ON THE CALCULATION OF LAMINAR BOUNDARY-LAYER FLOWS WITH MASS TRANSFER BY A SIMPLE INTEGRAL METHOD

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ABSTRACT

A new simple integral method has been developed and applied to boundary layer flow involving mass transfer. The original Karman-Pohlhausen was refined by effecting a second integration of the momentum equation in order to arrive at a basic differential equation for the determination of $\delta(x)$. A computer code (*SIM* code) has been developed and its application extended to a case of great interest; porous flat plate with similarity plowing using polynomial velocity profiles of constant- and variable-coefficients. Reasonable agreement with other existing results is obtained . This method is sufficiently simple to be of practical use.

NOMENCLATURE

C_b	blowing coefficient , v_v/U_o
C_i	coefficients of the polynomial velocity profiles
C_f	skin-friction coefficient , $\tau_v / \frac{1}{2}\rho U_o^2$
f	velocity profile, u/U_o
F_i	i=1,2 , see equations (9c) & (10c)
K_i	i=1,2 , see equations (9b) & (10b)
R_{ex}, R	Reynolds number based on x, $U_o x/\nu$
$R_{e\delta}$, Λ	Reynolds number based on x, $U_o \delta/\nu$
u, v	velocity components corresponding to (x,y)
v_v	blowing velocity at the wall
x, y	general orthogonal curvilinear coordinate
U_1, U_o	free stream velocity
β	blowing parameter
η	dimensionless coordinate, y/δ
δ	boundary-layer thickness
ν	kinematic viscosity
ρ	density of the fluid
τ_v	shear stress at the wall
ξ, ζ	blowing parameter for 2nd & 4th order polynomial profiles

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INTRODUCTION

The successful design of aerodynamic control surfaces relies heavily on knowledge of the attendant boundary-layer flows [1]. The ability of predicting and understanding the separation of such boundary layers is of vital importance in the design process, as boundary-layer separation is known to alter rather drastically the aerodynamic characteristics of the control surfaces. From practical point of view, therefore, the ability to effectively control the boundary layer is perhaps even more important [4].

A well-known method of boundary-layer control is the use of surface mass transfer. This method has other important applications in the modern technology of high-speed flight.

The primary purpose of this paper is to develop a new approximate method which can be used to study flows of this type. This method is sufficiently simple, accurate and reliable to be of practical use and warrant its development into a powerful and practical tool for tacking flows of complex nature.

This method is a new idea for improving the usual Karman - Pohlhausen integral method by effecting a second integration of the momentum equation in order to arrive at a basic differential equation for the determination of the boundary-layer thickness.

The new method is applied to boundary-layer flows with surface mass transfer. Some calculations were performed to obtain the friction on a porous plate with similarity using velocity profiles of polynomial form with either constant or variable coefficient.

Qualitatively correct results have been obtained and some favorable comparisons with other exiting results have been found.

GOVERNING EQUATIONS:

The general differential equations and boundary conditions describing plane, incompressible, laminar boundary-layer flows are as follows :

continuity : $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1a)$$

momentum : $$\frac{\partial}{\partial x} u^2 + \frac{\partial}{\partial y} uv = \nu \frac{\partial^2 u}{\partial y^2} + U_1 \frac{\partial U_1}{\partial x} \qquad (1b)$$

and
$$u(x,0) = 0$$
$$v(x,0) = v_w(x)$$
$$u(x,\delta) = U(x)$$

Here (x,y) forms an orthogonal curvilinear coordinate system with x measuring the distance along the body surface, and (u,v) is the corresponding velocity vector (fig. 1).

Because of the parabolic nature of the partial differential equation, an "*initial*" conditions on u is generally required to complete the formulation of the problem. *However in the present method, this initial condition is not essential and is generally replaced by a condition on the initial boundary-layer thickness, δ(0).*

integrating equation (1b) once from the wall, y=0, to some distance y and using equation (1a) we have :

$$\frac{\tau_w(x)}{\rho} = yU_1 \frac{dU_1}{dx} + \nu \frac{\partial u}{\partial y} + u \int_0^y \frac{\partial}{\partial x} u \, dy - uv_w - \int_0^y \frac{\partial}{\partial x} u^2 \, dy \qquad (2)$$

This equation, with the upper limit y replaced by δ(x) is the basic differential equation for determining δ(x) in Karaman method once a velocity profile u/U_1 is assumed.

