GENERALIZED LOCALLY PAIRWISE CLOSED SETS ON BITOPOLOGICAL SPACES AND SOME OF ITS PROPERTIES

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Abstract. In this paper, we introduce the notions of generalized locally pairwise closed sets, generalized locally pairwise closed star sets, generalized locally pairwise closed star star sets on bitopological spaces and study some of their properties. The properties of the space (X, τ_1, τ_2) are studied through the study of the space (X, τ_{12}) which is a supra topology associated to the bitopological space (X, τ_1, τ_2). Also, characterization of these sets have obtained. Finally, we introduce the notions of $GLPC-$ (resp. $GLPC^*, GLPC^{**}$) (continuous) irresolute functions which based on these sets and study some of their properties.

1   Introduction

(Levine, 1970) [1] introduced the fundamental concept of generalized closed sets. He defined a set A to be generalized closed if its closure contained in every open superset of A. A bitopological space (X, τ_1, τ_2) was introduced by Kelly [2] in 1963, as a method of generalizes topological space (X, τ). Every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$. In 1983, Mashhor et al. [3] introduced supra topological spaces by dropping only the intersection condition. In [4] Kandil et al. generated a supra topological space (X, τ_{12}) from the bitopological space (X, τ_1, τ_2) and they studied some properties of the space (X, τ_1, τ_2) via properties of the associated space (X, τ_{12}). Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting [4–9].

In general topology at 1988, Ganstern and Reilly [10] was studied the concepts of locally closed (LC) set and locally closed continuous (LC-continuous) functions. At 1996 Krishan, Palaniappan and Haruo [11] were studied the concept of generalized locally closed sets, generalized locally closed star sets, generalized locally closed star star sets and generalized locally closed continuous (GLC-continuous) functions. In this paper we introduce the notion of generalized locally pairwise closed sets, generalized locally pairwise closed star and generalized locally pairwise closed star star and study some of their properties. Also, we have used the previous concepts to define and study the concept of pairwise-submaximal and generalized pairwise submaximal P- submaximal. Finally,we study the concept of generalized locally pairwise closed ($GLPC$-) functions and some of their properties.

2   Preliminaries

This section contains the basic properties of generalized pairwise closed sets, generalized pairwise open sets, supra topological spaces and bitopological spaces.

Definition 2.1. *[2] A bitopological spaces (briefly, bts) is a triple (X, τ_1, τ_2) where τ_1 and τ_2 are arbitrary topologies on X.*

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Definition 2.2. *[6]. Let (X, τ_1, τ_2) be a bitopological space. Then, $A \subseteq X$ is said to be pairwise open (briefly, P-open) if $A = U_1 \cup U_2$, $U_i \in \tau_i$, $(i = 1, 2)$. A set A is called a P-closed if its complement A^c is P-open.*
Note that the notion of P-open sets as well as P-closed sets has studied in [4, 12] under the name of P^*-open and P^*-closed.

Definition 2.3. *[13]. A family $\eta \subseteq P(X)$ is said to be a supra topology if η contains X, ϕ and closed under arbitrary union. The elements of η are said to be supra open sets and their complements are said to be supra closed sets. The complement of any $A \subseteq X$, denoted by A^c.*

Proposition 2.1. *[12] Let (X, τ_1, τ_2) be a bts. The family of all P-open subsets of X, denoted by τ_{12} is a supra topology on X and (X, τ_{12}) is the supra topological space associated to the bts (X, τ_1, τ_2).*

Definition 2.4. *[4]. An operator $C : P(X) \longrightarrow P(X)$ is a supra closure operator if it satisfies the following conditions for all $A, B \subseteq X$.*

1. *$C(\phi) = \phi$.*

2. *$A \subseteq C(A)$.*

3. *$C(A \cup B) \supseteq C(A) \cup C(B)$.*

4. *$C(C(A)) = C(A)$.*

Proposition 2.2. *[12]. Let (X, τ_1, τ_2) be a bts. Then, the operator $cl_{12} : P(X) \longrightarrow P(X)$ defined by $cl_{12}(A) = \overline{A}^1 \cap \overline{A}^2$, is a supra closure operator such that $\tau_{12} = \{A \subseteq X : cl_{12}(A^c) = A^c\}$ where $\overline{A}^i$ is the closure of A with respect to τ_i and $i = 1, 2$.*

Proposition 2.3. *[12]. Let (X, τ_1, τ_2) be a bts. Then, the operator $int_{12} : P(X) \longrightarrow P(X)$ defined by, $int_{12}(A) = A^{o1} \cup A^{o2}$ is a supra interior operator such that $\tau_{12} = \{A \subseteq X : int_{12}(A) = A\}$, where $A^{oi}, (i = 1, 2)$ is the τ_i- interior with respect to τ_i.*

Proposition 2.4. *[12] . Let (X, τ_1, τ_2) be a bts and $A \subseteq X$. Then*

1. *$\tau_1, \tau_2 \subseteq \tau_{12}$.*

2. *$cl_{12}(A) = X \backslash int_{12}(X \backslash A)$.*

3. *$int_{12}(A) = X \backslash cl_{12}(X \backslash A)$.*

4. *A is P-open $\Leftrightarrow A = int_{12}(A)$.*

5. *A is P-closed $\Leftrightarrow A = cl_{12}(A)$.*

Definition 2.5. *[4]. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \gamma_1, \gamma_2)$ is said to be P^*- continuous function (briefly, P^*-cts) if $f^{-1}(B) \in \tau_{12}$ for all $B \in \gamma_{12}$.*

Definition 2.6. *[4]. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \gamma_1, \gamma_2)$ is said to be P^*- closed (P^*-open) if $f(A)$ is $\gamma_{12}-$ closed (open) set on Y for all τ_{12}-closed (open) set A in X.*

Definition 2.7. *[14] Let (X, τ_1, τ_2) be a bts and (X, τ_{12}) its associated supra topological space. Then, $A \subseteq X$ is called a generalized pairwise closed set (briefly, gp-closed) if $cl_{12}(A) \subseteq O$ whenever $A \subseteq O$, O is a P-open.*

Theorem 2.1. *[14] A set A is a gp-closed set if and only if $cl_{12}(A) \backslash A$ contains no non empty P-closed sets.*

Corollary 2.1. *[14] Let A is a gp-closed set. Then, A is P-closed if and only if $cl_{12}(A) \backslash A$ is a P-closed set.*

Remark 2.1. *[14]*

1. *If A is P-closed set, then A is gp-closed set.*

2. *If A and B are gp-closed sets, then $A \cup B$ and $A \cap B$ are not necessary gp-closed set.*

Definition 2.8. *[14] Let (X, τ_1, τ_2) be a bts, $A \subseteq X$ and $(A, \tau_1|_A, \tau_2|_A)$ be a bitopological subspace of (X, τ_1, τ_2). The collection $\tau_{12}(A)$ is defined as a following: $\tau_{12}(A) = \tau_1|_A \sqcup \tau_2|_A = \{U_1 \cup U_2 : U_1 \in \tau_1|_A, U_2 \in \tau_2|_A\}$ such that $\tau_i|_A = \{A \cap U_i : U_i \in \tau_i, i = 1, 2\}$.*

Theorem 2.2. *[14] Let (X, τ_1, τ_2) be a bts, $A \subseteq X$. Then, $\tau_{12}|_A = \tau_{12}(A)$, where $\tau_{12}|_A = \{A \cap U : U \in \tau_{12}\}$.*

Theorem 2.3. *[14] Let (X, τ_1, τ_2) be a bts. Suppose that $B \subseteq A \subseteq X$, B is a gp-closed set relative to A and A is a gp-closed set of X. Then, B is a gp-closed set relative to X.*

