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On the Oscillation of Impulsive Neutral First-order Differential Equations with Variable Arguments

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Abstract

Throughout the article, we study the oscillation of a general class of first-order neutral differential equations in presence of variable delays under the effect of impulses.

Due to its importance in applications, there are many papers concerning with the property of oscillation and non-oscillation of neutral delay differential equations. Although, a lot of works are concerning with the oscillation of neutral delay differential equations without impulse or impulsive neutral with constant delays, however few papers dealt with the impulsive neutral and those with variable delays. In this paper, we establish sufficient conditions of certain neutral equations with variable delay arguments. New oscillation criteria are deduced. Our results are based on using equivalence transformation and two useful lemmas to prove the obtained criteria.

The results of this paper improve those of [20] by adding several non-linear delay functions to the equations instead of having one delay term. Where it is assumed that the two variable delays satisfying a *Lipschitz* condition.

Moreover we discuss more general non-linear delay functions comparing with those used in [14]. Our results improve and extend some recent results in the literature. An illustrative example is given.

Keywords: Oscillation; impulsive; neutral first order differential equation; variable delay

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1. Introduction

Consider the following neutral impulsive equations:

$$\begin{cases}[r(\eta)-P(\eta)r(\lambda(\eta))]'+Q(\eta)\Omega\Big(r\big(\rho(\eta)\big)\Big)-V(\eta)\Psi\Big(r\big(\theta(\eta)\big)\Big)=0,\eta\neq\eta_j, j=1,2,\ldots;\\ r(\eta_j^+)=I_j(r(\eta_j), r(\lambda(\eta_j^+)=I_j\Big(r\Big(\lambda(\eta_j)\Big)\Big).\end{cases} \quad (1)$$

In which, $\{\eta_j\}_{j\in N}$ represents the moments of impulse, λ , ρ and θ are the delay functions, $P(\eta), Q(\eta)$ and $V(\eta)$ are continuous non-negative functions. Also, $P(\eta)\in PC([\eta_0,\infty),R^+)$,$Q(\eta), V(\eta)\in C([\eta_0,\infty),R^+)$, $\lambda(\eta), \rho(\eta)$ and $\theta(\eta)\in C([\eta_0,\infty),R)$ are increasing functions such that

$$lim_{\eta\to\infty}\lambda(\eta)=lim_{\eta\to\infty}\rho(\eta)=lim_{\eta\to\infty}\theta(\eta)=\infty.$$

Throughout the paper, we assume that:

(A_1) The functions $\Omega(\eta)$ and $\Psi(\eta)$ are non-decreasing functions and satisfying:

$r\Omega(z)>0,\ r\Psi(z)>0.$

(A_2) There exist some positive costants β_1 *and* β_2 such that

$|\Omega(r)|\geq\beta_1|r|$, and $|\Psi(r)|\geq\beta_2|r|$

(A_3) For $I_j\in C(R,R)$, there exists a sequence of positive real numbers b_j satisfying

$b_j\leq\frac{I_j(r)}{r}\leq 1.$

In the last decades, there has been a great interest in studying differential equations, see for example ([12,16, 24, 27]). Recently many authors were concerning with the oscillation of neutral impulsive differential equations (see [1, 2, 6, 8-11,13,17-19,21-23, 25]) due to its importance in some models of real phenomena which include derivatives on the part of the used variable. Although, many papers discussing the oscillation of linear and nonlinear impulsive equations with constant delays ([3, 4, 6]), however few of them dealt with variable delays ([5, 7, 8, 15, 26]).

In 2019, Santra et al. [20] discussed the oscillatory behavior of the non-linear neutral equations with constant delay:

$$\begin{cases}[r(\eta)-P(\eta)r(\eta-\lambda)]'+Q(\eta)\Omega\big(r(\eta-\rho)\big)=0,\eta\neq\eta_j,j=1,2,\dots;\\ \quad r(\eta_j^+)=I_j(r(\eta_j),r(\eta_j^+-\lambda)=I_j\Big(r\big(\eta_j-\lambda\big)\Big).\end{cases} \tag{2}$$

They also investigated several criteria of Eq. (2) with several ranges for the neutral coefficient $P(\eta)$.

