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PERFORMANCE OF STRAIGHT SIDED FINITE ELEMENTS IN SIMPLY SUPPORTED CURVED-EDGED THIN PLATES

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ABSTRACT

In this work we explain how can we use usual finite elements to solve curved thin plates which are simply supported. Numerous examples have been treated by different finite elements. Results are very satisfactory.

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FORMULATION OF THE PROBLEM

Consider a thin elastic plate with middle surface given by the domain $\Omega \subset R^2$ with boundary Γ and acted upon by the transversal load q , see Fig.1.

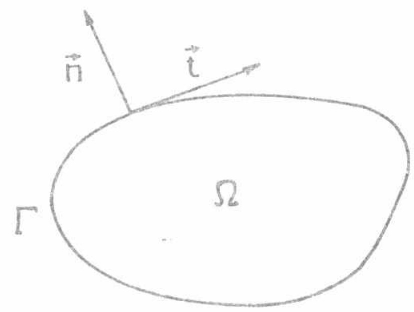


Fig.1 The middle surface of the plate

Assuming small deflection and a linearly isotropic elastic material, the plate problem is to find the deflection w which satisfies the fourth order partial differential equation :

$$D \Delta^2 w = q$$

where D is the plate rigidity.
together with certain specified boundary conditions.
To define the boundary conditions let $\vec{n}$ be the outward unit normal to Γ, $\vec{t}$. the unit tangent to Γ. Now let the boundary Γ be partitioned into three parts Γ_i, $i = 1,2,3$, and consider the following boundary conditions:

1) $w = \dfrac{\partial w}{\partial n} = 0$ on Γ_1 (clamped boundary)
2) $w = M_n = 0$ on Γ_2 (simply supported boundary)
3) $M_n = Q_n - \dfrac{\partial}{\partial s} M_{nt} = 0$ on Γ_3 (free boundary)

where $M_n = -D \left(\dfrac{\partial^2 w}{\partial n^2} + \nu \dfrac{\partial^2 w}{\partial t^2} \right)$

$$Q_n = - D \frac{\partial}{\partial n} \left(\frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 w}{\partial t^2} \right)$$

$$M_{nt} = D\,(1-\nu) \frac{\partial^2 w}{\partial n \partial t}$$

In this work, we are interested in the case of simply supported boundary conditions. It is important to realize that the first boundary condition $w = 0$

is an essential boundary condition which must be included in the finite element data. As for the second condition $M_n = 0$, we know that it is a 'natural' one, which means that it is not included in the data needed for finite elemenet solutions, and it will be satisfied naturally by the finite element solution [1] Let us now take a closer look to this condition, when the simply supported part of the boundary Γ_2 is straight, the first condition $w = 0$ necessitates $\frac{\partial w}{\partial t} = 0$ as well. The second condition $M_n = 0$ implies $\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} = 0$

Now, the first condition $w = 0$ implies $\frac{\partial^2 w}{\partial t^2} = 0$. and so the second condition becomes $\frac{\partial^2 w}{\partial n^2} = 0$ and no depenence on Poisson's ratio.

Let us now consider the case when the simply supported boundary is curved, if we replace the boundary Γ of the plate , as usually done by finite element solution, by a polygon $\tilde{\Gamma}$, we will face the following problems over the simply supported part :

1. Is w equal to zero over $\tilde{\Gamma}$? In particular will $\frac{\partial w}{\partial t}$ be forced to be zero in the finite element data?
2. What happens to the natural boundary condition $M_n = 0$?

The first problem does not exist with finite elements which have not $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ as degrees of freedom such as Morley element [2], and Nagtegaal and Slater element [3]. Unfortunately the majority of plate elements have these derivatives as degrees of freedom and one of the aims of this paper is to solve this problem.