Corollary 2.2. *[14] If A is a gp-closed set and F is P-closed set, then $A \cap F$ is a gp-closed set.*

Theorem 2.4. *[14] Let (X, τ_1, τ_2) be a bts and $A \subseteq X$. If A is a gp closed set and $A \subseteq B \subseteq cl_{12}(A)$. Then, B is a gp-closed set.*

Theorem 2.5. *[14] Let (X, τ_1, τ_2) be a bts and $A \subseteq Y \subseteq X$. Suppose that A is gp-closed in X. Then, A is gp-closed relative to Y.*

Theorem 2.6. *[14] Let (X, τ_1, τ_2) be a bts, τ_{12} be supra topology on X induced by τ_1, τ_2. Then, $\tau_{12} = \tau_{12}^c$ if and only if every subset of X is a gp- closed set.*

Definition 2.9. *[14] A set A is called a generalized pairwise open (briefly, gp-open) set if A^c is a gp-closed set.*

Theorem 2.7. *[14] Let (X, τ_1, τ_2) be a bts. Then, A set A is a gp-open set if and only if $F \subseteq int_{12}(A)$ whenever F is a P-closed set and $F \subseteq A$.*

Definition 2.10. *[7] Let (X, τ_1, τ_2) be a bts and $A \subseteq B \subseteq X$. Then, A and B are P^*- separated in X if $A \cap cl_{12}(B) = \phi$ and $cl_{12}(A) \cap B = \phi$.*

Not that if A and B are P^*-separated and $C \subseteq A, D \subseteq B$, then C and D are P^*- separated.

Theorem 2.8. *[14] Let (X, τ_1, τ_2) be a bts. If A and B are P^*-separated and gp-open sets, then $A \cup B$ is a gp-open set.*

Corollary 2.3. *[14] Let (X, τ_1, τ_2) be a bts and let A and B be two gp-closed sets and suppose that A^c, B^c are P^*- separated. Then, $A \cap B$ is a gp-closed set.*

Theorem 2.9. *[14] Let (X, τ_1, τ_2) be a bts. Then, a set A is gp- open if and only if $O = X$ whenever O is a P-open and $int_{12}(A) \cup A^c \subseteq O$.*

Theorem 2.10. *[14] Let (X, τ_1, τ_2) be a bts and $A, B \in P(X)$. If $int_{12}(A) \subseteq B \subseteq A$ and A is a gp-open set, then B is gp-open set.*

Theorem 2.11. *[14] Let (X, τ_1, τ_2) be a bts. Then, a set A is a gp-closed if and only if $cl_{12}(A) \setminus A$ is gp-open.*

Definition 2.11. *[14] Let (X, τ_1, τ_2) be a bts. A space (X, τ_1, τ_2) is said to be a pairwise symmetric (briefly, P^*-symmetric) if for all x and y in X, then $[x \in cl_{12}\{y\} \Longrightarrow y \in cl_{12}\{x\}]$.*

Theorem 2.12. *[14] A space (X, τ_1, τ_2) is a P^*-symmetric if and only if $\{x\}$ is a gp- closed $\forall x \in X$.*

Theorem 2.13. *[14] Let (X, τ_1, τ_2) and (Y, γ_1, γ_2) be two bitopological spaces and A be gp-closed set in (X, τ_1, τ_2). (X, τ_1, τ_2). Suppose that $f : X \longrightarrow Y$ is P^*-cts and f is P^*-closed function, then $f(A)$ is a gp-closed set.*

Remark 2.2. *[14] If f is P^*- continuous and P^*-closed function and A is gp-open set, then $f(A)$ is not necessary to be a gp-open set.*

Theorem 2.14. *[14] If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \upsilon_1, \upsilon_2)$ is P^*-cts and P^*-closed function, B is a gp-closed (gp-open) subset of Y, then $f^{-1}(B)$ is gp-closed (gp-open) set in X, respectively.*

3 Generalized locally pairwise closed [resp., closed star, closed star star] sets.

In this section we defined the concepts of locally pairwise closed sets, generalized locally pairwise closed sets, generalized locally pairwise closed star sets and generalized locally pairwise closed star star sets. Also we study some of their properties and find some relations between them.

Definition 3.1. *Let (X, τ_1, τ_2) be a bitopological space and $\tau_{12} = \{U \cup V : U \in \tau_1, V \in \tau_2\}$. A set S is called locally P-closed set if $S = G \cap F$ where G is P-open set and F is P-closed set relative to τ_{12}.*

Remark 3.1. *The following are well known:*

1. *A subset S of space (X, τ_{12}) is locally P-closed set if and only if its complement $X \setminus S$ is the union of a P-open set and P-closed set.*

2. *Every P-open (resp., P-closed) set is locally P-closed sets.*

3. *The complement of locally P-closed set need not be locally P-closed set which shown by the following example.*

Example 3.1. *Let $X = \{1, 2, 3, 4\}$, $\tau_1 = \{X, \phi, \{4\}\}$, $\tau_2 = \{X, \phi, \{1, 2, 4\}\}$, $\tau_{12} = \{X, \phi, \{4\}, \{1, 2, 4\}\}$, $\tau_{12}^c = \{X, \phi, \{3\}, \{1, 2, 3\}\}$. Then, $\{1, 2\}$ is locally P-closed set that is to say $\{1, 2\} = \{1, 2, 3\} \cap \{1, 2, 4\}$ but $\{1, 2\}^c = \{3, 4\}$ is not locally P-closed.*

Definition 3.2. *Let (X, τ_1, τ_2) be a bitopological space and τ_{12} is the supra topology generated by τ_1 and τ_2. A subset S in (X, τ_{12}) is called generalized locally pairwise closed set (briefly, GLPC set) if $S = G \cap F$ where G is $gp-$open set and F is $gp-$closed set.*

Remark 3.2. *Every $gp-$closed set (resp., $gp-$open set) is GLPC set.*

The collection of all (generalized) locally pairwise closed sets of (X, τ_1, τ_2) denoted by $GLPC$ [resp., LPC].

Definition 3.3. *Let (X, τ_1, τ_2) be a bitopological space. For a subset S of X, S is called generalized locally pairwise closed star set $(GLPC^*)$ if there exist a $gp-$open set G and P-closed set F such that $S = G \cap F$. [$S \in GLPC^*(X, \tau_{12})$].*

Definition 3.4. *Let (X, τ_1, τ_2) be a bitopological space. For a subset S of X, S is called generalized locally pairwise closed star star set $(GLPC^{**})$ if there exist a P-open set G and $gp-closed$ set F such that $S = G \cap F$. [$S \in GLPC^{**}(X, \tau_{12})$].*

Proposition 3.1. *Let S be a set in (X, τ_1, τ_2).*

1. *If $S \in LPC(X, \tau_{12})$, then $S \in GLPC^*(X, \tau_{12})$ and $S \in GLPC^{**}(X, \tau_{12})$. In other words, LPC $(X, \tau_1, \tau_2) \subseteq GLPC^*(X, \tau_1, \tau_2) \cap GLPC^{**}(X, \tau_1, \tau_2)$.*

2. *If $S \in GLPC^*(X, \tau_{12})$ or $S \in GLPC^{**}(X, \tau_{12})$, then S is generalized locally pairwise closed set. In other words, $GLPC^*(X, \tau_{12}) \cup GLPC^{**}(X, \tau_{12}) \subseteq GLPC(X, \tau_{12})$.*

Proof. Let (X, τ_1, τ_2) be a bitopological space. Then, (X, τ_{12}) is a supra topological space generated by τ_1 and τ_2.