The new approach is based on the refinement of this equation by effecting a second integration of the momentum equation (1b) in order to arrive at a basic differential equation for the determination of δ(x). Equation (2) is used as an integral representation of the skin-friction term. There for, equation (2) is integrated from the wall to δ(x) thus obtain

$$\frac{\tau_w(x)}{\rho} \delta = \frac{\delta^2}{2} U_1 \frac{dU_1}{dx} + \nu U_1 + \int_0^{\delta} u dy \frac{\partial}{\partial x} \int_0^{y} u \, dy - v_w \int_0^{\delta} u dy - \int_0^{\delta} dy \frac{\partial}{\partial x} \int_0^{y} u^2 dy \qquad (3)$$

From Equation (2) , with the upper limits of the integration replaced by $\delta(x)$, the skin-friction in equation (3) could be represented, under the condition that the shear stress vanishes at the outer edge and using Leibnitz rule;

$$\frac{\tau_v(x)}{\rho} = \delta U_1 \frac{dU_1}{dx} + U_1 \frac{d}{dx}\int_0^{\delta} u\,dy - U_1 v_v - \frac{d}{dx}\int_0^{\delta} u^2\,dy \qquad (4)$$

Combining equations (3) & (4) we have the basic equation for $\delta(x)$

$$\frac{1}{2}\delta U_1^2 \frac{dU_1}{dx} + U_1\delta \frac{d}{dx}\int_0^{\delta} u\,dy - U_1 v_v \delta - \delta \frac{d}{dx}\int_0^{\delta} u^2\,dy$$

$$= \nu U_1 + \int_0^{\delta} udy \frac{\partial}{\partial x}\int_0^{y} u\,dy - v_v \int_0^{\delta} udy - \int_0^{\delta} dy \frac{\partial}{\partial x}\int_0^{y} u^2 dy \qquad (5)$$

equation (5), is a frist-order nonlinear ordinary differential equation for $\delta(x)$ for any assumed velocity profile u/U_1.

Equation (4) & (5) form the basis of all our calculations in the selected cases to be investigated.

POROUS PLATE :

in a flow over flat plate, we have $U_1 = U_o$ = constant and zero pressure gradient, from which equation (4) and (5) become

$$\frac{1}{2} C_f = \frac{\tau_v(x)}{\rho U_o^2} = \frac{d}{dx}\int_0^{\delta} \frac{u}{U_o}\left(1 - \frac{u}{U_o}\right) dy - C_b \qquad (6)$$

and

$$\frac{d}{dx}\int_0^{\delta} \frac{u}{U_o}\,dy - C_b - \frac{d}{dx}\int^{\delta}\left(\frac{u}{U_o}\right)^2 dy = \frac{1}{R_{e\delta}} + \frac{1}{\delta}\int_0^{\delta} \frac{u}{U_o}\,dy \frac{\partial}{\partial x}\int_0^{y} \frac{u}{U_o}\,dy$$

$$-\frac{C_b}{\delta}\int_0^{\delta}\frac{u}{U_o}\,dy-\frac{1}{\delta}\int_0^{\delta}dy\quad\frac{\partial}{\partial x}\int_0^{y}\left(\frac{u}{U_o}\right)^2 dy \qquad (7)$$

VELOCITY PROFILE:

Assume the velocity profile to be

$$u/U_o = \sum_i C_i(x)\,\eta^i \qquad (8)$$

i) 4th order polynomial :

under the following boundary conditions :

$$u(x,0) = 0\ ,\ u(x,\delta) = U_o\ ,\ u_y(x,\delta) = 0\ \&\ u_{yy}(x,\delta) = 0$$

the profile is ;

$$\frac{u}{U_o} = f = \frac{1}{1+\zeta}\left[2\eta + 6\zeta\eta^2 + 2(1+4\zeta)\eta^3 + (1+3\zeta)\eta^4\right] \qquad (9a)$$