The second problem is a serious one. In fact, imposing $w = 0$. along $\tilde{\Gamma}$ makes $\frac{\partial^2 w}{\partial t^2} = 0$ on $\tilde{\Gamma}$ and the natural boundary condition $M_n = 0$ becomes $\frac{\partial^2 w}{\partial n^2} = 0$.

Thus, the obtained solution is a solution to the equation :

$$D \Delta^2 w = q \text{ with } w = \frac{\partial^2 w}{\partial n^2} = 0 \text{ along } \tilde{\Gamma}.$$

This is known as Babuška paradox [4]. We are not going to solve this problem directly , but we pay the attention to the fact that imposing $w = 0$ along $\tilde{\Gamma}$ occurs in the limit , and so we actually do not have $\frac{\partial^2 w}{\partial t^2} = 0$ and consequently $\frac{\partial^2 w}{\partial n^2}$ will not be zero.

HOW TO IMPOSE THE BOUNDARY CONDITIONS

The essential boundary condition for simply supported plates is $w = 0$ along the boundary. This condition necessitates $\frac{\partial w}{\partial t} = 0$.along the same boundary . When curved boundary is approximated by 'straight-edged' polygon ,the imposition of $\frac{\partial w}{\partial t} = 0$ at inter-element boundaries, means that at a typical boundary node A: $\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial t_2} = 0$ for two different directions $\vec{t_1}$ and $\vec{t_2}$ (see Fig.2). This implies $\frac{\partial w}{\partial n} = 0$ as well.Thus, we are actually satisfying the clamped boundary condition!. In order to avoid this situation, one could think of imposing $\frac{\partial w}{\partial t} = 0$ at a typical boundary node but differentiating this time in the ' exact' direction of tangent to boundary (see Fig 2) This approach will be named 'imposed', and leads - as will be seen later - to wrong results. This can be explained as follows: as the mesh is refined, at a typical node A, both w and $\frac{\partial w}{\partial t}$ are imposed to be zero , and both $\frac{\partial w}{\partial t_1}, \frac{\partial w}{\partial t_2}$ converge to zero as A_1 and A_2 approache A.Once more, clamped boundary conditions are achieved but this time in the limit.

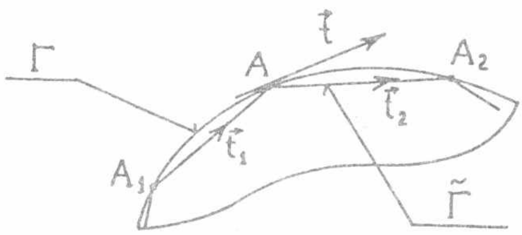


Fig.2 Approximation of curved boundary by 'straight-edged' polygon

Following Rhee [5], we are going to impose only $w = 0$ at boundary nodes. This means that the derivative degrees of freedom will be left free. According to this approach, we achieve $w = 0$ along the whole boundary only in the limit. Since the convergence of first (second)derivatives is one (two)order of magnitude less than that of the function [6,12], we expect that the condition $\frac{\partial^2 w}{\partial n^2} = 0$ will not be satisfied. This last approach of handling the simply support boundary condition will be named 'relaxed' where we impose only $w = 0$. The numerical results of our work supports this approach. It remains to mention that the use of curved finite elements[8,9]is the best way Unfortunately , the existing finite element codes do not contain such elements (we speak of course about the most common famous codes).One of the aims of this work is how to solve simply-supported curved-edged thin plates using these codes.