In [14], the authors established sufficient conditions of the oscillation of the neutral equations with variable coefficients

$$\begin{cases}\big[r(\eta)-P(\eta)r\big(\lambda(\eta)\big)\big]'+Q(\eta)r\big(\rho(\eta)\big)-V(\eta)r\big(\theta(\eta)\big)=0,\eta\neq\eta_j,j=1,2,\dots;\\ \quad r(\eta_j^+)+b_jr(\eta_j)=a_jr(\eta_j),\eta=\eta_j.\end{cases} \tag{3}$$

In this article, we discuss the oscillatory behavior of (1) in presence of positive and negative variable delay coefficients.

2. Preliminaries.

Now, we give two useful lemmas to prove our results.

Lemma 2.1.

Suppose that $r(\eta)$ be an eventually positive solution of (1) and for a continuous function $\gamma(\eta)$,

$$\varpi(\eta)=r(\eta)-P(\eta)r(\lambda(\eta))-\int_\eta^{\rho^{-1}(\gamma(\eta))}Q(\nu)\Omega\Big(r\big(\rho(\nu)\big)\Big)d\nu-\int_{\theta^{-1}(\gamma(\eta))}^{\eta}V(\nu)\Psi\Big(r\big(\theta(\nu)\big)\Big)d\nu, \tag{4}$$

where $\gamma(\eta)<t,\theta^{-1}\big(\gamma(\eta)\big)<t\ and\ \rho^{-1}\big(\gamma(\eta)\big)>\eta\ for\ \eta\in\big(\eta_j,\eta_{j+1}\big],0<\eta_0<\eta_1<\cdots<\eta_j\to\infty\ at\ j\to\infty$.

Let $\varpi(\eta)$ satisfies the following conditions:

(C_1) $\beta_1Q\Big(\rho^{-1}\big(\gamma(\eta)\big)\Big)[\rho^{-1}\big(\gamma(\eta)\big)]'-\beta_2V\Big(\theta^{-1}\big(\gamma(\eta)\big)\Big)\big[\theta^{-1}\big(\gamma(\eta)\big)\big]'\geq 0.$

$$(C_2)\begin{cases} P(\eta_j^+) \geq p(\eta_j), & \lambda(\eta_j) \neq \eta_{i.} \\ P(\eta_j^+) \geq {}^1/_{b_j}\, p(\eta_j), & \lambda(\eta_j) = \eta_i. \end{cases}$$

(C_3) If $\lim_{\eta\to\infty} \sup\{P(\eta) + \beta_1 \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(\upsilon)d\upsilon + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(\upsilon)d\upsilon \leq 1,$

then $\begin{cases} \varpi(\eta) > 0, & \text{for } \eta \geq \eta_0, \\ \varpi(\eta_j^+) \geq 0, & \text{for } \eta \in (\eta_j, \eta_{j+1}]. \end{cases}$

Proof.

Suppose that *r(η)* be an eventually positive solution of Eq. (1) such that $r(\eta)$, $r(\lambda(\eta)) > 0$, $r(\rho(\eta))$ and $r(\theta(\eta))$ also be positive. From (1) and (2), we get

$$\varpi'(\eta) = [r(\eta) - P(\eta)r(\lambda(\eta))]' - [Q(\eta)\Omega(r(\rho(\eta))]_{\eta}^{\rho^{-1}(\gamma(\eta))} - [V(\eta)\Psi(r(\theta(\eta)))]_{\theta^{-1}(\gamma(\eta))}^{\eta}$$

$$= -Q(\rho^{-1}(\gamma(\eta))\Omega(r(\rho(\rho^{-1}(\gamma(\eta))) - V(\theta^{-1}(\gamma(\eta)))\Psi(r(\theta(\theta^{-1}(\gamma(\eta)))). \quad (5)$$

By (A_1), we obtain

$$\varpi'(\eta) \leq -\beta_1 Q(\rho^{-1}(\gamma(\eta))r(\rho\left(\rho^{-1}(\gamma(\eta))\right) - \beta_2 V(\theta^{-1}(\gamma(\eta)))r(\theta(\theta^{-1}(\gamma(\eta))) \leq$$

$$-\beta_1 Q(\rho^{-1}(\gamma(\eta))[\rho^{-1}(\gamma(\eta))]' - \beta_2 V(\theta^{-1}(\gamma(\eta)))[\theta^{-1}(\gamma(\eta))]' r(\gamma(\eta)) \quad (6)$$