1. Suppose that S is locally P-closed set. Then $\exists$ P-open set G and P-closed set F such that, $S = G \cap F$. But, G is P-open set and therefore it is gp-open set, F is P-closed set, it follows that $S \in GLPC^*(X, \tau_{12})$. Similarly, the set F is P-closed set implies, F is gp-closed set. So, $S \in GLPC^{**}(X, \tau_{12})$.

2. Let $S \in GLPC^*(X, \tau_{12})$. Then, there exist gp-open G and P-closed set F such that $S = G \cap F$. But, every P-closed set is gp-closed set. Then, S is $GLPC$ set.
   Let $S \in GLPC^{**}(X, \tau_{12})$. Then, there exist P-open G and gp-closed set F such that $S = G \cap F$. But, every P-open set is gp-open set. Hence, S is a $GLPC$ set.

Example 3.2. *Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$ and $\tau_2 = \{X, \phi, \{a\}\}$. Then, $\tau_{12} = \{X, \phi, \{a\}\}$. We get $LPC(X, \tau_{12}) = \{X, \phi, \{a\}, \{b, c\}\}$ and $GLPC^*(X, \tau_{12}) = GLPC^{**}(X, \tau_{12}) = GLPC(X, \tau_{12}) = P(X)$.*

Example 3.3. *Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{c\}\}$. Then, $\tau_{12} = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$. We get, $LPC(X, \tau_{12}) = GLPC^*(X, \tau_{12}) = GLPC^{**}(X, \tau_{12}) = GLPC(X, \tau_{12}) = P(X)$.*

Theorem 3.1. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, for a subset S in (X, τ_{12}) the following are equivalent:*

1. *$S \in LPC(X, \tau_{12})$.*

2. *$S = G \cap cl_{12}(S)$ for some P-open set G.*

3. *$cl_{12}(S) \backslash S$ is P-closed set.*

4. *$S \cup (X \backslash cl_{12}(S))$ is P-open set.*

5. *$S \subseteq int_{12}(S \cup X \backslash cl_{12}(S))$.*

Proof. $(1) \Longrightarrow (2)$. Let $S \in LPC(X, \tau_{12})$. Then, $\exists$ P-open set G and P-closed set F such that $S = G \cap F$. Since, $S \subseteq G$ and $S \subseteq F$ and F is P-closed set which implies that $cl_{12}(S) \subseteq F$. Then, $S \subseteq G \cap cl_{12}(S)$. Conversely, $S = G \cap F \supseteq G \cap cl_{12}(S)$. Hence, $S = G \cap cl_{12}(S)$.

$(2) \Longrightarrow (1)$. Since G is P-open set and $cl_{12}(S)$ is P-closed set, then $G \cap cl_{12}(S) \in LPC(X, \tau_{12})$ by Definition 3.1.

$(2) \Longrightarrow (3)$. Let $S = P \cap cl_{12}(S)$ and P be a P-open set. Then, $cl_{12}(S) \backslash S = cl_{12}(S) \cap [P^c \cup (cl_{12}(S))^c] = cl_{12}(S) \cap P^c \cup \phi = cl_{12}(S) \cap P^c$. We have $cl_{12}(S)$ is P-closed set and P^c is P- closed set then, $cl_{12}(S) \cap P^c$ is P-closed set. Hence, $cl_{12}(S) \backslash S$ is a P-closed set.

$(3) \Longrightarrow (2)$. We want to find a set P which is P-open set such that $S = P \cap cl_{12}(S)$. Take $P = [cl_{12}(S) \backslash S]^c$ which is P- open set and $S = [cl_{12}(S) \backslash S]^c \cap cl_{12}(S)$.

$(3) \Longrightarrow (4)$. Let $cl_{12}(S) \backslash S$ be a $P-$closed set. Then, $(cl_{12}(S) \backslash S)^C = [cl_{12}(S)]^c \cup S$ is a $P-$ open set. Hence, $X \backslash cl_{12}(S) \cup S$ is a $P-$ open set.

$(4) \Longrightarrow (3)$. Let $S \cup X \backslash cl_{12}(S)$ be a P-open set. Then, $[S \cup X \backslash cl_{12}(S)]^c = S^c \cap cl_{12}(S) = cl_{12}(S) \backslash S$ is a $P-$ closed set.

$(4) \Longrightarrow (5)$. Let $S \cup (X \backslash cl_{12}(S))$ be a P-open set. Then, $S \subseteq S \cup (X \backslash cl_{12}(S))$. But, $int_{12}(S \cup (X \backslash cl_{12}(S)) = S \cup (X \backslash cl_{12}(S))$. Hence, $S \subseteq int_{12}(S \cup X \backslash cl_{12}(S))$.

$(5) \Longrightarrow (4)$. Let $S \subseteq int_{12}(S \cup X \backslash cl_{12}(S))$. Then, $S \cup [X \backslash cl_{12}(S)] \subseteq int_{12}(S \cup X \backslash cl_{12}(S)) \cup [X \backslash cl_{12}(S)] = int_{12}(S \cup X \backslash cl_{12}(S) \cup int_{12}[X \backslash cl_{12}(S)] \subseteq int_{12}(S \cup X \backslash cl_{12}(S) \cup [X \backslash cl_{12}(S)]) = int_{12}(S \cup X \backslash cl_{12}(S))$. Then, $(S \cup X \backslash cl_{12}(S)) \subseteq int_{12}(S \cup X \backslash cl_{12}(S))$. Hence, $S \cup X \backslash cl_{12}(S)$ is a P-open set.

Theorem 3.2. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, for a subset S of (X, τ_{12}) the following are equivalent:*

1. *$S \in GLPC^*(X, \tau_{12})$.*

2. $S = P \cap cl_{12}(S)$ for some gp-open set P.

3. $cl_{12}(S) \backslash S$ is gp-closed set.

4. $S \cup (X \backslash cl_{12}(S))$ is gp-open set.

Proof. $(1) \Longrightarrow (2)$. Let $S \in GLPC^*(X, \tau_{12})$. Then $\exists$ gp-open set P and P-closed set F such that $S = P \cap F$. Since, $S \subseteq P$ and $S \subseteq F$ and F is P-closed set which implies that $cl_{12}(S) \subseteq F$. Then, $S \subseteq P \cap cl_{12}(S)$. Conversely, $S = P \cap F \supseteq P \cap cl_{12}(S)$. Hence, $S = P \cap cl_{12}(S)$.

$(2) \Longrightarrow (1)$. Since P is a gp-open set and $cl_{12}(S)$ is a P-closed set, then, $P \cap cl_{12}(S) \in GLPC^*(X, \tau_{12})$ by Definition 3.3.

$(2) \Longrightarrow (3)$. Let $S = P \cap cl_{12}(S)$ where P is a gp-open set. Then, $cl_{12}(S) \backslash S = cl_{12}(S) \cap [P^c \cup (cl_{12}(S))^c] = cl_{12}(S) \cap P^c \cup \phi = cl_{12}(S) \cap P^c$. We have $cl_{12}(S)$ is P-closed set and P^c is gp-closed set then, by using Corollary 2.2, we get $cl_{12}(S) \cap P^c$ is a gp-closed set. Hence, $cl_{12}(S) \backslash S$ is a gp-closed set.

$(3) \Longrightarrow (2)$. We want to find a set P which is gp-open set such that $S = P \cap cl_{12}(S)$. Take $P = [cl_{12}(S) \backslash S]^c$ which is gp- open set and $S = [cl_{12}(S) \backslash S]^c \cap cl_{12}(S)$.

$(3) \Longrightarrow (4)$. Let $cl_{12}(S) \backslash S$ be a gp$-$closed set. Then, $(cl_{12}(S) \backslash S)^C = [cl_{12}(S)]^c \cup S$ is a gp$-$ open set. Hence, $X \backslash cl_{12}(S) \cup S$ is a gp$-$ open set.