APPLICATIONS

We are going to discuss three cases:
case 1. a simply supported circular plate
case 2. a. simply supported elliptical plate
case 3. a parabolic plate simply supported along its curved part

Case 1 A Simply Supported Circular Plate (Fig.3):

The analytical solution of a uniformly loaded simply supported circular plate is well known [10]:

$$w = q\frac{(a^2 - r^2)}{64\,D}\left[\left(\frac{5+\nu}{1+\nu}\right)a^2 - r^2\right]$$

The finite element solution of this problem is given in Table 1

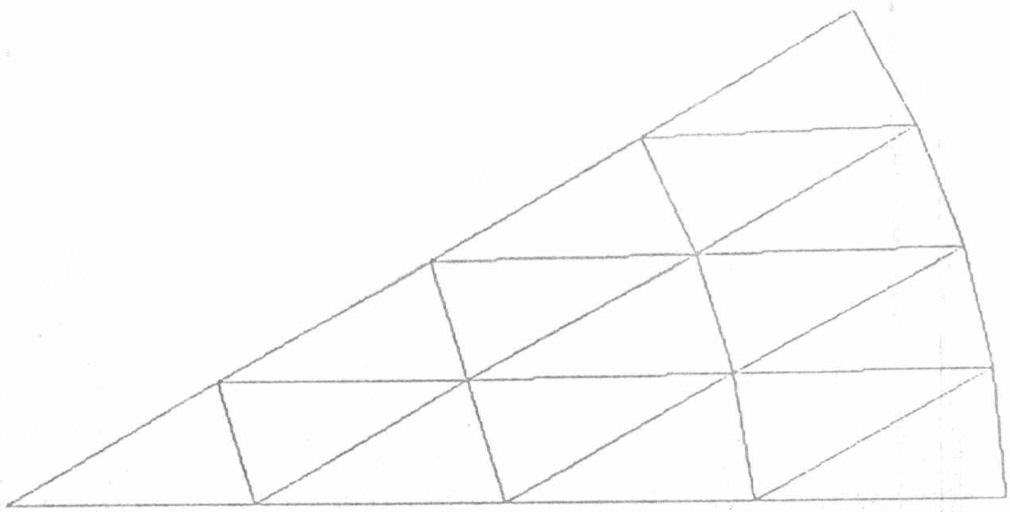


Fig.3 Simply supported circular plate.
A sector of vertix angle $\pi/6$ is discretized and symmetry is exploited

Due to the fact that the deflection w is independent on the angle θ , the imposition of $\frac{\partial w}{\partial t} = 0$ along the boundary ('imposed' boundary condition) will not influence the results. As it is clear from Table 1 and Fig.4 , convergence to the exact solution occurs without difficulty.

