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LARGE DEFLECTION AND STABILITY ANALYSIS OF COMPOSITE LAYERED PLATE USING A NEW HIGH-ORDER ELEMENT

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ABSTRACT

This paper presents a finite element analysis of the geometrically nonlinear and stability behaviours of composite layered plates using a high-order facetted shell element. Element equations are derived based on Reissner theory and, the lateral deflection is modelled using Conforming or Non-conforming Hermition shape functions. Green's strain equations are used to represent nonlinear terms associated with geometrical nonlinearity. To avoid the unsymmetry problem of the stiffness matrix due to large deformation effect, the part of the stiffness equation dealing with the large deformation effect is represented separately as a force vector. A programming package based on the developed element was designed. Several case studies have been investigated and package results were compared with published theoretical and/or experimental results. The effect of some effective parameters on the behaviour of the composite plates has been investigated with some case studies, and the results have proved that the developed package can be useful tool for the optimisation of composite layered plate structure.

1. INTRODUCTION

The technology of composite materials has experienced a rapid development. The main reason for this development is requirements for high performance materials, especially in military applications, aircraft's and aerospace structures. Plates are the most commonly structural forms used in these applications. The transverse shear strains in composite layered plates are very effective parameters to estimate an accurate deflection, stresses, natural frequencies and critical buckling loads. Geometrical nonlinearity and stability analyses need an understanding of the behaviour of structures under large deformation and inplane load to select a proper size of the structures to achieve an economical design.

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There are many finite element publications for geometrically nonlinear and stability analysis of plates in the literature. Geometrically nonlinear analysis of plates was presented by Pica et. al. [1], who used a Mindlin formulation with the assumption of small rotations.

Reddy [2,3] introduced a refined nonlinear theory of plates with transverse shear deformation, and a simple higher order theory for laminated composite plates. Tessler [4] derived a two dimensional laminated theory for linear elastic analysis of thick composite plates with the equivalent single-layer assumptions for the displacements, transverse shear strain, and transverse normal stress. By the same assumptions of Mindlin theory, Balamurugan et. al. [5] ivestigated the dynamic instability of anisotropic laminated composite plates considering geometric nonlinearity. Attia and El-Zafrany [6] introduced a family of high-order facetted shell elements for linear and nonlinear stress and vibration analyses of composite layered plate and shell structures. Nonlinear terms associated with geometrical nonlinearity are also derived using a practical approach based upon the actual components of strain.

The stability analysis of plates has a long history and there are many publications in the literature. Hiroyuki [7] analyzed the natural frequencies and buckling stress of cress-ply laminated composite plates by taking into consideration the effects of shear deformation, thickness change and rotary inertia.

Many of the element derivations presented in the literature are based on:

- Hypothetical nodal parameters, which are difficult to handle with different types of boundary conditions.
-
  represented the effect of the geometrical nonlinearity.
- Most of the derivation are based on Mindlin or Kirchhoff assumptions.

This paper presents a finite element analysis of the geometrically nonlinear and stability behaviours of composite layered plates using a high-order facetted shell element. Element equations are derived based on Reissner theory and, the lateral deflection is modelled using Conforming or Non-
strain equations are used to represent nonlinear terms associated with geometrical nonlinearity. To avoid the unsymmetry problem of the stiffness matrix due to large deformation effect, the part of the stiffness equation dealing with the large deformation effect is represented separately as a force vector. A programming package based on the developed element was designed. Several case studies have been investigated and package results were compared with published theoretical and/or experimental results. The effect of some effective parameters on the behaviour of the composite plates has been investigated with some case studies, and the results have proved that the developed package can be useful tool for the optimisation of composite layered plate structure.

2. DISPLACEMENT FORMULATION

A composite layered plate is consisting of a number of orthotropic layers, and usually defined in terms of a midplane and thickness distribution. The thickness (h) is much smaller than other dimensions and is measured in the direction (z) normal to midplane. It may be classified into thin and thick according to the value of the ratio of the thickness to span length.

2.1. Transverse Strains

The boundary conditions around the total thickness can be summarised as; there is no transverse shear strain at lower and upper surfaces. If the transverse shear strains are approximated as quadratic function in z, then by using a three point Lagrangian Interpolation, it can be proved that:

$$\underline{\gamma} = \frac{5}{4}\left(1 - \frac{4z^2}{h^2}\right)\bar{\underline{\gamma}}$$

where $\bar{\underline{\gamma}} = \{\bar{\gamma}_{xz}\ \bar{\gamma}_{yz}\}^T$ is the average transverse shear strains and $\underline{\gamma} = \{\gamma_{xz}\ \gamma_{yz}\}^T$ is the shear strain components.

