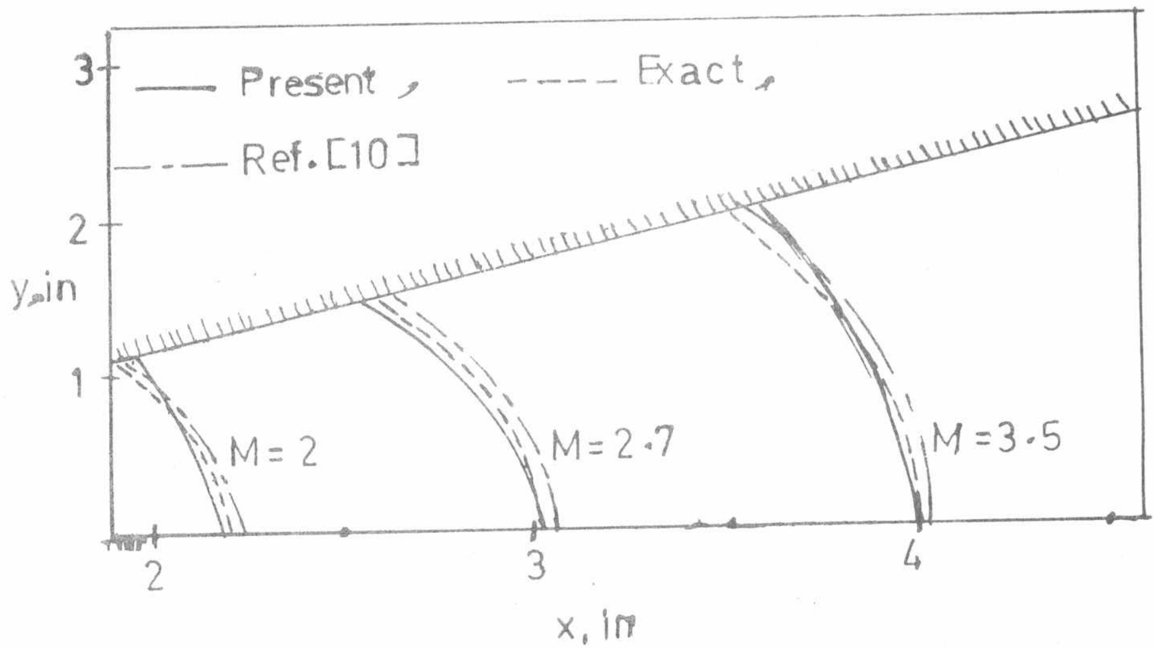

Fig.2. Mach number contours for the source flow

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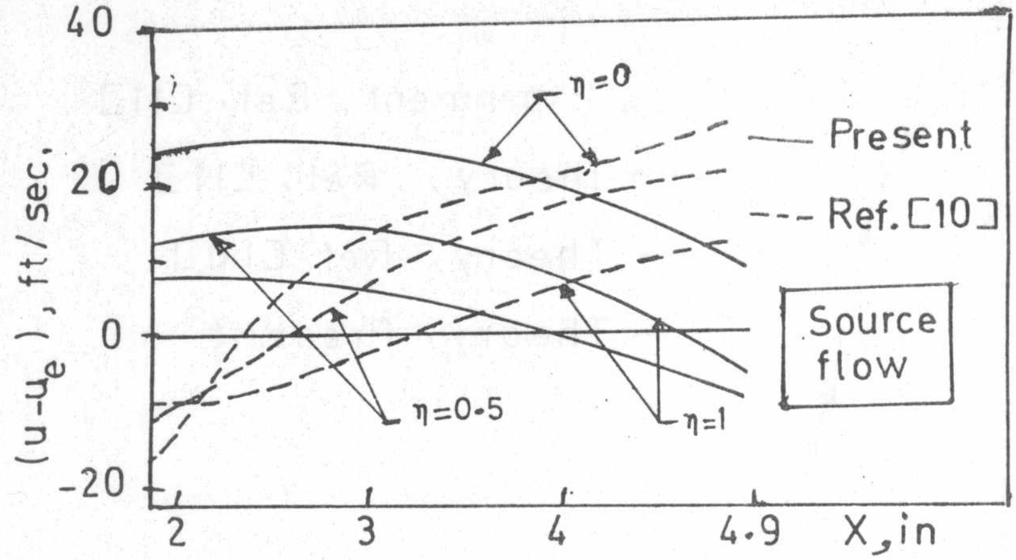