Where $\zeta = \dfrac{C_b R_e \delta}{6}$

$$K_1 = \delta_1/\delta = 3+4\zeta/10(1+\zeta) \qquad (9b)$$

$$F_1 = \delta_2/\delta = (0.117+0.247\zeta+0.114\zeta^2)/(1+\zeta)^2 \qquad (9c)$$

ii) 2nd order polynomial :

under the following boundary conditions :

$$u(x,0) = 0\ ,\ u(x,\delta) = U_o$$

the profile is

$$\frac{u}{U_o} = f = \frac{1}{1+\xi}\left[\eta + \xi\eta\right] \qquad (10a)$$

Where $\xi = \dfrac{C_b R_{e\delta}}{2}$

$$K_2 = \delta_1/\delta = 3+4\zeta/6(1+\zeta) \qquad (10b)$$

$$F_2 = \delta_2/\delta = (0.166+0.333\xi+0.133\xi^2)/(1+\xi)^2 \qquad (10c)$$

For ζ and ξ equal zero, equations (9a) & (10a) are reduced to the constant coefficient 4th degree and linear profiles.

DERIVATION OF BASIC EQUATIONS:

Substituting the profile, equation (8), into equation (7) and introduce Reynolds number R_{ex} (or R) we obtain after some algebra

$$\frac{1}{2} C_f R^{1/2} = R^{1/2} \frac{d\delta_2}{dx} - c_b R^{1/2} \qquad (11)$$

$$\int_0^1 d\eta \frac{\partial}{\partial x} \int_0^\eta \Lambda f^2 d\eta - \int_0^1 f d\eta \frac{\partial}{\partial x} \int_0^\eta \Lambda f \, d\eta = \frac{1}{\Lambda} - \frac{d}{dR} (\Lambda F_i + C_b K_i) \qquad (12)$$

SIMILARITY BLOWING:

In this case we know that $C_b = \beta R^{-1/2}$ and $\Lambda \propto R^{-1/2}$ [4] therefore ζ or ξ becomes constant. This constant takes on a fixed value only for a fixed value of β ,i.e. , it not a universal constant.

i) 4th degree polynomial :

Subtitling equation (9a) into equation (12) we obtain

$$\frac{d\Lambda^2}{dR} = \{\frac{1}{2} \frac{1}{(1+\zeta)^2} [.058+.1387\zeta + .0676 \zeta^2]\} / [1 + .6\zeta \frac{3+4\zeta}{1+\zeta}]$$

Assuming $R_{e\delta}$ 0) = Λ(0) =0 , hence the solution is

$$\frac{\Lambda^2}{R} = 2 (1+\zeta)^2 [1 + .6 \zeta \frac{3+4\zeta}{1+\zeta}] / [.058+.1387\zeta + .0676 \zeta^2] \qquad (13)$$

Assuming $R_{e\delta}(0) = \Lambda(0) = 0$, hence the solution is

$$\frac{\Lambda^2}{R} = 2\,(1+\zeta)^2\,[1 + .6\,\zeta\,\frac{3+4\zeta}{1+\zeta}\,] \,/\, [.058+.1387\zeta + .0676\,\zeta^2\,] \qquad (13)$$

from which $\delta = x(\beta)^{1/2} / R_{ex}^{1/2}$

The conventional blowing parametter , β is then found

$$\beta = 6\zeta\left(\frac{\Lambda^2}{R}\right)^{-1/2} \qquad (14)$$

Finally, the skin friction is obtained by combining equations (9c), (11) and (13)

$$\frac{1}{2}\,C_f\,R^{1/2} = F_1\left(\frac{\Lambda^2}{R}\right)^{1/2} - \beta \qquad (15)$$

Results of the calculation are given in figure (2).

i) 2nd degree polynomial :

Exactly the same procedure can be followed to give the results of this case

$$\frac{\Lambda^2}{R} = 2(1+\zeta)\,[1 + 2\zeta + .75\zeta^2\,] \,/\, [.125+.266\,\zeta + .116\,\zeta^2\,] \qquad (16)$$

$$\beta = 6\zeta\left(\frac{\Lambda^2}{R}\right)^{-1/2} \qquad (14)$$

$$\frac{1}{2}\,C_f\,R^{1/2} = F_1\left(\frac{\Lambda^2}{R}\right)^{1/2} - \beta \qquad (15)$$

Results of the calculation are given in figure (3).