2.2 Displacement Components

The displacement components at a general point may be resolved into u, v (in-plane displacements) and w (out-of plane displacement). From the transverse shear strains equations, the displacement components can be expressed as,

$$u = u_0 + z\,\theta_y + f(z)\,\psi_y$$
$$v = v_0 - z\,\theta_x - f(z)\,\psi_x$$
$$w = w_0$$

where

u_0, v_0, w_0 are the displacement components of the midplane along the x, y, and z respectively

θ_x , θ_y represent average rotation angles in the x and y directions, respectively

ψ_x, ψ_y additional rotation in y and x direction equivalent to $-\bar{\gamma}_{yz}$, $\bar{\gamma}_{xz}$, respectively

and $f(z) = \frac{1}{4}\left(Z - \frac{20\,z^3}{3\,h^2}\right)$

3. NODAL PARAMETERS AND INTERPOLATED DISPLACEMENT COMPONENTS

For an n-node element, the degrees of freedom at any point on the midplane are defined in terms of in-plane displacements, out-of plane displacements, and transverse shear strain. The nodal displacement vector of the element (at time t) will be defined as follows:

$$\underline{\delta} = \{\underline{\delta}_0\ \ \underline{\delta}_b\ \ \underline{\delta}_t\}$$

where

$$\underline{\delta}_0 = \{(u_0)_1 \quad (v_0)_1 \quad (u_0)_2 \quad (v_0)_2 \quad \dots \quad (u_0)_n \quad (v_0)_n\}$$

$$\underline{\delta}_b = \{(w_0)_1 \quad (\theta_{x0})_1 \quad (\theta_{y0})_2 \quad (\theta_{z0})_2 \quad \dots \quad (w_0)_n \quad (\theta_{x0})_n \quad (\theta_{y0})_n \quad (\theta_{z0})_n\}$$

$$\underline{\delta}_0 = \{(\psi_x)_1 \quad (\psi_y)_1 \quad (\psi_x)_2 \quad (\psi_y)_2 \quad \dots \quad (\psi_x)_n \quad (\psi_y)_n\}$$

For an n-node element, Lagrangian interpolation is used with in-plane and transverse shear components,

$$u_0(x,y) = \sum_{i=1}^{n} N_i(x,y)\ u_i \qquad v_0(x,y) = \sum_{i=1}^{n} N_i(x,y)\ v_i$$

$$\psi_x(x,y) = \sum_{i=1}^{n} N_i(x,y)\ (\psi_x)_i \qquad \psi_y(x,y) = \sum_{i=1}^{n} N_i(x,y)\ (\psi_y)_i$$

The out-of plane components are expressed in terms of w (x, y) which is interpolated by means of Hermitian interpolation. Two types of interpolation, non-conforming and conforrning, based upon El-Zafarany and Cookson [8, 9] are emplyed where w(x,y) is interpolated as follows:

$$w(x,y) = \sum_{i=1}^{n}\left[F_i(\xi,\eta)\, w_i + H_i(\xi,\eta)\,(\theta_x)_i - G_i(\xi,\eta)\,(\theta_y)_i + P_i(\xi,\eta)\,(\theta_z)_i\right] + \left[-H_i(\xi,\eta)\,(\psi_x)_i + G_i(\xi,\eta)\,(\psi_y)_i\right]$$

where N_i, F_i, G_i, H_i, and P_i represent Lagrangian and Hermitian shape function of node i. Notice aslo that P_i dose not exist for the non-conforming Hermitian interpolation.

4. STRAIN-DISPLACEMENT RELATIONS

For simplification of derivations, the strains are defined in terms of two separate vectors:
(i) x-y components vector $\underline{\varepsilon} = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\}^T$
(ii) Transverse shear vector $\underline{\gamma} = \{\gamma_{xz} \quad \gamma_{yz}\}^T$

ations the vector of x-y strain components can be partitioned as follows:

$$\underline{\varepsilon} = \underline{\varepsilon}_S + \underline{\varepsilon}_L$$

where the subscript s represents small strains and L represents large strains.

$$\underline{\varepsilon}_S = \underline{\varepsilon}_0 - z\ \underline{\varepsilon}_b + g(z)\ \underline{\varepsilon}_t$$

$$\underline{\varepsilon}_L = \underline{\varepsilon}_m + \underline{\varepsilon}_w + z^2\ \underline{\varepsilon}_\theta + g^2(z)\ \underline{\varepsilon}_\psi - z\ \underline{\varepsilon}_{m\theta} + g(z)\ \underline{\varepsilon}_{m\psi} + z\ g(z)\ \underline{\varepsilon}_{\theta\psi}$$

where $g(z) = z + f(z)$

The infinitesimal strain components can be defined in terms of nodal displacements and strain shape function matrices $\underline{B}$ as follows:

$$\hat{\underline{\gamma}} = \underline{B}_\gamma(x,y)\ \underline{\delta}_t \qquad \underline{\varepsilon}_0 = \underline{B}_0(x,y)\ \underline{\delta}_0$$