Hence by the condition (C_1), we conclude that

$$\varpi'(\eta) \leq 0, \quad (7)$$

for all $\eta \in (\eta_j, \eta_{j+1}]$. First, we show that $\varpi(\eta_j^+) \leq \varpi(\eta_j)$. Since

$$\varpi(\eta_j^+) = r(\eta_j^+) - P(\eta_j^+)r(\lambda(\eta_j^+))$$
$$- \int_{\eta_j}^{\rho^{-1}(\gamma(\eta_j))} Q(\nu)\Omega\left(r(\rho(\nu))\right)d\nu - \int_{\theta^{-1}(\gamma(\eta_j))}^{\eta_j} V(\nu)\Psi\left(r(\theta(\nu))\right)d\nu$$

$$= I_j\left(r(\eta_j)\right) - P(\eta_j^+)I_j(r(\lambda(\eta_j)))$$
$$- \int_{\eta_j}^{\rho^{-1}\left(\gamma(\eta_j)\right)} Q(\nu)\Omega\left(r\left(\rho(\nu)\right)\right)d\nu - \int_{\theta^{-1}\left(\gamma(\eta_j)\right)}^{\eta_j} V(\nu)\Psi\left(r\left(\theta(\nu)\right)\right)d\nu, \qquad (8)$$

For $\lambda(\eta_j) = \eta_i$, applying the condition (C_2)

$$\varpi(\eta_j^+) \le r(\eta_j) - b_j P(\eta_j^+) r(\lambda(\eta_j))$$
$$- \int_{\eta_j}^{\rho^{-1}\left(\gamma(\eta_j)\right)} Q(\nu)\Omega\left(r\left(\rho(\nu)\right)\right)d\nu - \int_{\theta^{-1}\left(\gamma(\eta_j)\right)}^{\eta_j} V(\nu)\Psi\left(r\left(\theta(\nu)\right)\right)d\nu$$
$$\le r(\eta_j) - P(\eta_j) r(\lambda(\eta_j))$$
$$- \int_{\eta_j}^{\rho^{-1}\left(\gamma(\eta_j)\right)} Q(\nu)\Omega\left(r\left(\rho(\nu)\right)\right)d\nu - \int_{\theta^{-1}\left(\gamma(\eta_j)\right)}^{\eta_j} V(\nu)\Psi\left(r\left(\theta(\nu)\right)\right)d\nu.$$

So,

$$\varpi(\eta_j^+) \le \varpi(\eta_j) \qquad (9)$$

For $(\eta_j) \ne \eta_i$, we get

$$\varpi(\eta_j^+) = I_j\left(r(\eta_j)\right) - P(\eta_j^+) r(\lambda(\eta_j))$$
$$- \int_{\eta_j}^{\rho^{-1}\left(\gamma(\eta_j)\right)} Q(\nu)\Omega\left(r\left(\rho(\nu)\right)\right)d\nu - \int_{\theta^{-1}\left(\gamma(\eta_j)\right)}^{\eta_j} V(\nu)\Psi\left(r\left(\theta(\nu)\right)\right)d\nu$$
$$\le r(\eta_j) - P(\eta_j) r(\lambda(\eta_j))$$
$$- \int_{\eta_j}^{\rho^{-1}\left(\gamma(\eta_j)\right)} Q(\nu)\Omega\left(r\left(\rho(\nu)\right)\right)d\nu - \int_{\theta^{-1}\left(\gamma(\eta_j)\right)}^{\eta_j} V(\nu)\Psi\left(r\left(\theta(\nu)\right)\right)d\nu$$
$$\le r\left(\lambda(\eta_j)\right),$$

which implies that $\varpi(\eta_j^+) \le \varpi(\eta_j)$. (10)

Hence $-\infty \le h < \infty, |h| = \sup\{\varpi(\eta_j^+), \lim_{j\to\infty} \varpi(\eta_j)\}, \eta \in [\eta_h, \infty)$ for $h \ge \eta_0$.