DISCUSSION AND CONCLUSIONS

It is clear in figure (6) , that the results from the new method using variable-coefficient polynomial profiles agree very closely with those using constant-coefficient polynomials up to fairly strong blowing, $\beta \cong 0.4$. .This is applies to both the 4th and the 2nd degree (linear profile in the case of constant-coefficient polynomial) polynomial. The most important advantage of the new method is that the results are insensitive to the choice of the profiles vise versa to the results obtained by the Karman-Pohlhausen method , Figure (5), which are highly sensitive to the choice of the velocity profile.This sensitivity suggests the lake of reliability of the Karaman method, and the large error in the prediction of skin friction obviously reflects the inadequacy of the method compared with the new method as shown in figures (2) and (4),

At a higher blowing rates, results from the variable -coefficient profile and those from the constant-coefficient profiles begin to show noticeable difference ,Fig. (6). The variable coefficient profiles seem to offer better prediction for blow-off (β_c = 0.625 for the 2nd degree polynomial and β_c = 0.532 for the 4th degree polynomial). The results are compared to the exact results [2] (of β_c = 0.619) and it show distinctly the agreement with it as illustrated in fig. (3). The results obtained are believed to be sufficiently accurate for engineering purposes, in light of its insensitivity to the choice of the velocity profile.

The new method and its application to cases involving both pressure gradient and mass transfer is now carried out .

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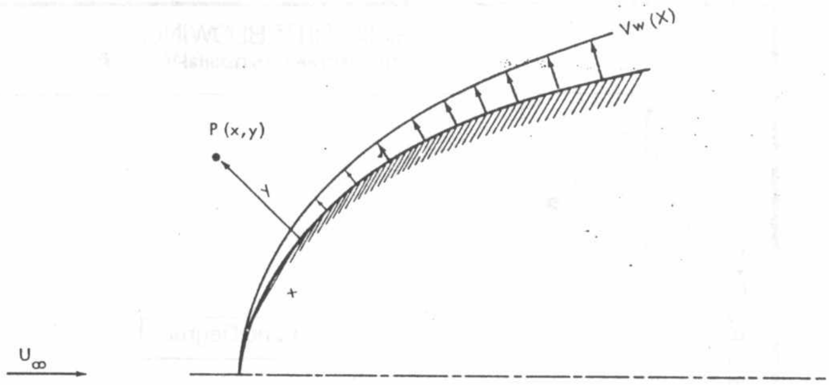