$(4) \Longrightarrow (3)$. Let $S \cup X \backslash cl_{12}(S)$ be a gp-open set. Then, $[S \cup X \backslash cl_{12}(S)]^c = S^c \cap cl_{12}(S) = cl_{12}(S) \backslash S$ which is gp$-$ closed set.

Remark 3.3. *In Example 3.2, if we take $S = \{a, b\}$, then $S \in GLPC^*(X, \tau_{12})$. But, $int_{12}(S \cup (X \backslash cl_{12}(S)))= int_{12}\{a, b\} = \{a\} \not\supseteq S$ (by Theorem 3.2) .*

Theorem 3.3. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, for a subset S in (X, τ_{12}). If $S \in GLPC^{**}(X, \tau_{12})$, then $P \cap int_{12}(S) \subseteq S$ for some gp-closed set P.*

Proof. Let $S \in GLPC^{**}(X, \tau_{12})$. Then $\exists$ gp-closed set P and P-open set G such that $S = P \cap G$. Since, $S \subseteq P$, then $int_{12}(S) \subseteq P$, which implies that $P \cap int_{12}(S) \cap S \subseteq S \cap P = S$. Hence, $P \cap int_{12}(S) \subseteq S$.

Definition 3.5. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, a set A is pairwise dense (P$-$ dense) in X if $cl_{12}(A) = X$.*

Definition 3.6. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, a space X is called pairwise-submaximal [briefly P-submaximal] if every P-dense subset is P-open set.*

Definition 3.7. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. A space X is called generalized pairwise-submaximal if every P-dense subset is gp-open set.*

Corollary 3.1. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, $P(X) = LPC(X, \tau_{12})$ if and only if the space X is P-submaximal.*

Proof. Necessity: Let S be a P-dense subset of (X, τ_{12}). Then, $cl_{12}(S) = X$ and by Theorem3.1 (4) we get, $S = S \cup (X \backslash cl_{12}(S))$. So, $S \in LPC(X, \tau_{12})$ then, S is P-open set. Hence, X is P-submaximal.
Sufficiency: Let $S \in P(X)$ and take $U = S \cup (X \backslash cl_{12}(S))$. Then, $cl_{12}(U) = cl_{12}(S) \cup cl_{12}[cl_{12}(S)]^c$. So, $cl_{12}(U) = X$ then we get, U is P- open set. But, $S = U \cap cl_{12}(S)$ and $cl_{12}(S)$ is P-closed set. This implies that $S \in LPC(X, \tau_{12})$. Therefore, $P(X) \subseteq LPC(X, \tau_{12})$ but in general we have, $LPC(X, \tau_{12}) \subseteq P(X)$. Hence, $LPC(X, \tau_{12}) = P(X)$.

Corollary 3.2. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, $P(X) = GLPC^*(X, \tau_{12})$ if and only if the space X is gp-submaximal.*

Proof. Necessity: Let S be a P-dense subset of (X, τ_{12}). Then, $cl_{12}(S) = X$ and by Theorem3.2 (4) we get, $S = S \cup (X \backslash cl_{12}(S))$. So, $S \in GLPC^*(X, \tau_{12})$ then, S is gp-open set. Hence, X is a gp-submaximal.
Sufficiency: Let $S \in P(X)$ and take $U = S \cup (X \backslash cl_{12}(S))$. Then, $cl_{12}(U) = cl_{12}(S) \cup cl_{12}[cl_{12}(S)]^c$. So, $cl_{12}(U) = X$ then we get, U is gp- open set. But, $S = U \cap cl_{12}(S)$ and $cl_{12}(S)$ is P-closed set. This implies that $S \in GLPC^*(X, \tau_{12})$ therefore, $P(X) \subseteq GLPC^*(X, \tau_{12})$ but in general we have, $GLPC^*(X, \tau_{12}) \subseteq P(X)$. Hence, $GLPC^*(X, \tau_{12}) = P(X)$.

Remark 3.4. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. If the space X is P-submaximal, then it is gp$-$submaximal but the converse is not true. (See Example 3.2 which shows $P(X) = GLPC^*(X, \tau_{12})$ but $LPC(X, \tau_{12}) \neq P(X)$).*

Theorem 3.4. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Suppose A and B be subset of X.*

1. *If $A \in GLPC^{**}(X, \tau_{12})$ and B is P-closed , then $A \cap B \in GLPC^{**}(X, \tau_{12})$.*

2. *If $A \in GLPC(X, \tau_{12})$ and B is P-closed , then, $A \cap B \in GLPC(X, \tau_{12})$.*

Proof. (1) Let $A \in GLPC^{**}(X, \tau_{12})$. Then, $\exists$ P-open set Q and gp- closed set P such that $A = P \cap Q$. Take $B \subseteq X$ which is P-closed set this implies that $A \cap B = Q \cap P \cap B$. But, $P \cap B$ is $gp-$ closed set (by Corollary 2.2). Hence, $A \cap B \in GLPC^{**}(X, \tau_{12})$.
(2) Let $A \in GLPC(X, \tau_{12})$. Then, $\exists gp$- open set F and gp- closed set G such that $A = F \cap G$. Take $B \subseteq X$ which is P-closed set therefore, $B \cap G$ is gp- closed set. Hence, $A \cap B \in GLPC(X, \tau_{12})$.

Remark 3.5. *In Theorem 3.4 we can note that:*
(1) if B is P-open set, then $A \cap B$ is not necessary belong to $GLPC^{**}(X, \tau_{12})$.
(2) if B is P-open set, then $A \cap B$ is not necessary belong to $GLPC(X, \tau_{12})$, (see Example 3.4).

Example 3.4. *Let $X = \{1, 2, 3, 4, 5\}$, $\tau_1 = \{X, \phi, \{1, 2, 3\}\}$ and $\tau_2 = \{X, \phi, \{3, 4, 5\}\}$. Then, $\tau_{12} = \{X, \phi, \{1, 2, 3\}, \{3, 4, 5\}\}$, $\tau_{12}^{c} = \{X, \phi\{4, 5\}, \{1, 2\}\}$. Take $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$. So, $A \in GLPC^{**}$ (X, τ_{12}), $B \in \tau_{12}$ but, $A \cap B = \{3\} \notin GLPC(X, \tau_{12})$ and $A \cap B \notin GLPC^{**}$ (X, τ_{12}).*

Theorem 3.5. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Suppose A and B are subsets of X and $A \subseteq B$. If B is gp-closed set in (X, τ_{12}) and $A \in GLPC^{**}(B, \tau_{12}|_B)$, then, $A \in GLPC^{**}(X, \tau_{12})$.*

Proof. Let B be a gp- closed set in X and $A \in GLPC^{**}(B, \tau_{12}|_B)$. Then, there exist P-open set Q of $(B, \tau_{12}|_B)$ and gp-closed set M of $(B, \tau_{12}|_B)$ such that, $A = Q \cap M$ but we have $Q = B \cap S$ for some $S \in \tau_{12}$ and M is gp- closed set in $\tau_{12}|_B$, then M is gp- closed set in τ_{12} (Theorem 2.3). So, $A = (B \cap S) \cap M = S \cap M$ since, $A \subseteq B, M \subseteq B$. Hence, $A \in GLPC^{**}(X, \tau_{12})$.