$$\underline{\varepsilon}_b = \underline{B}_b(x,y)\ \underline{\delta}_b + \underline{B}_{bt}(x,y)\ \underline{\delta}_t \qquad \underline{\varepsilon}_t = \underline{B}_t(x,y)\ \underline{\delta}_t$$

where $$\underline{B}_\gamma(x,y)=\begin{bmatrix}0 & -N_1 & \dots & \dots\\ N_1 & 0 & \dots & \dots\end{bmatrix},\quad \underline{B}_0(x,y)=\begin{bmatrix}\frac{\partial N_1}{\partial x} & 0 & \dots & \dots\\ 0 & \frac{\partial N_1}{\partial y} & \dots & \dots\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \dots & \dots\end{bmatrix}$$

$$\underline{B}_b(x,y)=\begin{bmatrix}\frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 H_1}{\partial x^2} & -\frac{\partial^2 G_1}{\partial x^2} & \frac{\partial^2 P_1}{\partial x^2} & \dots & \dots\\ \frac{\partial^2 F_1}{\partial y^2} & \frac{\partial^2 H_1}{\partial y^2} & -\frac{\partial^2 G_1}{\partial y^2} & \frac{\partial^2 P_1}{\partial y^2} & \dots & \dots\\ 2\frac{\partial^2 F_1}{\partial x\partial y} & 2\frac{\partial^2 H_1}{\partial x\partial y} & -2\frac{\partial^2 G_1}{\partial x\partial y} & 2\frac{\partial^2 P_1}{\partial x\partial y} & \dots & \dots\end{bmatrix}$$

$$\underline{B}_{bt}(x,y)=\begin{bmatrix}-\frac{\partial^2 H_1}{\partial x^2} & \frac{\partial^2 G_1}{\partial x^2} & \dots & \dots\\ -\frac{\partial^2 H_1}{\partial y^2} & \frac{\partial^2 G_1}{\partial y^2} & \dots & \dots\\ -2\frac{\partial^2 H_1}{\partial x\partial y} & 2\frac{\partial^2 G_1}{\partial x\partial y} & \dots & \dots\end{bmatrix},\quad \underline{B}_t=\begin{bmatrix}0 & \frac{\partial N_1}{\partial x} & \dots & \dots\\ -\frac{\partial N_1}{\partial y} & 0 & \dots & \dots\\ -\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \dots & \dots\end{bmatrix}$$

The finite strain components can also be expressed in terms of rotation vectors $\underline{\theta}$ and $\underline{A}$ as follows:

$$\underline{\varepsilon}_m=\frac{1}{2}\underline{A}_m\,\underline{\theta}_m=\frac{1}{2}\underline{A}_m\,\underline{G}_m\,\underline{\delta}_0,\qquad \underline{\varepsilon}_w=\frac{1}{2}\underline{A}_w\,\underline{\theta}_w=\frac{1}{2}(\underline{A}_w\,\underline{G}_w\,\underline{\delta}_b+\underline{A}_w\,\underline{G}_{wt}\,\underline{\delta}_t)$$

$$\underline{\varepsilon}_\theta=\frac{1}{2}\underline{A}_\theta\,\underline{\theta}_\theta=\frac{1}{2}(\underline{A}_\theta\,\underline{G}_\theta\,\underline{\delta}_b+\underline{A}_\theta\,\underline{G}_{\theta t}\,\underline{\delta}_t),\qquad \underline{\varepsilon}_\psi=\frac{1}{2}\underline{A}_\psi\,\underline{\theta}_\psi=\frac{1}{2}\underline{A}_\psi\,\underline{G}_\psi\,\underline{\delta}_t$$

$$\underline{\varepsilon}_{m\theta}=\underline{A}_\theta\,\underline{\theta}_m=\underline{A}_\theta\,\underline{G}_m\,\underline{\delta}_0=\underline{A}_m\,\underline{\theta}_\theta=\underline{A}_m\,\underline{G}_\theta\,\underline{\delta}_b+\underline{A}_m\,\underline{G}_{\theta t}\,\underline{\delta}_t,\qquad \underline{\varepsilon}_{m\psi}=\underline{A}_\psi\,\underline{\theta}_m=\underline{A}_\psi\,\underline{G}_m\,\underline{\delta}_0=\underline{A}_m\,\underline{\theta}_\psi=\underline{A}_m\,\underline{G}_\psi\,\underline{\delta}_t$$

$$\underline{\varepsilon}_{\theta\psi}=-\underline{A}_\theta\,\underline{\theta}_\psi=-\underline{A}_\theta\,\underline{G}_\psi\,\underline{\delta}_t=-\underline{A}_\psi\,\underline{\theta}_\theta=-\underline{A}_\psi\,\underline{G}_\theta\,\underline{\delta}_b-\underline{A}_\psi\,\underline{G}_{\theta t}\,\underline{\delta}_t$$