FIG.1 COORDINATE SYSTEM

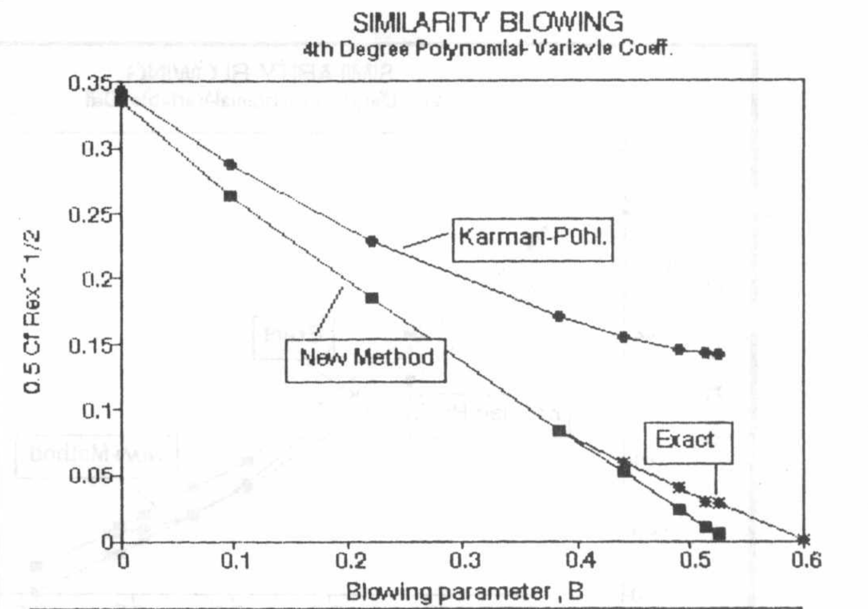

Fig.. (2) Skin Friction on a Porous Plate with similarity blowing

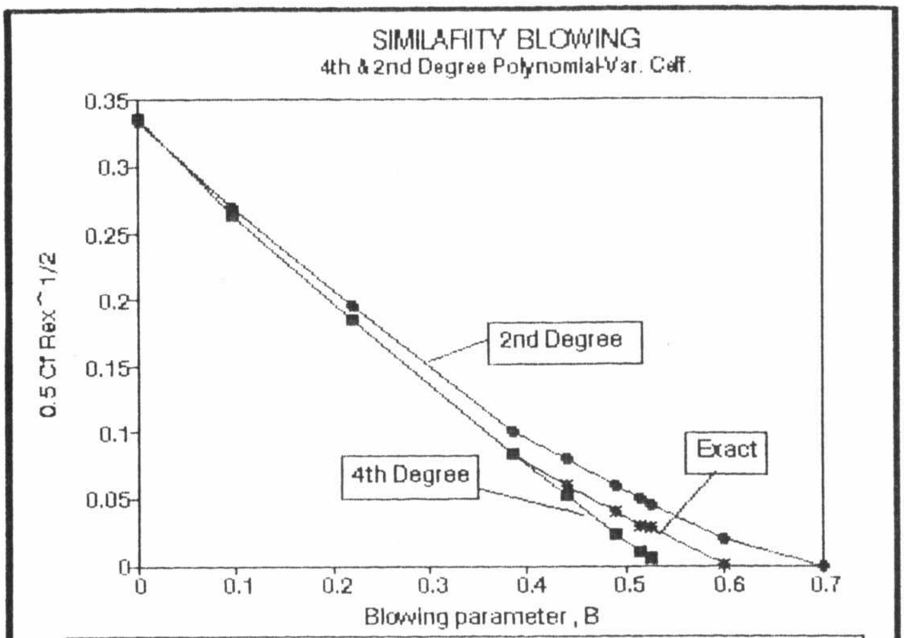

Fig. (3) Skin Friction on a Porous Plate , Comparison with Exact Solution

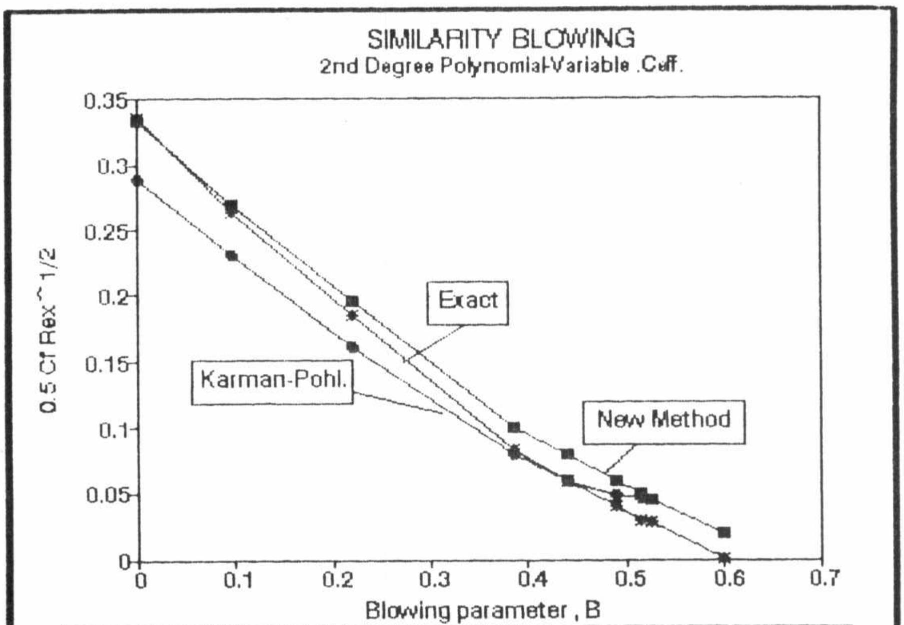