Definition 3.8. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then,*

1. *A set S is called locally pairwise open set (briefly, LPO - set) if $S^c \in LPC(X, \tau_{12})$. The family of all locally pairwise open set denoted by $LPO(X, \tau_{12})$.*

2. *A set S is called generalized locally pairwise open set (briefly, GLPO- set) if $S^c \in GLPC(X, \tau_{12})$. The family of all generalized locally pairwise open set denoted by $GLPO(X, \tau_{12})$.*

3. *A set S is called generalized locally pairwise open star set (briefly, GLPO*- set) if $S^c \in GLPC^*(X, \tau_{12})$. The family of all generalized locally pairwise open star set denoted by $GLPO^*(X, \tau_{12})$.*

4. *A set S is called generalized locally pairwise open star star set (briefly, GLPO**- set) if $S^c \in GLPC^{**}(X, \tau_{12})$. The family of all generalized locally pairwise open star stare set denoted by $GLPO^{**}$ (X, τ_{12}).*

Proposition 3.2. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, $S \in LPO(X, \tau_{12})$ implies that $\exists$ P-open set G and P-closed set F such that $S = G \cup F$.*

Proof. It is follows from definition of $LPC(X, \tau_{12})$.

Proposition 3.3. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, $S \in GLPO(X, \tau_{12})$ implies that $\exists$ gp-open set G and gp-closed set F such that $S = G \cup F$.*

Proof. It is follows from definition of $GLPC(X, \tau_{12})$.

Proposition 3.4. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, $S \in GLPO^*(X, \tau_{12})$ implies that $\exists$ P-open set G and gp-closed set F such that $S = G \cup F$.*

Proof. It is follows from definition of $GLPC^*(X, \tau_{12})$.

Proposition 3.5. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) is the supra topological space generated by τ_1 and τ_2. Then, $S \in GLPO^{**}(X, \tau_{12})$ implies that $\exists$ gp-open set G and P-closed set F such that $S = G \cup F$.*

Proof. It is straightforward from definition of $GLPC^{**}(X, \tau_{12})$.

Proposition 3.6. *Let S be a subset in (X, τ_1, τ_2).*

1. *If S is locally pairwise open set, then $S \in GLPO^*(X, \tau_{12})$ and $S \in GLPO^{**}(X, \tau_{12})$. In other words, $LPO(X, \tau_1, \tau_2) \subseteq GLPO^*(X, \tau_1, \tau_2) \cap GLPO^{**}(X, \tau_1, \tau_2)$.*

2. *If $S \in GLPO^*(X, \tau_{12})$ or $S \in GLPO^{**}(X, \tau_{12})$, then S is generalized locally pairwise open set. In other words, $GLPO^*(X, \tau_{12}) \cup GLPO^{**}(X, \tau_{12}) \subseteq GLPO(X, \tau_{12})$.*

Proof. (1). Let $S \in LPO(X, \tau_{12})$. Then, $S^c \in LPC(X, \tau_{12})$. By Proposition 3.1 (1), $S^c \in GLPC^*(X, \tau_{12}) \cap GLPC^{**}(X, \tau_{12})$. Hence, $S \in GLPO^*(X, \tau_{12}) \cap GLPO^{**}(X, \tau_{12})$.

(2). Let $S \in GLPO^*(X, \tau_{12})$. Then, $S^c \in GLPC^*(X, \tau_{12})$. By Proposition 3.1 (2), $S^c \in GLPC(X, \tau_{12})$, then $S \in GLPO(X, \tau_{12})$. Let $S \in GLPO^{**}(X, \tau_{12})$. Then, $S^c \in GLPC^{**}(X, \tau_{12})$. By Proposition 3.1 (2), $S^c \in GLPC(X, \tau_{12})$, then $S \in GLPO(X, \tau_{12})$. Note that the family of $LPO(X, \tau_{12})$ may be not a topology and may be not supra topology on X as shown in the following Example 3.5.

Example 3.5. *Let* $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{b, c\}\}$. *Then,* $\tau_{12} = \{X, \phi, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, $\tau_{12}^c = \{X, \phi\{c, d\}, \{a, d\}, \{d\}\}$ *and* $LPO(X, \tau_{12}) = \{X, \phi, \{a, b\}, \{b, c\}, \{a, b, c\}, \{c, d\}, \{a, d\}, \{d\}, \{a, b, d\}, \{b, c, d\}\}$. *We have,* $\{a, b\} \cap \{a, d\} = \{a\} \notin LPO(X, \tau_{12})$ *and* $\{c, d\} \cup \{a, d\} = \{a, c, d\} \notin LPO(X, \tau_{12})$. *Therefore,* $LPO(X, \tau_{12})$ *neither topology not supra topology on* X.

Theorem 3.6. *Let* (X, τ_1, τ_2) *be a bitopological space and* (X, τ_{12}) *is the supra topological space generated by* τ_1 *and* τ_2. *Then, for a subset* S *in* (X, τ_{12}) *the following are equivalent:*

1. $S \in LPO(X, \tau_{12})$.

2. $S = F \cup int_{12}(S)$ *for some P-closed set* F.

3. $S^c \cup int_{12}(S)$ *is P-open set.*

4. $S \setminus int_{12}(S)$ *is P-closed set.*

5. $S \supseteq cl_{12}(S \setminus int_{12}(S))$.

Proof. (1) $\implies$ (2). Let $S \in LPO^*(X, \tau_{12})$. Then $S^c \in LPC(X, \tau_{12})$. By Theorem 3.1, $S^c = G \cap cl_{12}(X \setminus S)$, G is P-open set. Then, $S = G^c \cup (cl_{12}(X \setminus S))^c$. Hence, $S = G^c \cup int_{12}(S)$.

(2) $\implies$ (1). Let $S = F \cup int_{12}(S)$, F is P-closed set and we have $int_{12}(S)$ is P-open set. Hence, $S \in LPO(X, \tau_{12})$.

(2) $\implies$ (3). Let $S = F \cup int_{12}(S)$ for some P-closed set F. Then, $S^c \cup int_{12}(S) = (F \cup int_{12}(S))^c \cup int_{12}(S) = (F^c \cap (int_{12}S)^c) \cup int_{12}(S)$ $= (F^c \cup int_{12}S) \cap (((int_{12}S)^c) \cup int_{12}S) = (F^c \cup int_{12}S) \cap X = F^c \cup int_{12}(S)$. Since, F^c is P-open and $int_{12}(S)$ is P-open, then $S^c \cup int_{12}S$ is a P-open set.

(3) $\implies$ (2). Let $S^c \cup int_{12}(S)$ be P-open set. Then, $(S^c \cup int_{12}(S))^c = S \cap (int_{12}S)^c$ is P-closed set. Take $F = S \cap (int_{12}S)^c$. Now, $F \cup int_{12}S = S \cap (int_{12}S)^c \cup int_{12}S = (S \cup int_{12}(S)) \cap (int_{12}S)^c \cup int_{12}S = (S \cup int_{12}(S)) = S$. Hence, $S = F \cup int_{12}S$.

(3) $\implies$ (4). Let $S^c \cup int_{12}S$ be P-open set. Then, $(S^c \cup int_{12}S)^c$ is P-closed. Hence, $S \cap (int_{12}(S))^c = S \setminus int_{12}S$ is a P-closed set.

(4) $\implies$ (3). Let $S \setminus int_{12}S$ be P-closed set. Then, $(S \cap (int_{12}S)^c)^c = S^c \cup int_{12}S$ is a P-open set.

(4) $\implies$ (5). Let $S \setminus int_{12}S$ be P-closed set. Then, $S \supseteq S \setminus int_{12}$. But $cl_{12}(S \setminus int_{12}S) = S \setminus int_{12}S$. This implies that $S \supseteq cl_{12}(S \setminus int_{12}(S))$.