where $$\underline{\theta}_m=\left\{\frac{\partial u_0}{\partial x}\ \ \frac{\partial v_0}{\partial x}\ \ \frac{\partial u_0}{\partial y}\ \ \frac{\partial v_0}{\partial y}\right\},\quad \underline{\theta}_\psi=\left\{\frac{\partial \psi_y}{\partial x}\ \ -\frac{\partial \psi_x}{\partial x}\ \ \frac{\partial \psi_y}{\partial y}\ \ -\frac{\partial \psi_x}{\partial y}\right\}$$

$$\underline{\theta}_\theta=\left\{\frac{\partial^2 w}{\partial x^2}\ \ \frac{\partial^2 w}{\partial x\partial y}\ \ \frac{\partial^2 w}{\partial x\partial y}\ \ \frac{\partial^2 w}{\partial^2 y}\right\},\quad \underline{\theta}_w=\left\{\frac{\partial w}{\partial x}\ \ \frac{\partial w}{\partial y}\right\}$$

$$\underline{A}_m = \begin{bmatrix} \frac{\partial u_0}{\partial x} & \frac{\partial v_0}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial u_0}{\partial y} & \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} & \frac{\partial v_0}{\partial y} & \frac{\partial u_0}{\partial x} & \frac{\partial v_0}{\partial x} \end{bmatrix} , \quad \underline{A}_\theta = \begin{bmatrix} \frac{\partial w^2}{\partial x^2} & \frac{\partial w^2}{\partial x \partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial w^2}{\partial x \partial y} & \frac{\partial w^2}{\partial y^2} \\ \frac{\partial w^2}{\partial x \partial y} & \frac{\partial w^2}{\partial y^2} & \frac{\partial w^2}{\partial x^2} & \frac{\partial w^2}{\partial x \partial y} \end{bmatrix}$$

$$\underline{A}_\psi = \begin{bmatrix} \frac{\partial \psi_y}{\partial x} & -\frac{\partial \psi_x}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial \psi_y}{\partial y} & -\frac{\partial \psi_x}{\partial y} \\ \frac{\partial \psi_y}{\partial y} & -\frac{\partial \psi_x}{\partial y} & \frac{\partial \psi_y}{\partial x} & -\frac{\partial \psi_x}{\partial x} \end{bmatrix} , \quad \underline{A}_w = \begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \end{bmatrix}$$

Due to an interesting property of the rotation vectors $\underline{\theta}$ and matrices $\underline{A}$, it is easy to verify that

$$\underline{A}^T \underline{\sigma} = \underline{S}\,\underline{\theta}$$

where $\underline{S}$ is a matrix contains the value of the stress and is defined as follows:

$$\underline{S} = \begin{bmatrix} \sigma_x & 0 & \tau_{xy} & 0 \\ 0 & \sigma_x & 0 & \tau_{xy} \\ \tau_{xy} & 0 & \sigma_y & 0 \\ 0 & \tau_{xy} & 0 & \sigma_y \end{bmatrix} \qquad \underline{S}_w = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

where the integrated stress can be defined as

$$\left[\underline{S}_{mm}, \underline{S}_{\theta\theta}, \underline{S}_{\psi\psi}, \underline{S}_{m\theta}, \underline{S}_{m\psi}, \underline{S}_{\theta\psi}\right] = \sum_{m=1}^{N} \int_{Z_L(m)}^{Z_U(m)} \left[1, z^2, g^2, z, g, zg\right] \underline{\sigma}^{(m)}\, dz$$

The rotation vectors $\underline{\theta}$ can be related to the nodal parameters as

$$\underline{\theta}_m = \underline{G}_m \underline{\delta}_0 \qquad \underline{\theta}_w = \underline{G}_w \underline{\delta}_b + \underline{G}_{wt} \underline{\delta}_t$$

$$\underline{\theta}_\psi = \underline{G}_\psi \underline{\delta}_t \qquad \underline{\theta}_\theta = \underline{G}_\theta \underline{\delta}_b + \underline{G}_{\theta t} \underline{\delta}_t$$

where
$$\underline{G}_m = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \cdots & \cdots \\ 0 & \frac{\partial N_1}{\partial x} & \cdots & \cdots \\ \frac{\partial N_1}{\partial y} & 0 & \cdots & \cdots \\ 0 & \frac{\partial N_1}{\partial y} & \cdots & \cdots \end{bmatrix} , \qquad \underline{G}_\psi = \begin{bmatrix} 0 & \frac{\partial N_1}{\partial x} & \cdots & \cdots \\ -\frac{\partial N_1}{\partial x} & 0 & \cdots & \cdots \\ 0 & \frac{\partial N_1}{\partial y} & \cdots & \cdots \\ -\frac{\partial N_1}{\partial y} & 0 & \cdots & \cdots \end{bmatrix}$$