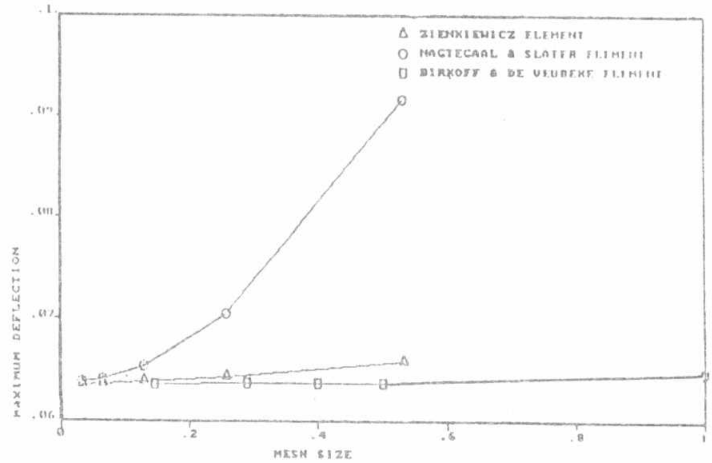


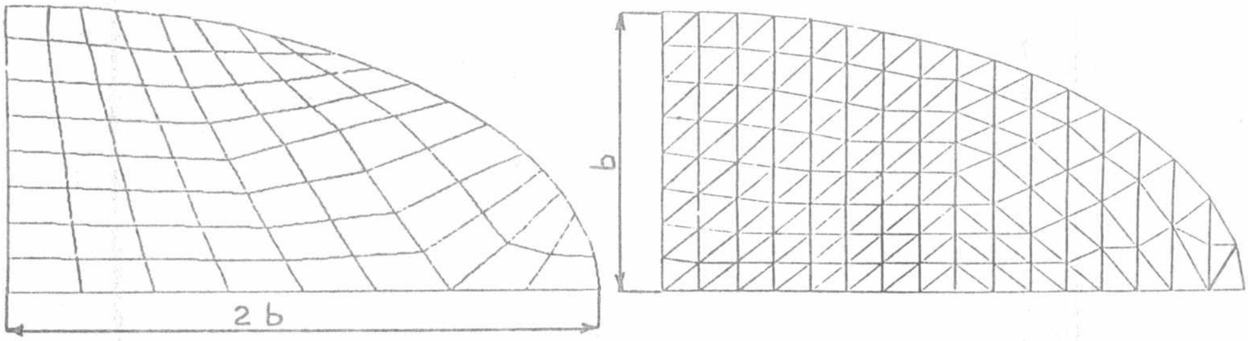
Fig.4 Maximum deflection of circular plate

Table 1 Maximum deflection and bending moments of a uniformly loaded simply supported circular plate

($w = \alpha q a^4 / D$, $M_x = \beta a^2 / D$, $M_y = \gamma a^2 / D$,a is the circle radius)

a) Triangular finite element of Zienkiewicz[11]			
No. of elements	α	β	γ
1	0.065908	0.21475	0.2088
4	0.064337	0.21412	0.2124
16	0.063867	0.21138	0.2108
64	0.063709	0.21008	0.2107
b) Quadrilateral element of Birkoff & DeVeubeke [11,4]			
No. of elements	α	β	γ
1	0.064829	0.22205	0.2108
4	0.063815	0.21055	0.2077
8	0.063780	0.21085	0.2078
14	0.0637373	0.20800	0.2068
56	0.0636518	0.20662	0.2062
c) Triangular element of Nagtagaal & Slater [3]			
No. of elements	α	β	γ
1	0.091575	0.116955	0.04971
4	0.070639	0.183567	0.16677
16	0.065442	0.200700	0.19650
64	0.064144	0.204950	0.20392
256	0.063836	0.206067	0.32207
Exact[10]	0.06370192	0.20625	0.20625

Case 2. Simply Supported Elliptical Plate (Fig.5):



a) Quadrilateral elements b) Triangular elements

Fig.5 Finite element discretization of elliptical plate.
(Due to symmetry only one quarter of the plate is used)

As we see from Table2 there is a slight difference between 'imposed' and 'relaxed' boundary conditions.
An important remark can be seen from Fig 6 : Convergence occurs to a solution which is close to the exact one but not to the 'exact' value. From engineering point of view the difference is small (less than 1.7 %).

Table 2 Maximum deflection of a uniformly loaded simply supported elliptical plate ($w = \alpha q b^4 / D$)

a) Using triangular finite element of Zienkiewicz

No.of elements	Relaxed	Imposed
1	0.14160640	0.14161
3	0.14424524	0.14382
12	0.14339542	0.14347
48	0.14270549	0.14271
192	0.14241722	0.14221
195	0.14232000	0.14210

b) Using quadrilateral element of Birkoff & DeVeubeke

No of elements	Relaxed	Imposed
1	0.14277	0.14401
3	0.14401	0.14277
16	0.14288	0.14288
17	0.14559	0.14541
68	0.14234	0.14234