(5) $\implies$ (4). Let $S \supseteq cl_{12}(S \setminus int_{12}(S))$. Then, $S \cap (int_{12}S)^c \supseteq cl_{12}(S \setminus int_{12}S) \cap (int_{12}(S))^c \supseteq cl_{12}(S \setminus int_{12}S) \cap cl_{12}(int_{12}(S))^c \supseteq cl_{12}((S \setminus int_{12}S) \cap (int_{12}(S))^c)$ but, $S \cap (int_{12}S)^c \subseteq cl_{12}(S \cap (int_{12}S)^c)$. Hence, $cl_{12}(S \cap (int_{12}S)^c) = S \cap (int_{12}S)^c$. Consequentially, $S \cap (int_{12}S)^c = (S \setminus int_{12}S)$ is a P-closed. Hence, the result.

Theorem 3.7. *Let* (X, τ_1, τ_2) *be a bitopological space and* (X, τ_{12}) *is the supra topological space generated by* τ_1 *and* τ_2. *Then, for a subset* S *in* (X, τ_{12}) *the following are equivalent:*

1. $S \in GLPO^*(X, \tau_{12})$.

2. $S = F \cup int_{12}(S)$ *for some gp-closed set* F.

3. $S^c \cup int_{12}(S)$ *is gp-open set.*

4. $S \setminus int_{12}(S)$ *is gp-closed set.*

Proof. (1) $\implies$ (2). Let $S \in GLPO^*(X, \tau_{12})$. Then $S^c \in GLPC^*(X, \tau_{12})$. By Theorem 3.2, $S^c = P \cap cl_{12}(S)$, for some gp-open set P. Therefore, $S = P^c \cup (cl_{12}(X \setminus S))^c$. Hence, $S = P^c \cup int_{12}(S)$.

(2) $\implies$ (1). Let $S = F \cup int_{12}(S)$, F is P-closed set and we have $int_{12}(S)$ is P-open set. Hence, $S \in LPO(X, \tau_{12})$.

(2) $\implies$ (3). Let $S = F \cup int_{12}(S)$ for some gp-closed set F. Then, $S^c \cup int_{12}(S) = (F \cup int_{12}(S))^c \cup int_{12}(S) = (F^c \cap (int_{12}S)^c) \cup int_{12}(S)$ $= (F^c \cup int_{12}S) \cap (((int_{12}S)^c) \cup int_{12}S) = (F^c \cup int_{12}S) \cap X = F^c \cup int_{12}(S)$. Since, F^c is gp-open and $int_{12}(S)$ is P-open. Now, F is $gp-$closed set and $(int_{12}S)^c$ is P-closed set. Then by Corollary 2.2, $F \cap (int_{12}S)^c$ is gp-closed set and therefore $F^c \cup int_{12}S$ is gp-open set. Hence, $S^c \cup int_{12}S$ is a gp-open set.

(3) $\implies$ (2). Taking the complement of no(3). From(3) we have $(S^c \cup int_{12}(S))^c = S \cap (int_{12}S)^c = F$ which is gp-closed set. Now, $F \cup int_{12}S = (S \cap (int_{12}S)^c) \cup int_{12}S = (S \cup int_{12}S) \cap X = S \cup int_{12}(S)$.

(3) $\implies$ (4). Let $S \cup int_{12}S$ be gp-open set. Then, $(S^c \cup int_{12}S)^c$ is gp-closed. Hence, $S \cap (int_{12}(S))^c = S \backslash int_{12}S$ is a gp-closed set.

(4) $\implies$ (3).Let $S \backslash int_{12}S$ be gp-closed set. Then, $(S \cap (int_{12}S)^c)^c = S^c \cup int_{12}S$ is gp-open set. Hence, the result.

Theorem 3.8. *Let (X, τ_1, τ_2) be a bitopological space and (X, τ_{12}) be a supra topological space generated by τ_1 and τ_2. Suppose A and B be subsets of X.*

1. *If $A \in GLPO^{**}(X, \tau_{12})$ and B is P-open , then $A \cup B \in GLPO^{**}(X, \tau_{12})$.*

2. *If $A \in GLPO(X, \tau_{12})$ and B is P-open , then $A \cup B \in GLPO(X, \tau_{12})$.*

Proof. (1). Let $A \in GLPO^{**}(X, \tau_{12})$. Then, $\exists$ P-closed set Q and gp- open set P such that $A = P \cup Q$. Take $B \subseteq X$ which is P-open set this implies that $A \cup B = Q \cup P \cup B$. But, $P \cup B$ is gp-open set. Hence, $A \cup B \in GLPO^{**}(X, \tau_{12})$.
(2). Let $A \in GLPO(X, \tau_{12})$. Then, $\exists gp$- open set G and gp- closed set F such that $A = F \cup G$. Take $B \subseteq X$ which is P-open set, then $B \cup G$ is gp- open set. Hence, $A \cup B \in GLPO(X, \tau_{12})$.

Remark 3.6. *In Theorem 3.4 we can note that:*
(1) if B is P-closed set, then $A \cup B$ is not necessary belong to $GLPO^{**}(X, \tau_{12})$.
(2) if B is P-closed set, then $A \cup B$ is not necessary belong to $GLPO(X, \tau_{12})$. In Example 3.4 we have $A = \{4,5\} \in GLPO^{**}$ and $B = \{1,2\} \in \tau_{12}^c$ but, $A \cup B = \{1, 2, 4, 5\} \notin GLPO(X, \tau_{12})$ and $A \cup B \notin GLPO^{**}(X, \tau_{12})$.

4   Generalized locally pairwise closed-Functions and some of their properties

Let $f : X \longrightarrow Y$ be a function between two bitopolgical spaces (X, τ_1, τ_2) and (Y, ν_1, ν_2). We use the notations in this section LPC- continuity, LPC-irresoluteness and sub-LPC- continuity. After that we define generalization of LPC-irresolute functions, LPC-continuous function and sub-LPC-continuous function and study some of their properties.

Definition 4.1. *Let $f : X \longrightarrow Y$ be a function between two bitopolgical spaces (X, τ_1, τ_2) and (Y, ν_1, ν_2). Suppose (X, τ_{12}) and (Y, ν_{12}) are the supra topological spaces associated to (X, τ_1, τ_2) and (Y, ν_1, ν_2) ,respectively. Then, a function f is called*

1. *LPC-irresolute if $f^{-1}(A) \in LPC(X, \tau_{12})$ $\forall A \in LPC(Y, \nu_{12})$.*

2. *LPC- continuous if $f^{-1}(G) \in LPC(X, \tau_{12})$, $\forall G \in \nu_{12}$.*

3. *Sub- LPC-continuous if there is a base β for (Y, ν_{12}) such that $f^{-1}(V) \in LPC(X, \tau_{12})$, $\forall V \in \beta$.*

Definition 4.2. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \nu_1, \nu_2)$ be a function. Suppose (X, τ_{12}) and (Y, ν_{12}) are the supra topological spaces associated to (X, τ_1, τ_2) and (Y, ν_1, ν_2), respectively. Then, a function f is called GLPC-irresolute [resp., GLPC*-irresolute , GLPC**- irresolute] if $f^{-1}(V) \in GLPC(X, \tau_{12})$ [resp., $f^{-1}(V) \in GLPC^*(X, \tau_{12})$, $f^{-1}(V) \in GLPC^{**}(X, \tau_{12})]$, $\forall V \in GLPC(Y, \nu_{12})$ [resp., $\forall V \in GLPC^*(Y, \nu_{12})$, $\forall V \in GLPC^{**}(Y, \nu_{12})]$.*