Now, we claim that $\varpi(\eta_j) \geq 0$ for *j=h, h+1*,…. If not, there exists $m \geq h$ such that:

$$\varpi(\eta_m) < 0 \text{ for } \eta \geq \eta_m.$$

Since $\varpi(\eta)$ is non-increasing on $[\eta_0, \infty)$, then there exists a positive constant ε satisfying $\varpi(\eta_m) = -\varepsilon < 0$.

Further from (1), we get

$$r(\eta) \leq -\varepsilon + P(\eta)r\big(\lambda(\eta)\big) + \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(\nu)\Omega\Big(r\big(\rho(\nu)\big)\Big)\,d\nu - \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(\nu)\Psi\Big(r\big(\theta(\nu)\big)\Big)\,d\nu. \tag{11}$$

There exist two cases:

Case 1: If $r(\eta)$ is unbounded then $\exists$ a sequence of points $\{S_n\}$ such that $S_n \geq \eta_m$, then $\lim_{n\to\infty} r(S_n) = \infty$ and $r(S_n) = \max\{r(\eta), \eta_m \leq \eta \leq S_n\}$.

Then the inequality (11) becomes

$$\begin{aligned} r(S_n) &\leq -\varepsilon + P(S_n)r\big(\lambda(S_n)\big) \\ &\quad + \int_{S_n}^{\rho^{-1}(\gamma(S_n))} Q(\nu)\Omega\Big(r\big(\rho(\nu)\big)\Big)\,d\nu + \int_{\theta^{-1}(\gamma(S_n))}^{S_n} V(\nu)\Psi\Big(r\big(\theta(\nu)\big)\Big)\,d\nu \\ &\leq -\varepsilon + \{P(S_n) + \beta_1 \int_{S_n}^{\rho^{-1}(\gamma(S_n))} Q(\nu)d\nu + \beta_2 \int_{\theta^{-1}(\gamma(S_n))}^{S_n} V(\nu)d\nu\} r(S_n) \end{aligned}$$

Then $$r(S_n) \leq -\mu + r(S_n), \tag{12}$$

which implies that $0 \leq -\varepsilon$. This is a contradiction with the assumption that $\varepsilon > 0$.

Case 2: If $r(\eta)$ is bounded such that $\lim_{\eta\to\infty} \sup r(\eta) = M < \infty$.

Here**,** we can take the sequence of points $\{S_n\}$ as $\lim_{n\to\infty} r(S_n) = M$ and $r\big(\xi(S_n)\big) = max\{r(t): \delta_1(S_n) \leq \eta \leq \delta_2(S_n)\}$,

where $\delta_1(S_n) = \min\{\lambda(S_n), \rho(S_n)\}$ and $\delta_2(S_n) = \max\{\lambda(S_n), \theta(S_n)\}$. It is clear that $\lim_{n\to\infty} \xi(S_n) = \infty$ and $\lim_{n\to\infty} \sup r\big(\xi(S_n)\big) \leq M$. Then we get,

$$r(S_n) \leq -\varepsilon + P(S_n) r\big(\lambda(S_n)\big) + \int_{S_n}^{\rho^{-1}(\gamma(S_n))} Q(\nu)\Omega\Big(r\big(\rho(\nu)\big)\Big)\, d\nu + \int_{\theta^{-1}(\gamma(S_n))}^{S_n} V(\nu)\Psi\Big(r\big(\theta(\nu)\big)\Big)\, d\nu$$

$$\leq -\varepsilon + \{P(S_n) + \beta_1 \int_{S_n}^{\rho^{-1}(\gamma(S_n))} Q(\nu)d\nu + \beta_2 \int_{\theta^{-1}(\gamma(S_n))}^{S_n} V(\nu)d\nu\} r(\xi(S_n))$$

$$\leq -\varepsilon + r\big(\xi(S_n)\big). \qquad (13)$$

Taking limit sup. at $n \to \infty$, we obtain $0 \leq -\varepsilon$ which is again a contradiction. From the two cases, we can see that $\varpi(\eta_j) \geq 0$ for $\eta \in \big(\eta_j, \eta_{j+1}\big]$, *j=h, h+1,....*

Finally, to prove that $\varpi(\eta) > 0$ for *j=1, 2*,... If it not satisfied, suppose that there exists $m \geq 0$ at which $\varpi(\eta_m) = 0$.