Fig. 3. Error in axial velocity (u_e : exact velocity)

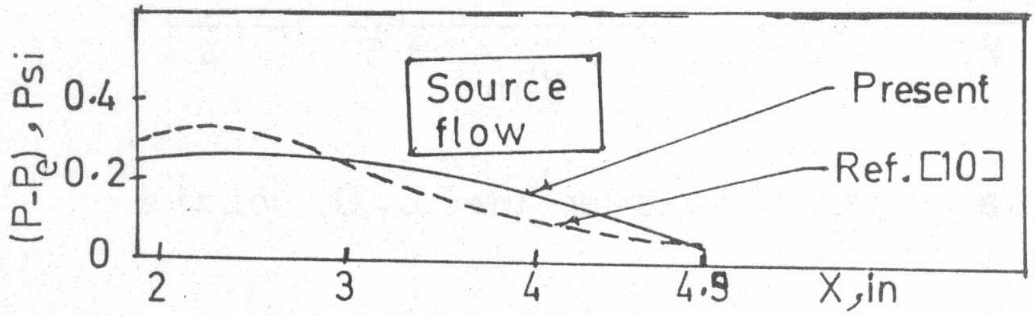

Fig. 4. Error in wall pressure (P_e: exact pressure)

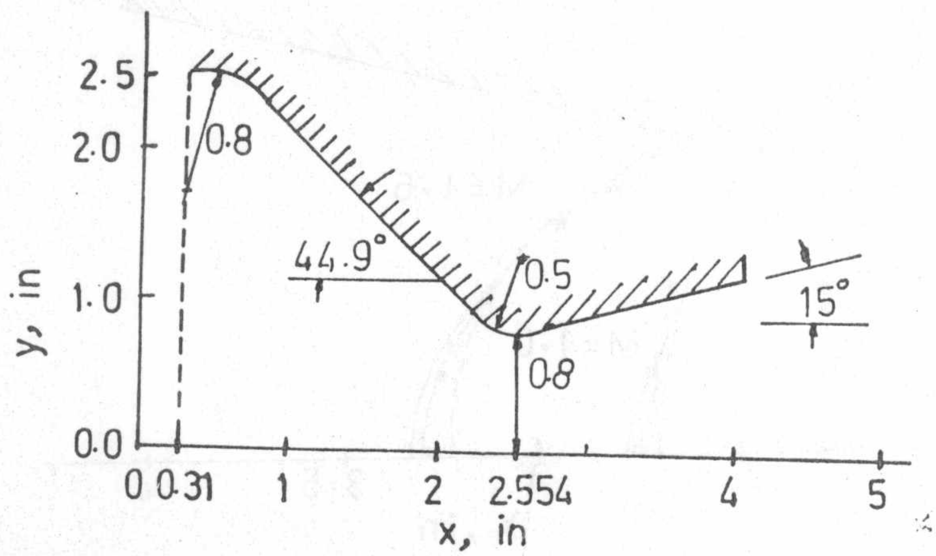