Fig. (4) Skin Friction for a Porous Plate based on 2nd Polynomial Profiles

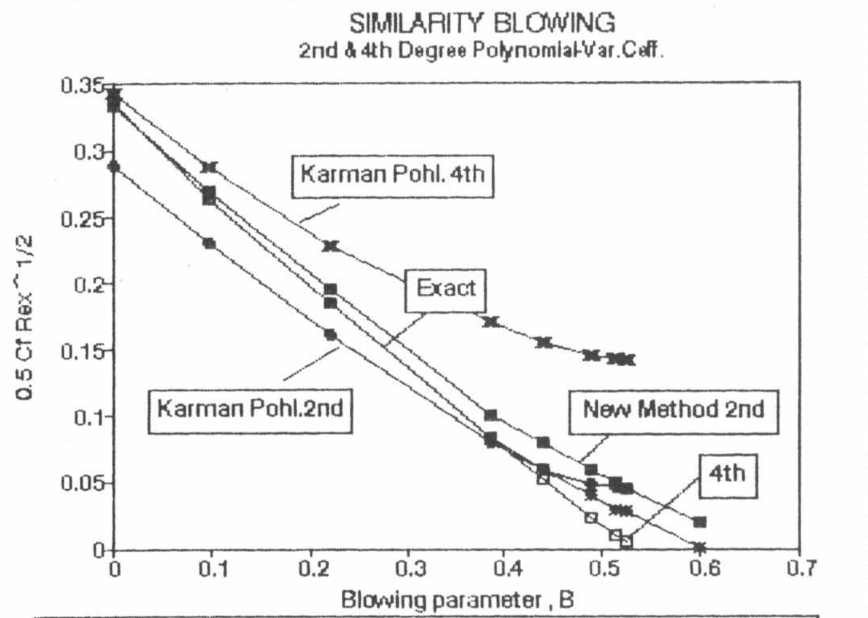

Fig. (5) A comparison Between the New Method and Karman Method

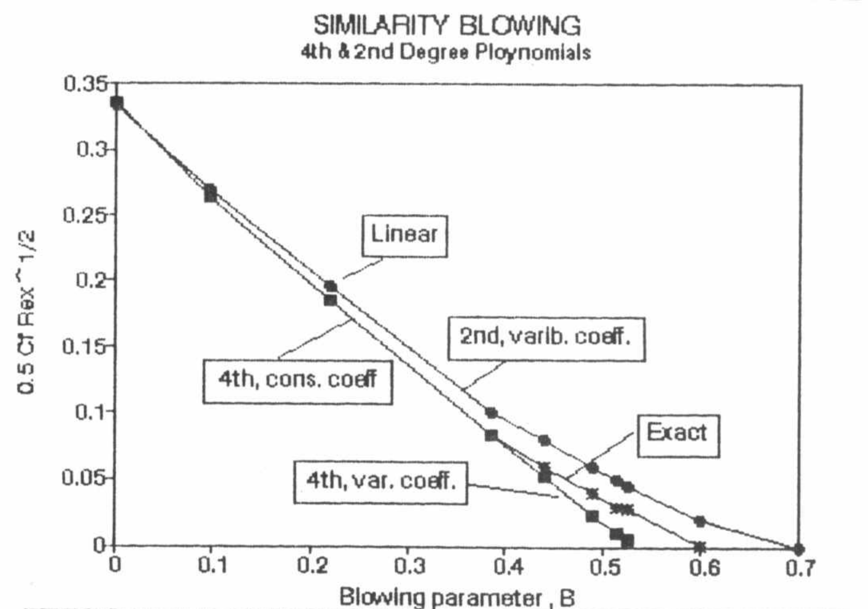

Fig. (6) A Comparison Between Variable- and Constant-Coefficient Plynomial