Integrating (7) from η_m *to* η_{m+1}, we get

$$\varpi(\eta_{m+1}) =$$

$$\varpi(\eta_m^+) - \int_{\eta_m}^{\eta_{m+1}} \{\beta_1 Q(\rho^{-1}(\gamma(\eta))\big[\rho^{-1}\big(\gamma(\eta)\big)\big]' - \beta_2 V(\theta^{-1}(\gamma(\eta)))[\theta^{-1}\big(\gamma(\eta)\big)]'\} r\big(\gamma(\eta)\big) d\eta$$

$$< \varpi(\eta_m^+) \leq \varpi(\eta_m) = 0, \qquad (14)$$

which is a contradiction. Ultimately, $\varpi(\eta) > 0$ for $\eta \geq \eta_0$.

Lemma 2.2.

Suppose that the functions $\phi(t)$and $B(t) \in C(R^+, R^+)$ such that $B(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} B(t) = \infty$. If the function $A(t)$ satisfies

$$\lim_{t\to\infty} inf \int_{B(t)}^{t} \phi(s)ds > \frac{1}{e}, \qquad (15)$$

then the inequality $r'(t) + \phi(t) r(B(t)) \leq 0$ has no eventually positive solution.

3. Main Results.

Now, we give sufficient conditions for the oscillation of Eq. (1).

Theorem 3.1.

Let $(C_1)-(C_3)$ hold. If

$$\lim_{\eta\to\infty} inf \int_{\gamma(\eta)}^{\eta} \{\beta_1 Q(\rho^{-1}(\gamma(\eta))\left[\rho^{-1}\big(\gamma(\eta)\big)\right]' - \beta_2 V(\theta^{-1}(\gamma(\eta)))[\theta^{-1}\big(\gamma(\eta)\big)]'\}\,.$$

$$\left\{1 + P\big(\gamma(s)\big) + \beta_1 \int_{\gamma(\eta)}^{\rho^{-1}\left(\gamma((\gamma(\eta)))\right)} Q(\nu)d\nu + \beta_2 \int_{\theta^{-1}\left(\gamma((\gamma(\eta)))\right)}^{\gamma(\eta)} V(\nu)d\nu\right\} ds > \frac{1}{e}, \quad (16)$$

then Eq. (1) is oscillatory, where $\theta^{-1}\big(\gamma(\eta)\big) < \eta$, and $\rho^{-1}\big(\gamma(\eta)\big) > \eta$.

Proof. Suppose that $r(\eta)$ be an eventually positive solution of Eq. (1).

But since $\varpi(\eta) \le r(\eta)$, then by Lemma 2.1, we have

$$r(\eta) = \varpi(\eta) + P(\eta) r\big(\lambda(\eta)\big) +$$
$$\int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(\nu)\Omega\left(r\big(\rho(\nu)\big)\right) d\nu + \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(\nu)\Psi\left(r\big(\theta(\nu)\big)\right) d\nu$$

$$r(\eta) \ge \varpi(\eta) + P(\eta) r\big(\lambda(\eta)\big) + \beta_1 \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(\nu)\varpi\big(\rho(\nu)\big)d\nu + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(\nu)\varpi\big(\theta(\nu)\big)d\nu$$

$$\ge \varpi(\eta) + P(\eta)\varpi\big(\lambda(\eta)\big) + \beta_1 \varpi\big(\gamma(\eta)\big) \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(\nu)d\nu + \beta_2 \varpi\big(\theta(\eta)\big) \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(\nu)d\nu$$

$$\ge \varpi(\eta) + P(\eta)\varpi\big(\lambda(\eta)\big) + \beta_1 \varpi(\eta) \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(\nu)d\nu + \beta_2 \varpi(\eta) \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(\nu)d\nu$$

$$\ge \varpi(\eta)\{1 + P(\eta) + \beta_1 \int_{\eta}^{\rho^{-1}(\gamma(\eta))} Q(\nu)d\nu + \beta_2 \int_{\theta^{-1}(\gamma(\eta))}^{\eta} V(\nu)d\nu\}.$$