$$\underline{G}_W = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial H_1}{\partial x} & -\frac{\partial G_1}{\partial x} & \frac{\partial P_1}{\partial x} & \cdots & \cdots \\ \frac{\partial F_1}{\partial y} & \frac{\partial H_1}{\partial y} & -\frac{\partial G_1}{\partial y} & \frac{\partial P_1}{\partial y} & \cdots & \cdots \end{bmatrix} , \quad \underline{G}_{Wt} = \begin{bmatrix} -\frac{\partial H_1}{\partial x} & \frac{\partial G_1}{\partial x} & \cdots & \cdots \\ -\frac{\partial H_1}{\partial y} & \frac{\partial G_1}{\partial y} & \cdots & \cdots \end{bmatrix}$$

$$\underline{G}_\theta = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x^2} & -\frac{\partial^2 G_1}{\partial x^2} & \frac{\partial^2 P_1}{\partial x^2} & \cdots & \cdots \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial x \partial y} & -\frac{\partial^2 G_1}{\partial x \partial y} & \frac{\partial^2 P_1}{\partial x \partial y} & \cdots & \cdots \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial x \partial y} & -\frac{\partial^2 G_1}{\partial x \partial y} & \frac{\partial^2 P_1}{\partial x \partial y} & \cdots & \cdots \\ \frac{\partial^2 F_1}{\partial y^2} & \frac{\partial^2 F_1}{\partial y^2} & -\frac{\partial^2 G_1}{\partial y^2} & \frac{\partial^2 P_1}{\partial y^2} & \cdots & \cdots \end{bmatrix} , \quad \underline{G}_{\theta t} = \begin{bmatrix} -\frac{\partial^2 H_1}{\partial x^2} & \frac{\partial^2 G_1}{\partial x^2} & \cdots & \cdots \\ -\frac{\partial^2 H_1}{\partial x \partial y} & \frac{\partial^2 G_1}{\partial x \partial y} & \cdots & \cdots \\ -\frac{\partial^2 H_1}{\partial x \partial y} & \frac{\partial^2 G_1}{\partial x \partial y} & \cdots & \cdots \\ -\frac{\partial^2 H_1}{\partial y^2} & \frac{\partial^2 G_1}{\partial y^2} & \cdots & \cdots \end{bmatrix}$$

5. STRAIN ENERGY EQUATIONS

Due to a differential displacement field, the corresponding change of strain energy density is,

$$d\overline{U} = d\underline{\varepsilon}^T \underline{\sigma} + d\underline{\gamma}^T \underline{\tau}$$

Hence, the strain energy density can be written as follows

$$d\overline{U} = d\underline{\varepsilon}_S{}^T \underline{\sigma}_S + d\underline{\varepsilon}_S{}^T \underline{\sigma}_L + d\underline{\varepsilon}_L{}^T \underline{\sigma} + d\underline{\gamma}^T \underline{\tau}$$

which can be rewritten as

$$d\overline{U} = d\overline{U}_S + d\overline{U}_L + d\overline{U}_{SL}$$

where $d\overline{U}_S$ contains infinitesimal strain effects only, and $d\overline{U}_L, d\overline{U}_{SL}$ contain finite strain effects.

The strain energy per unit area is defined as,

$$U' = \int_{-h/2}^{h/2} \overline{U}\, dz \qquad \text{i.e. } \delta U' = \int_{-h/2}^{h/2} \delta\overline{U}\, dz$$

The stress strain relations for a composite laminate plate can be written as,

$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon} \quad , \underline{\tau} = \underline{\mu}\,\underline{\gamma}$$

where $\underline{D}$ and $\underline{\mu}$ are the material stress-strain matrices in the laminate coordinate system. Generally, the material constant matrices for a muli layer composite are defined as,

For Transverse shear strain, $\underline{\mu}$ is defined by $\underline{\mu}_{\gamma\gamma} = \sum_{m=1}^{N} \int_{Z_L(m)}^{Z_U(m)} f(z)\, \underline{\mu}^{(m)}\, dz$

For x-y plane infinitesimal strain, the integrated $\underline{D}$ matrices are defined by

$$\left[\underline{D}_{oo},\underline{D}_{bb},\underline{D}_{tt},\underline{D}_{ob},\underline{D}_{ot},\underline{D}_{bt}\right]=\sum_{m=1}^{N}\int_{Z_L(m)}^{Z_U(m)}\left[1,z^2,g^2,z,g,zg\right]\underline{D}^{(m)}\ dz$$