Definition 4.3. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \nu_1, \nu_2)$ be a mapping. Suppose (X, τ_{12}) and (Y, ν_{12}) are the supra topological spaces associted to (X, τ_1, τ_2) and (Y, ν_1, ν_2), respectively. Then, a function f is called GLPC$-$ continuous [resp., GLPC*$-$ continuous, GLPC**$-$ continuous] if $f^{-1}(V) \in GLPC(X, \tau_{12})$ [resp., $f^{-1}(V) \in GLPC^*(X, \tau_{12})$, $f^{-1}(V) \in GLPC^{**}(X, \tau_{12})$) $]\forall V \in \nu_{12}$.*

Theorem 4.1. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \nu_1, \nu_2)$ be a mapping. Suppose (X, τ_{12}) and (Y, ν_{12}) are the supra topological spaces associated to by (X, τ_1, τ_2) and (Y, ν_1, ν_2), respectively.*

1. *If f is LPC-continuous, then it is GLPC*-continuous and GLPC**-continuous.*

2. *If f is GLPC*-continuous or GLPC**-continuous, then it is GLPC-continuous.*

3. *If f is GLPC-irresolute (resp. GLPC*-irresolute, GLPC**-irresolute), then it is GLPC-continuous (resp. GLPC*-continuous, GLPC**-continuous)*

4. *If f is P*-continuous function and P*$-$ closed function, then f is GLPC-irresolute function, GLPC*-irresolute function and GLPC**-irresolute function.*

Proof.

1. Suppose that f is LPC- continuous. Let $V \in \nu_{12}$. Then, $f^{-1}(V) \in LPC(X, \tau_{12})$. By using Proposition 3.1 we get $f^{-1}(V) \in GLPC^*(X, \tau_{12})$ and $f^{-1}(V) \in GLPC^{**}(X, \tau_{12})$. Hence, f is a $GLPC^*$- continuous and $GLPC^{**}$- continuous.

2. It is follows from Proposition 3.1. Since every P-open set is gp -open set and every P-closed set is gp - closed set.

3. Also can be done like no.(2)

4. Let f be P^*-continuous and P^* closed function. Suppose that $A \in GLPC^*(Y, \nu_{12})$ we want to prove that $f^{-1}(A) \in GLPC^*(X, \tau_{12})$. Then, $\exists$ P-closed set F and gp- open set G such that $A = G \cap F$ by using Theorem 2.14 we get $f^{-1}(G)$ is gp-open set and $f^{-1}(F)$ is P-closed set this implies that, $f^{-1}(A) = f^{-1}(G \cap F) = f^{-1}(G) \cap f^{-1}(F)$. Hence, $f^{-1}(A) \in GLPC^*(X, \tau_{12})$. Similarly, we can show that f is $GLPC$-irresolute function and $GLPC^{**}$- irresolute function.

Remark 4.1. *The converse of Theorem 4.1 need not be true in general as shown in the following examples.*
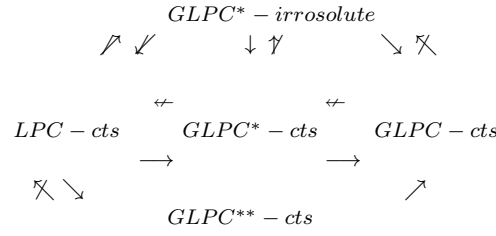
Example 4.1. *Let* $X = Y = \{1, 2, 3\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \{X, \phi, \{1\}\}$, $\nu_1 = \{X, \phi\}$ *and* $\nu_2 = P(Y)$. *Then,* $\tau_{12} = \{X, \phi, \{1\}\}$, $\nu_{12} = P(Y)$. *Take* $f : X \longrightarrow Y; f(x) = x \ \forall x \in X$ *which is the identity function.* $GLPC(X, \tau_{12}) = GLPC^*(X, \tau_{12}) = GLPC^{**}(X, \tau_{12}) = P(X)$. *So,* $LPC(X, \tau_{12}) = \{X, \phi, \{1\}, \{2, 3\}\}$. $GLPC(Y, \nu_{12}) = GLPC^*(Y, \nu_{12}) = GLPC^{**}(Y, \nu_{12}) = LPC(Y, \nu_{12}) = P(Y)$. *We have* f *is not* $LPC-$ *continuous, but it is* $GLPC^*-$ *continuous,* $GLPC^{**}-$ *continuous and* $GLPC^*-$ *irresolute.*

Example 4.2. *Let* $X = Y = \{1, 2, 3\}$, $\tau_1 = \{X, \phi, \{1\}\}$, $\tau_2 = \{X, \phi, \{1, 2\}\}$, $\nu_1 = \{X, \phi\}$ *and* $\nu_2 = \{X, \phi, \{1\}\}$. *Then,* $\tau_{12} = \{X, \phi, \{1\}, \{1, 2\}\}$, $\nu_{12} = \{X, \phi, \{1\}\}$. *Take* $f : X \longrightarrow Y; f(1) = f(3) = 1, f(2) = 2$. *Hence,* $GLPC(Y, \nu_{12}) = GLPC^*(X, \nu_{12}) = GLPC^{**}(X, \nu_{12}) = P(X)$. *Then,* f *is not* $GLPC^*-$ *continuous but it is* $GLPC-$ *continuous and it is* $GLPC^{**}-$ *continuous.*

Example 4.3. *Let* $X = Y = \{1, 2, 3\}$, $\tau_1 = \{X, \phi, \{1\}\}$, $\tau_2 = \{X, \phi, \{1, 2\}\}$, $\nu_1 = \{X, \phi\}$ *and* $\nu_2 = \{X, \phi, \{1\}\}$. *Then,* $\tau_{12} = \{X, \phi, \{1\}, \{1, 2\}\}$, $\nu_{12} = \{X, \phi, \{1\}\}$. *Take* $f : X \longrightarrow Y; f(x) = x$ *which is the identity function. Then,* $\{1, 3\} \in GLPC^*(y, \nu_{12})$ *but,* $\{1, 3\} \notin GLPC^*(X, \tau_{12})$. *Hence,* f *is not* $GLPC^*-$ *irresolute but it is* $LPC-$ *continuous.*

On account of Theorem 4.1 and Examples 4.1 , 4.2 and 4.3 we have the following corollary.

Corollary 4.1. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \nu_1, \nu_2)$ *be a mapping. Then, we have the following diagram:*

$$GLPC^* - irrosolute$$

$$\nearrow \swarrow \qquad \downarrow \Upsilon \qquad \searrow \nwarrow$$

$$LPC - cts \xleftarrow{\ \ \ } GLPC^* - cts \xleftarrow{\ \ \ } GLPC - cts$$

$$\longrightarrow \qquad\qquad \longrightarrow$$

$$\nwarrow \searrow \qquad\qquad\qquad \nearrow$$

$$GLPC^{**} - cts$$

Theorem 4.2. *If* $f : X \longrightarrow Y$ *is* $GLPC^{**}-$ *continuous and* B *is P-closed set in* (X, τ_{12}), *then the restriction of* f *to* B *say* $f|_B : (B, \tau_{12}|_B) \longrightarrow (Y, \nu_{12})$ *is* $GLPC^{**}-$ *continuous.*

Proof. Let V be a P-open set of (Y, ν_{12}). Then, $f^{-1}(V) = G \cap F$ such that $G \in \tau_{12}$ and F is gp- closed set in (X, τ_{12}). So, $f^{-1}|_B(V) = (G \cap B) \cap (F \cap B) \in GLPC^{**}(B, \tau_{12}|_B)$. By Theorem 2.2 and Corollary 2.2 we have $(F \cap B)$ is gp-closed set in $(B, \tau_{12}|_B)$ and $(G \cap B)$ is P-open in $(B, \tau_{12}|_B)$. Hence, $f|_B$ is a $GLPC^{**}-$ continuous.

Remark 4.2.