Fig. 5. Nozzle geometry

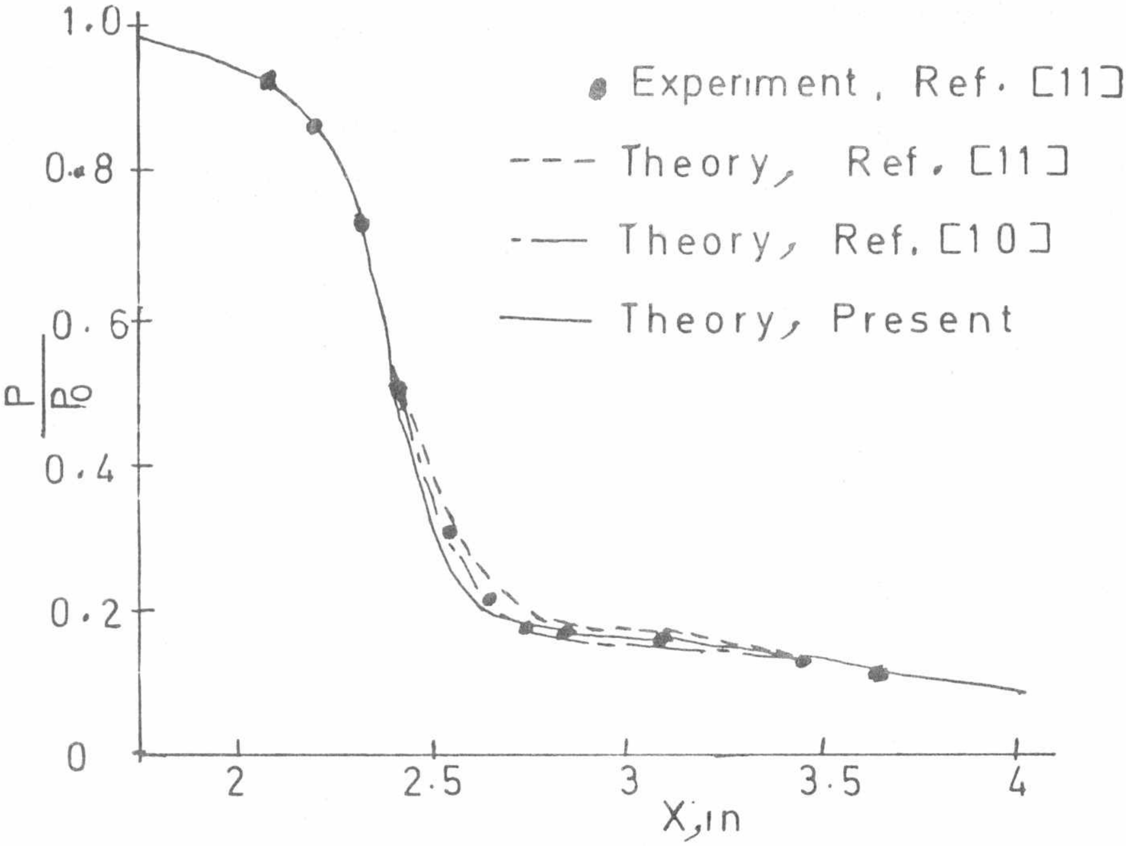

Fig. 6. Wall pressure ratio for the C-D nozzle

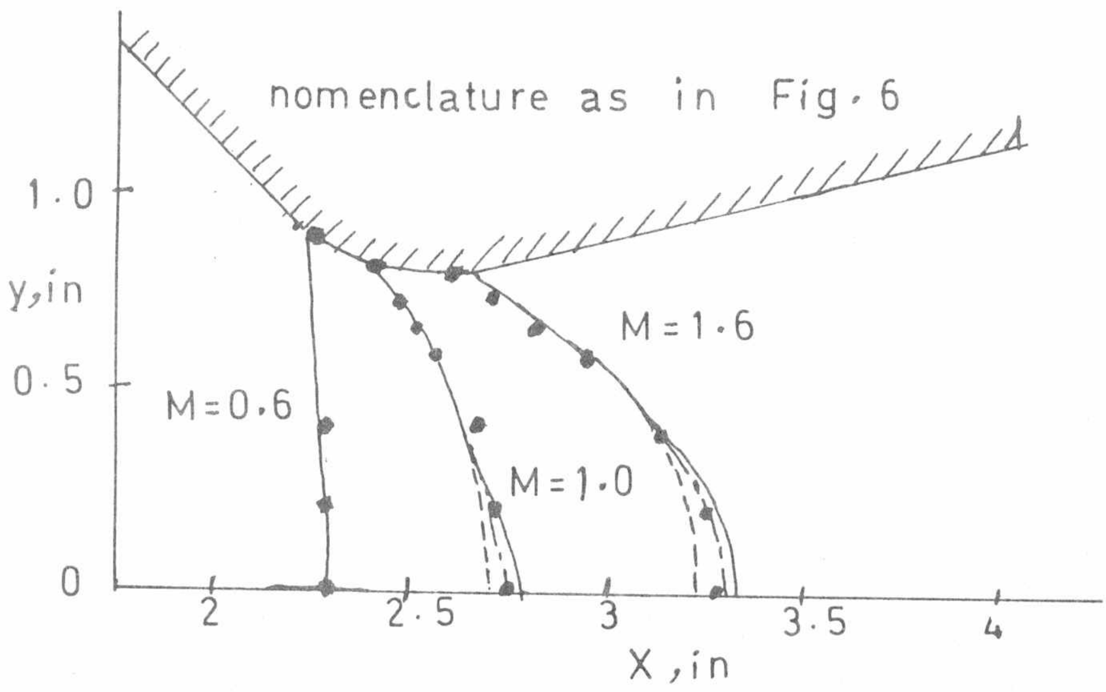

Fig. 7. Mach number contours for the C-D nozzle