For x-y plane finite strain, the integrated $\underline{D}$ matrices are defined by

$$\left[\underline{D}_{\theta b},\underline{D}_{\theta t},\underline{D}_{\theta\theta\psi},\underline{D}_{\theta\psi},\underline{D}_{\theta\theta},\underline{D}_{\psi b},\underline{D}_{\psi\theta\psi},\underline{D}_{\psi t},\underline{D}_{\psi\psi}\right]=\sum_{m=1}^{N}\int_{Z_L(m)}^{Z_U(m)}\left[z^3,z^2g,z^3g,z^2g^2,z^4,zg^2,zg^3,g^3,g^4\right]\underline{D}^{(m)}\ dz$$

where m is the layer number and N is the number of layers

6. INFINITESIMAL STIFFNESS MATRIX $\underline{K}$

By integrating the infinitesimal strain energy $d\overline{U}_S$ over the midplane, it is clear that,

$$dU_S = d\underline{\delta}^T\ \underline{K}\ \underline{\delta}$$

where $\underline{K}$ represents the infinitesimal stiffness matrix, which is defined as follows:

$$\underline{K}=\left[\begin{array}{c|c|c} \underline{K}_{o,o} & -\underline{K}_{o,b} & \underline{K}_{o,t}-\underline{K}_{o,bt} \\ \hline -\underline{K}_{b,o} & \underline{K}_{b,b} & \underline{K}_{b,bt}-\underline{K}_{b,t} \\ \hline \underline{K}_{t,o}-\underline{K}_{bt,o} & \underline{K}_{bt,b}-\underline{K}_{t,b} & \underline{K}_{\gamma,\gamma}+\underline{K}_{bt,bt}+\underline{K}_{t,t}-\underline{K}_{bt,t}+\underline{K}_{t,bt} \end{array}\right]$$

where $\underline{K}_{\theta_1\theta_2,\theta_3\theta_4} = \iint_{element}\underline{B}^T_{\theta_1\theta_2}\ \underline{D}_{\theta_1\theta_3}\ \underline{B}_{\theta_3\theta_4}\ dx\ dy$ $(\theta_1,\theta_2,\theta_3,\text{and}\ \theta_4$ equivalent to $o, b, t, \text{and}\ \gamma)$

7. STRESS STIFFNESS MATRIX $\underline{K}^{\sigma}$

By integrating the infinitesimal strain energy $d\overline{U}_L$ over the midplane, it is clear that,

$$dU_L = d\underline{\delta}^T\ \underline{K}^{\sigma}\ \underline{\delta}$$

where $\underline{K}^{\sigma}$ represents the stress stiffness matrix, which is defined as follows:

$$\underline{K}^{\sigma}=\left[\begin{array}{c|c|c} \underline{K}^{\sigma}_{m,m} & \underline{K}^{\sigma}_{m,\theta} & \underline{K}^{\sigma}_{m,\psi}+\underline{K}^{\sigma}_{m,\theta t} \\ \hline \underline{K}^{\sigma}_{\theta,m} & \underline{K}^{\sigma}_{w,w}+\underline{K}^{\sigma}_{\theta,\theta} & \underline{K}^{\sigma}_{w,wt}+\underline{K}^{\sigma}_{\theta,\theta t}-\underline{K}^{\sigma}_{\theta,\psi} \\ \hline \underline{K}^{\sigma}_{\psi,m}+\underline{K}^{\sigma}_{\theta t,m} & \underline{K}^{\sigma}_{wt,w}+\underline{K}^{\sigma}_{\theta t,\theta}-\underline{K}^{\sigma}_{\psi,\theta} & \underline{K}^{\sigma}_{wt,wt}+\underline{K}^{\sigma}_{\theta t,\theta t}+\underline{K}^{\sigma}_{\psi\psi}-\underline{K}^{\sigma}_{\theta t,\psi}-\underline{K}^{\sigma}_{\psi,\theta t} \end{array}\right]$$

where $\underline{K}^{\sigma}_{\theta_1\theta_2,\theta_3\theta_4} = \iint_{element}\underline{G}^T_{\theta_1\theta_2}\ \underline{S}_{\theta_1\theta_3}\ \underline{G}_{\theta_3\theta_4}\ dx\ dy$ $(\theta_1,\theta_2,\theta_3,\text{and}\ \theta_4$ equivalent to $m,\theta,t,\text{and}\ \psi)$

8. FINITE FORCE VECTOR $\underline{F}_L$

By integrating the strain energy $d\overline{U}_{SL}$ over the midplane, it can be deduced that,

$$dU_{SL} = d\underline{\delta}^T\ \underline{F}_L$$

where $\underline{F}_L$ represents the finite force vector, which is defined as follows:

$$\underline{F}_L = \begin{bmatrix} \underline{F}_0^L \\ \hline \underline{F}_b^L \\ \hline \underline{F}_t^L + \underline{F}_{bt}^L \end{bmatrix}$$

where $\underline{F}^L_{\theta_1\theta_2} = \iint_{element} \underline{B}^T_{\theta_1\theta_2}\, \underline{\sigma}_{\theta_1}\, dx\, dy$ (θ_1, and θ_2 equivalent to o, and b)