Exact [10] $w = 0.14469\, q\, b^4 / D$

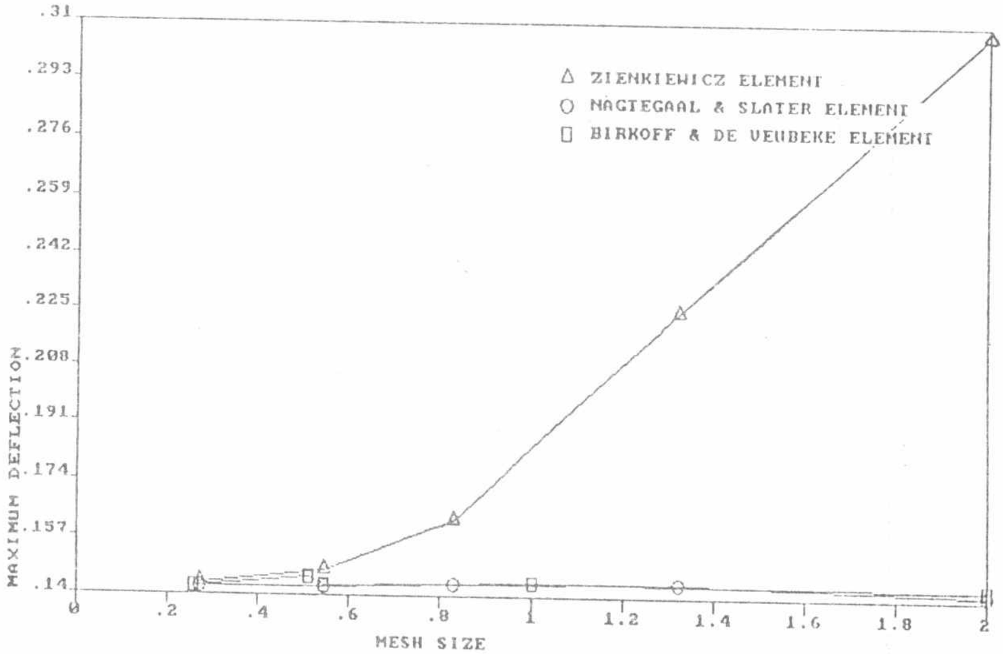
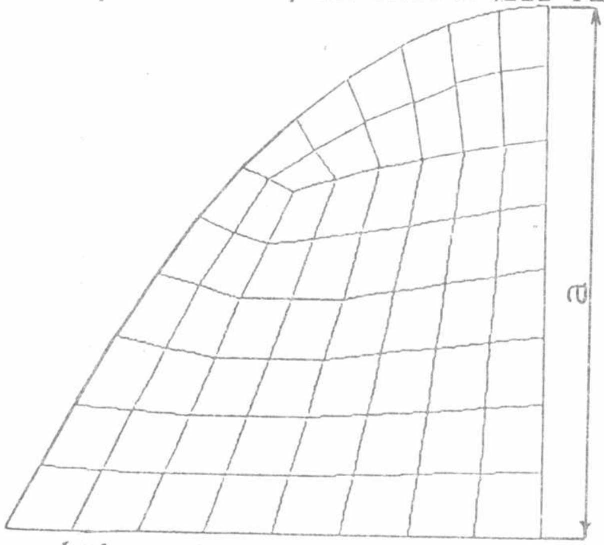


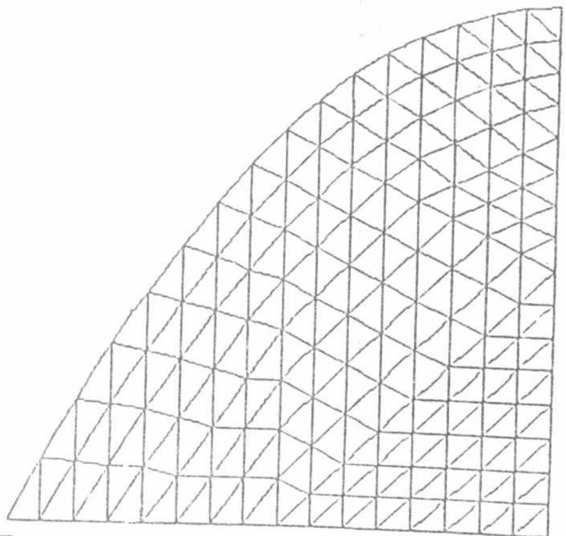
Fig.6 Maximum deflection of elliptical plate