Then

$$r(\gamma(\eta)) \ge \varpi(\gamma(\eta))\{1 + P\big(\gamma(\eta)\big) + \beta_1 \int_{\eta}^{\rho^{-1}\left(\gamma(\gamma(\eta))\right)} Q(\nu)d\nu + \beta_2 \int_{\theta^{-1}\left(\gamma(\gamma(\eta))\right)}^{\eta} V(\nu)d\nu\}. \quad (17)$$

Substituting from (17) into (7), we obtain

$$\varpi'(\eta) \le -\{\beta_1 Q(\rho^{-1}\big(\gamma(\eta)\big)\left[\rho^{-1}\big(\gamma(\eta)\big)\right]' - \beta_2 V\left(\theta^{-1}\big(\gamma(\eta)\big)\right)[\theta^{-1}\big(\gamma(\eta)\big)]'\}.$$

$$\left\{1+P\big(\gamma(s)\big)+\beta_1\int_{\gamma(\eta)}^{\rho^{-1}\left(\gamma((\gamma(\eta))\right)}Q(\nu)d\nu+\beta_2\int_{\theta^{-1}\left(\gamma((\gamma(\eta))\right)}^{\gamma(\eta)}V(\nu)d\nu\right\}ds\le 0. \qquad (18)$$

But from (15) and (16), we obtain that (18) has no positive solution. This is a contradiction, then the proof is complete.

Corollary 3.2. Let the conditions$(C_1)-(C_3)$ be hold. If

$$\{\beta_1 Q(\rho^{-1}\big(\gamma(\eta)\big)\big[\rho^{-1}\big(\gamma(\eta)\big)\big]'-\beta_2 V\left(\theta^{-1}\big(\gamma(\eta)\big)\right)\big[\theta^{-1}\big(\gamma(\eta)\big)]'\}$$

$$\ge\left[e\times min_{\eta\ge\eta_0}\{\eta-\gamma(\eta)\}\right]^{-1}\ge e^{-1}, \qquad (19)$$

then Eq. (1) oscillates.

Proof.

Going through as in Corollary 3.2. of [14], the proof is complete.

4. Example.

Consider the impulsive neutral equation:

$$\begin{cases}\left[r(\eta)-\frac{1+e^{-\eta}}{9}r(\eta-2\pi)\right]'+\frac{8-e^{-\eta}}{9}r\left(\eta-\frac{5\pi}{2}\right)(1+|r(\eta-2)|^{\mu})\\ \quad -\frac{e^{-\eta}}{9}r(\eta-2\pi)(1+|r(\eta-2)|^{\mu})=0,\eta\ne\eta_j,j=1,2,\dots; \\ \qquad r\big(\eta_j^+\big)=\frac{j}{j+1}r\big(\eta_j\big),\ \ r\big(\eta_j^+-\lambda\big)=I_j\big(r\big(\eta_j\big)-\lambda\big),\end{cases} \qquad (20)$$

where $I_j=j/j+1$ and $\mu>0$. Taking $\gamma(\eta)=t-9\pi/4, P\big(\eta_j\big)=1/20j$ and $b_j=j+1/j$.

Since $\Omega(r\big(\rho(\eta)\big))\ge r(\eta-5\pi/4)(1+|r(\eta-2)|^{\mu})$ and $\Psi(r\big(\theta(\eta)\big))\ge r(\eta-2\pi)(1+|r(\eta-2)|^{\mu})$, we can take $\beta_1=\beta_2=1$. Now since

$$\beta_1 Q\left(\rho^{-1}\big(\gamma(\eta)\big)\right)[\rho^{-1}\big(\gamma(\eta)\big)]'-\beta_2 V\left(\theta^{-1}\big(\gamma(\eta)\big)\right)\left[\theta^{-1}\big(\gamma(\eta)\big)\right]'=\frac{8}{9}-\frac{1}{9}(e^{-\pi/4}+e^{\pi/4})>0.$$