9. GEOMETRICAL NONLINEAR STATIC ANALYSIS

An equivalent nodal force $\underline{F}$ can be defined such that the work done by the actual applied forces due to a virtual displacement field is the same as that done by $\underline{F}$, i.e.

$$dW = d\underline{\delta}^T\ \underline{F}$$

The corresponding change of strain energy can be deduced from the strain energy as follows:

$$dU = d\underline{\delta}^T\left(\underline{K} + \underline{K}^\sigma\right)\underline{\delta} + d\underline{\delta}^T\ \underline{F}_L$$

Applying the principle of virtual work, then:

$$d\underline{\delta}^T\left(\underline{K} + \underline{K}^\sigma\right)\underline{\delta} + d\underline{\delta}^T\ \underline{F}_L - d\underline{\delta}^T\ \underline{F} = 0$$

Hence, $d\underline{\delta}$ contains arbitrary values, then

$$\left(\underline{K} + \underline{K}^\sigma\right)\underline{\delta} + \underline{F}_L - \underline{F} = 0$$

which represents the generalised equations of equilibrium. Let $\underline{\delta} + \Delta\underline{\delta}$ represent the exact solution of the equilibrium equation, then

$$\left(\underline{K} + \underline{K}^\sigma\right)\Delta\underline{\delta} = \underline{F} - \left(\underline{K} + \underline{K}^\sigma\right)\underline{\delta} - \underline{F}_L = \underline{R}$$

where the residual nodal force vector $\underline{R}$ can be defined as

$$\underline{R} = \underline{F} - \left(\underline{K} + \underline{K}^\sigma\right)\underline{\delta} - \underline{F}_L$$

which can be solved by means of iterative algorithm until acceptable value of error.

10. STABILITY ANALYSIS

In general, the critical load is the load corresponding to large deflection, and $\underline{K}^\sigma$ is proportional to the stress level. Thus, a small deflection analysis can be carried out with a small load representing the distribution of actual load, and has equivalent nodal loading vector $\underline{F}_0$. Just before instability, the strains can always be considered infinitesimal, and if instability occurs at $\underline{F} = \lambda\ \underline{F}_0$, where λ denotes the scale load factor on stresses necessary to achieve neutral equilibrium. This means that $\left(\underline{K} + \lambda\,\underline{K}^\sigma\right)\underline{\delta} \Rightarrow \underline{0}$ which leads to the eigenproblem, $\left|\underline{K} + \lambda\,\underline{K}^\sigma\right| = 0$ from which λ can be obtained.

11. VALIDATION AND NUMERICAL RESULTS

To ensure the accuracy of the present algorithm, two simple case studies of isotropic and laminated anisotropy materials have been solved. The results have been compared with published results and with ABAQUS software.

11.1 Static Analysis of Clamped Isotropic Plate

A clamped square plate, which analysed before in several works [10,11] has been solved. Two different meshes were attempted; a coarse mesh with 16 elements (4x4), and a fine mesh with 64-element (8x8), and the element type used in these meshes is 4-node quadrilateral element. A nondimensional parameter W= w(0,0)/t is obtained by different elements and meshes. Fig.1 shows the variation of W with different distributed load levels. From the Figure, the results of the present element are in good agreement with the published and ABAQUS results. Also, it is clear that the two meshes gave results close to each other. Hence it was decided to use the coarse mesh for the further analysis.

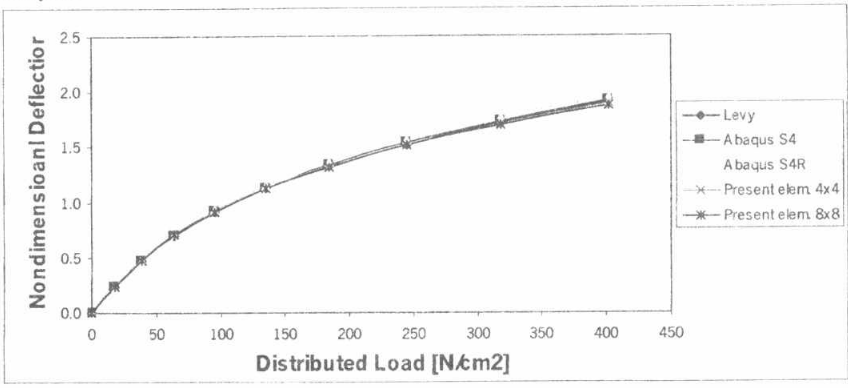

Fig.1. The Variation of Nondimensional Displacement Viruses Load Distribution

11.2 Static and Stability Analysis of Orthotropic Plate

The aim of this investigation is testing the ability of the programs to accept the materials in a composite form and to validate the ability of these programs to perform static and stability analysis of orthotropic materials. A cantilever plate with geometry and material properties as indicated in Ref. [12] was used.