Case 3 Simply Supported Parabolic Plate (Fig.7):

In this example, we have no analytical solution to compare with. But as seen from Fig.8 convergence occurs by all used elements to one result. It is also clear that 'imposed' simply supported boundary conditions leads to the results of clamped boundary conditions (see Table3).



a) quadrilateral elements



b) Triangular elements

Fig.7 Finite element discretization of parabolic plate

Table 3 Maximum deflection of a uniformly loaded simply supported parabolic plate ($w = \alpha q a^4 / D$)

a) Using triangular element of Zienkiewicz

No of elements	Relaxed	Imposed	Clamped
1	0.16006	0.114350	0.0084731
4	0.14149	0.030037	0.0150640
16	0.13578	0.023808	0.0167690
64	0.13489	0.019009	0.0173950
256	0.13468	0.017929	0.0176060

b) Using quadrilateral element of Birkoff & DeVeubeke

No of elements	Relaxed	Imposed	Clamped
1	0.14586	0.0409174	0.0138926
4	0.13643	0.0253047	0.0167629
15	0.13532	0.0227140	0.0174200
60	0.13481	0.0200890	0.0176310

c) Using triangular element of Nagtegaal & Slater

No. of elements	Simply Supported	Clamped
1	0.15110	0.059520
4	0.14737	0.034680
16	0.13830	0.022842
64	0.13565	0.019136

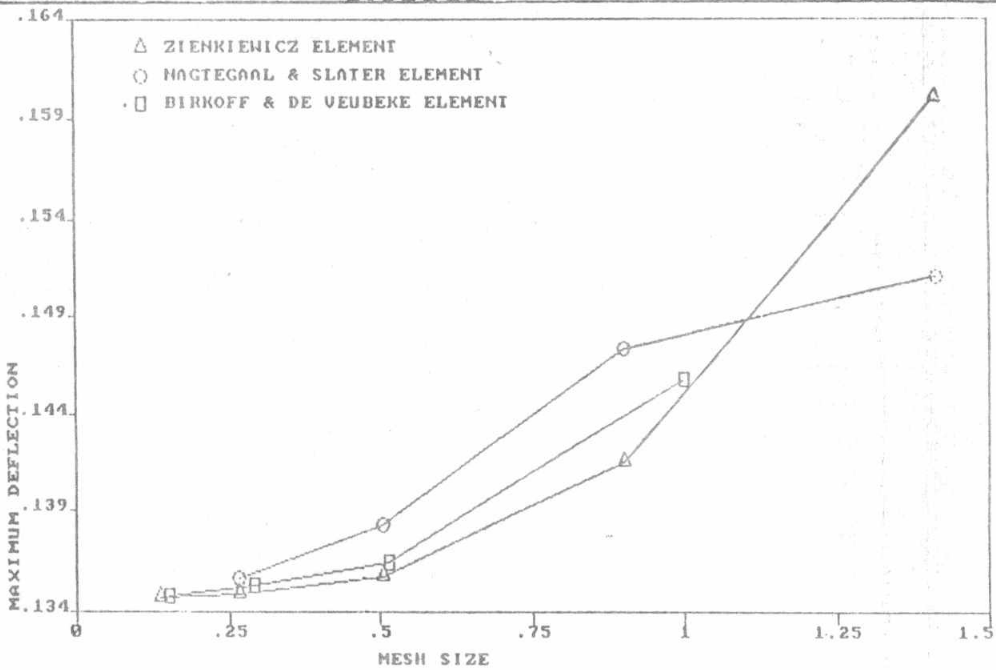


Fig.8.Maximum deflection of parabolic plate

CONCLUSION

It has been shown that the imposition of only w = 0 along the simply supported curved boundary is quiet sufficient to get very good results in case of thin plates. From engineering point of view we can, by this way, avoid the wrong results predicted by Babuška paradox and a safe use of existing codes is expected.

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