For $\eta_j=j$, then $P\big(\eta_j^+\big)=\lim_{\eta\to j^+}\frac{1+e^{-\eta}}{9}\ge\frac{1}{9}$ and $\frac{1}{b_j}P(\eta)=\frac{j+1}{j}\times\frac{1}{20j}=\frac{j+1}{20j^2}$ which satisfies the condition (C_2). Finally, we check the condition

$$\lim_{\eta\to\infty} inf \int_{\gamma(\eta)}^{\eta} \{\beta_1 Q(\rho^{-1}(\gamma(\eta))[\rho^{-1}(\gamma(\eta))]' - \beta_2 V(\theta^{-1}(\gamma(\eta)))[\theta^{-1}(\gamma(\eta))]'\}.$$

$$\left\{1 + P(\gamma(s)) + \beta_1 \int_{\gamma(\eta)}^{\rho^{-1}(\gamma((\gamma(\eta))))} Q(\nu)d\nu + \beta_2 \int_{\theta^{-1}(\gamma((\gamma(\eta))))}^{\gamma(\eta)} V(\nu)d\nu\right\} ds$$

$$\lim_{\eta\to\infty} inf \int_{\eta-9\pi/4}^{\eta} [\frac{8}{9} - \frac{1}{9}(e^{-s-\frac{\pi}{4}} + e^{-s+\frac{\pi}{4}})] \times [1 + \frac{1}{9} + \frac{1}{9}e^{-s+\frac{\pi}{4}} + \frac{2\pi}{9} - \frac{2e^{-s+\frac{9\pi}{4}}}{9} + \frac{e^{-s+\frac{10\pi}{4}}}{9} + \frac{e^{-s+2\pi}}{9}] ds \geq \frac{1}{e}.$$

Using Theorem 3.1, then Eq. (20) oscillates.

Conclusion. The aim of this paper is to discuss the oscillation of a general class of first-order impulsive neutral equations with positive and negative delays (1). New results are given to improve and extend some recent papers like [14] and [20]. An example is given to illustrate our results.

References

[1] R.P. Agarwal, S. R. Grace and D. O Regan, Oscillation theory for difference and functional differential equations, Kluwer academic publishers: (2000), Dordrecht

[2] R.P. Agarwal, F. Karakoc, A survey on oscillation of impulsive delay differential equations, Comput Math Appl 60: (2010) 1648–1685

[3] D. D. Bainov and P. S. Simeonov, Impulsive differential equations, Asymptotic properties of the solutions, Series on Advances in Mathematics for Applied Sciences 28, World Scientific: (1995) , Singapore

[4] D.D Bainov, M.B. Dimitrova, A.B. Dishilev, Oscillation of the solutions of impulsive differential equations and inequalities with a retarded argument, Rocky Mountain J Math 28: (1998) 25-40

[5] R.S. Dahiya, T. Candan, Oscillation behavior of arbitrary order neutral differential equations, Appl Math Lett 8: (2004) 953-958

[6] J. R. Graef, J. H. Shen and I. P. Stavroulakis, Oscillation of impulsive neutral delay differential equations, J Math Anal Appl 268: (2002) 310-333

[7] K.Z. Guan, J.H. Shen, Hille type oscillation criteria for a class of first order neutral Pantograph differential equations of Euler type, Commun Math Anal 1: (2007) 27-35

[8] K.Z. Guan, J.H. Shen, On first-order neutral differential equations of Euler form with unbounded delay, Appl Math Comput 189 : (2007) 1419-1427

[9] R. Guo, Q. Huang and H. Tian, " Nonoscillation and oscillation criteria for a class of second-order nonlinear neutral delay differential equations with positive and negative coefficients ", Axioms 11, 281 : (2022), https://doi.org/10.3390/ axioms11060281

[10] I. Gyori and G. Ladas, Oscillation theory of delay differential equations with applications, Clarendon Press, Oxford science applications (1991), England

[11] J. K. Hale, Theory of Functional Differential Equations, Applied Mathematical Sciences, (1977), New York

[12] A.E. Hamza, G. A. F. Ismail , D. Ahmed, Many Types of Stability of Abstract First and Second Order Linear Dynamic Equations on Time Scales, Journal of Scientific Research in Science 34 part1: (2017) 19-32

[13] T.S. Hassan, O. Moaaz, A. Nabih, M.B. Mesmouli, A.M.A. El-Sayed, " New Sufficient Conditions for Oscillation of Second-Order Neutral Delay Differential Equations", Axioms 10, 281:(2021), https://doi.org/10.3390/axioms10040281