With respect to the static analysis, the present results were compared with analytical and ABAQUS results as shown in Fig.2. Table 1 shows also the load factor λ for the stability analysis. This validation shows a good agreement between all elements in the case of orthotropic representation for static or stability analysis.

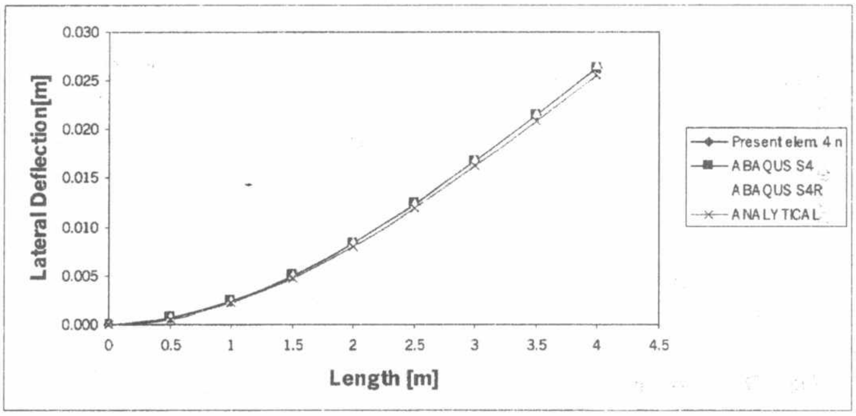

Fig.2. The Variation Of Lateral Displacement Along The Plate Length

Table 1 Instability factor for cantilever plate

	ABAQUS Elem.	Present Elem.	Analytical
Orthotropic material	393.33	391.11	385.33

11.3 Effect of Degree of Orthotropy

Three types of fibre-reinforced epoxy materials having different degrees of orthotropy are considered as indicate in Ref. [13]. These types are considered to represent all levels of orthotropy (weak & strong). The effect of degree of orthotropy was applied on a cantilever laminate composite plate consisting of 4 layers with angles of orientation θ (45, -45, 45, - 45). Fig. 3 shows the variation of lateral displacement all over the length of the plate with different types of Epoxy. The results show a good agreement between the present element and ABAQUS element at different degrees of orthotropy. Table 2 includes also the factor of stability against the degree of orthtropy, which show a good agreement with ABAQUS results. From these, there is a conclusion that the degree of orthotropy is in an inverse proportional to the deflection and the stability factor.

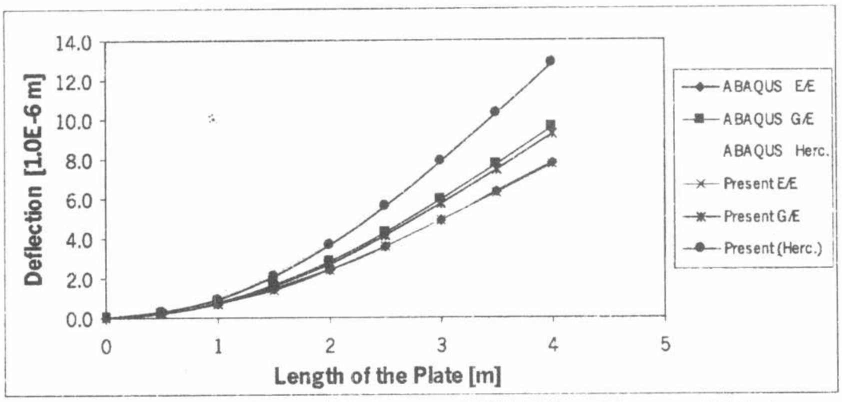

Fig.3.The Variation of Lateral Displacement With Different Degree of Orthotropy

Table 2 Instability Factor For Cantilever Plate

Type of material	E_1/E_2	ABAQUS Elem.	Present Elem.
E/E	2.44	4.21E+05	4.22269E+05
G/E	11.6	3.36E+05	3.45539E+05
Herc.	15.5	2.39E+05	2.43658E+05

12. CONCLUSIONS

It is clear from the previous case studies that the developed element consider transverse shear effects, it dose not suffer from shear locking as do Mindlin type elements, and reduced integration techniques are not required for the element developed in this work. The coupling between bending and membrane behaviour is introduced through the study of geometrical non-linearity problem. A stability analysis of structures under in plane load is discussed. The effect of mesh generation and degree of orthotropy have been studied for different case studies with different boundary conditions. The accuracy of the proposed element and algorithm has been verified through the comparison of the results with published results and results obtained through ABAQUS. The verification provides a good agreement for all case studies.

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