[14] A. Hussain, F. Aqeel, Property of oscillation of first order impulsive neutral differential equations with positive and negative coefficients, Journal of Al-Qadishiyah for computer science and mathematics 11(1) : (2019) 45-51

[15] B. Karpuz , O. Ocalan, Oscillation criteria for a class of first-order forced differential equations under impulsive effects, Advanced in dynamical system and applications 7(2) : (2012) 205-218

[16] V. Lakshmikantham, D.D.Bainov, P.S.Simieonov, Oscillation theory of impulsive differential equations, World Scientific, (1989), Singapore

[17] Zh. G. Luo, X.Y. Lin, J.H. Shen, Oscillation of impulsive neutral differential equations with positive and negative coefficients, Indian J Pure Appl Math 7: (2000) 753-766

[18] S. Pandian and Y. Balachandran, Some oscillation criteria for first order impulsive neutral differential equations, International Journal of Mathematics and Scientific Computing 1(2): (2011)

[19] S. Pandian, G. Purushothaman, Oscillation of impulsive neutral differential equation with several positive and negative coefficients, J Math Comput Sci. 2: (2012) 241–254

[20] S.S. Santra, A.K. Tripathy, On oscillatory first order nonlinear neutral differential equations with nonlinear impulses, J Appl Math Comput 59: (2019) 257–270

[21] S.S. Santra, K.M. Khedher, S. W. Yao, "New Aspects for Oscillation of Differential

Systems with Mixed Delays and Impulses", Symmetry 13, 780: (2021), https:// doi.org/10.3390/sym13050780

[22] J.H. Shen, Z.C. Wang, Oscillation and asymptotic behavior of solutions of delay differential equations with impulses, Ann Differ Eqs 10(1): (1994) 61–68

[23] J. Shen, Z. Zou, Oscillation criteria for first order impulsive differential equations with positive and negative coefficients, J Comput Appl Math 217: (2008) 28–37

[24] A.K.Tripathy, S.S.Santra, Necessary and sufficient conditions for oscillation of a class of first order impulsive differential equations, Funct Differ Equ 22: (2015) 149–167

[25] A. K. Tripathy, S.S. Santra, Necessary and Sufficient Conditions for Oscillations to a Second-Order Neutral Differential Equations with Impulses, Kragujevac Journal of Mathematics 47(1): (2023) 81–93

[26] Q.R. Wang, Oscillation criteria for first-order neutral differential equations, Appl Math Lett 8: (2002) 1025-1033

[27] X. Wu, S. Chen and H. Tang, Oscillation of a class of second order nonlinear ordinary differential equations with impulses, Appl Math Comp 138: (2003) 181-188

الملخص العربي

حول السلوك التذبذبي لمجموعة من المعادلات التفاضلية المحايدة من الرتبة الأولي تحتوى على معاملات تأخير متغيرة وتحت تأثير نبضات.

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الملخص العربي

في هذه المقالة ، قمنا بدراسة خاصية التذبذب لفئة عامة من المعادلات التفاضلية المحايدة من الدرجة الأولى في وجود معاملات تأخير متغيرة تحت تأثير النبضـــات. نظرًا لأهميتها في التطبيقات، هناك العديد من الأبحاث المتعلقة بخاصـــية التذبذب وعدم التذبذب في المعادلات التفاضـــلية للتأخير المحايد. على الرغم من أن الكثير من المقالات تتعلق بدراســـة تذبذب المعادلات التفاضـلية ذات معاملات للتأخير المحايد دون وجود اندفاع أو في حالة وجود تأخيرات ثابتة ، إلا أن القليل من الأبحاث تناولت المعادلات التفاضلية المحايدة ذات التأخيرات المتغيرة.

في هذه المقالة ، قدمنا شروطًا كافية للسلوك التذبذبى لبعض المعادلات المحايدة ذات معاملات تأخير متغيرة. تعمل هذه النتائج على تحسين نتائج سابقة بإضافة عدة دوال تأخير غير خطية إلى المعادلات بدلاً من وجود معامل تأخير واحد. علاوة على ذلك اسـتخدمنا دوال تأخير غير خطية أكثر عمومية مقارنة بتلك المسـتخدمة في [14]. تعمل نتائجنا على تحسـين وتوسـيع بعض النتائج الحديثة، أيضا تم اعطاء مثال توضيحي.