1. *The composition of two GLPC-irresolute (resp. GLPC**-irresolute, GLPC**-irresolute) functions is clearly GLPC-irresolute (resp. GLPC*-irresolute, GLPC**-irresolute) function.*

2. *The composition of* $g \circ f$ *GLPC-continuous [resp., GLPC**-continuous, GLPC**-continuous] function* f *and* P^*-continuous function g *is clearly GLPC-continuous [resp., GLPC**-continuous, GLPC**-continuous] function.*

Definition 4.4. *Let* $f : X \longrightarrow Y$ *be a mapping between bitoplogical spaces* (X, τ_1, τ_2) *and* (Y, ν_1, ν_2). *Suppose* (X, τ_{12}) *and* (Y, ν_{12}) *are supra topological spaces generated by* (X, τ_1, τ_2) *and* (Y, ν_1, ν_2), *respectively. Then, a function* f *is called*

1. *LPO-irresolute if* $f^{-1}(A) \in LPO(X, \tau_{12}) \ \forall A \in LPO(Y, \nu_{12})$.

2. *LPO- continuous if* $f^{-1}(G) \in LPO(X, \tau_{12})$, $\forall G \in \nu_{12}^c$.

Definition 4.5. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \nu_1, \nu_2)$ *be a mapping. Suppose* (X, τ_{12}) *and* (Y, ν_{12}) *are supra topological spaces associated to* (X, τ_1, τ_2) *and* (Y, ν_1, ν_2), *respectively. Then,* f *is called GLPO-irresolute [resp., GLPO**-irresolute , GLPO**- irresolute] if $f^{-1}(V) \in GLPO(X, \tau_{12})$[resp., $f^{-1}(V) \in GLPO^*(X, \tau_{12})$, $f^{-1}(V) \in GLPO^{**}(X, \tau_{12})$], $\forall V \in GLPO(Y, \nu_{12})$ [resp., $\forall V \in GLPO^*(Y, \nu_{12})$], $\forall V \in GLPO^{**}(Y, \nu_{12})$.*

Definition 4.6. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \nu_1, \nu_2)$ *be a mapping. Suppose* (X, τ_{12}) *and* (Y, ν_{12}) *are supra topological spaces associated to* (X, τ_1, τ_2) *and* (Y, ν_1, ν_2), *respectively. Then,* f *is called GLPO$-$ continuous [resp., GLPO$^*-$ continuous, GLPO$^{**}-$ continuous] if $f^{-1}(V) \in GLPO(X, \tau_{12})$ [resp., $f^{-1}(V) \in GLPO^*(X, \tau_{12})$, $f^{-1}(V) \in GLPO^{**}(X, \tau_{12})$] $\forall V \in \nu_{12}^c$.*

Theorem 4.3. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \nu_1, \nu_2)$ be a mapping. Suppose (X, τ_{12}) and (Y, ν_{12}) are supra topological spaces associated to (X, τ_1, τ_2) and (Y, ν_1, ν_2), respectively.*

1. *If f is LPO-continuous, then it is GLPO*-continuous and GLPO**-continuous.*

2. *If f is GLPO*-continuous or GLPO**-continuous, then it is GLPO-continuous.*

3. *If f is GLPO-irresolute [resp., GLPO*-irresolute, GLPO**-irresolute], then it is GLPO-continuous [resp., GLPO*-continuous, GLPO**-continuous].*

4. *If f is P*-continuous function and P*− open function, then f is GLPO-irresolute function, GLPO*-irresolute function and GLPO**-irresolute function.*

Proof.

1. Suppose that f is *LPO*- continuous. Let $V \in \nu_{12}$. Then, $f^{-1}(V)$ is locally pairwise open set in (X, τ_{12}). By using Proposition 3.1 we get $f^{-1}(V) \in GLPO^*(X, \tau_{12})$ and $f^{-1}(V) \in GLPO^{**}(X, \tau_{12})$. Hence, f is a $GLPO^*$- continuous and $GLPO^{**}$- continuous.

2. It is trivial by definitions. Since, every P-open set is gp -open set and every P-closed set is gp - closed set.

3. Also can be done like no.(2)

4. Let f be P^*-continuous function and P^* open function. Suppose that $A \in GLPO^*(Y, \nu_{12})$ we want to prove that $f^{-1}(A) \in GLPO^*(X, \tau_{12})$. Then, $\exists F$ gp-closed set and G is P- open set such that $A = G \cup F$ by using Theorem 2.14 we get $f^{-1}(F)$ is gp-closed set and $f^{-1}(G)$ is P-open set this implies that, $f^{-1}(A) = f^{-1}(G \cup F) = f^{-1}(G) \cup f^{-1}(F)$. Hence, $f^{-1}(A) \in GLPO^*(X, \tau_{12})$. Similarly, we can show that f is $GLPO$- irresolute function and $GLPO^{**}$- irresolute function.

Remark 4.3. *The converse of Theorem 4.3 need not be true in general as shown by the following examples.*
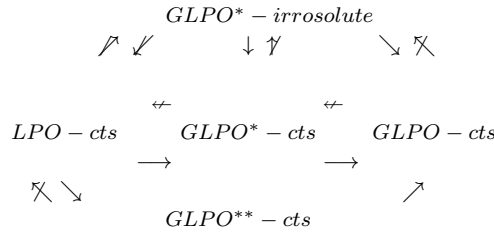
Example 4.4. *Let $X = Y = \{1, 2, 3\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \{X, \phi, \{1\}\}$, $\nu_1 = \{X, \phi\}$ and $\nu_2 = P(Y)$. Then, $\tau_{12} = \{X, \phi, \{1\}\}$, $\nu_{12} = P(Y)$. Take $f : X \longrightarrow Y; f(x) = x$ $\forall x \in X$ which is the identity function. $GLPO(X, \tau_{12}) = GLPO^*(X, \tau_{12}) = GLPO^{**}(X, \tau_{12}) = P(X)$. So, $LPO(X, \tau_{12}) = \{X, \phi, \{1\}, \{2, 3\}\}$. $GLPO(Y, \nu_{12}) = GLPO^*(Y, \nu_{12}) = GLPO^{**}(Y, \nu_{12}) = LPO(Y, \nu_{12}) = P(Y)$. We have f is not $LPO-$ continuous, but it is $GLPO^*-$ continuous, $GLPO^{**}-$ continuous and $GLPO^*-$ irresolute.*

Example 4.5. *Let $X = Y = \{1, 2, 3\}$, $\tau_1 = \{X, \phi, \{1\}\}$, $\tau_2 = \{X, \phi, \{1, 2\}\}$, $\nu_1 = \{X, \phi\}$ and $\nu_2 = \{X, \phi, \{1\}\}$. Then, $\tau_{12} = \{X, \phi, \{1\}, \{1, 2\}\}$, $\nu_{12} = \{X, \phi, \{1\}\}$. Take $f : X \longrightarrow Y; f(1) = f(3) = 1, f(2) = 2$. Hence, $GLPO(Y, \nu_{12}) = GLPO^*(X, \nu_{12}) = GLPO^{**}(X, \nu_{12}) = P(X)$. Then, f is not $GLPO^*-$ continuous but it is $GLPO-$ continuous and it is $GLPO^{**}-$ continuous.*

Example 4.6. *Let $X = Y = \{1, 2, 3\}$, $\tau_1 = \{X, \phi, \{1\}\}$, $\tau_2 = \{X, \phi, \{1, 2\}\}$, $\nu_1 = \{X, \phi\}$ and $\nu_2 = \{X, \phi, \{1\}\}$. Then, $\tau_{12} = \{X, \phi, \{1\}, \{1, 2\}\}$, $\nu_{12} = \{X, \phi, \{1\}\}$. Take $f : X \longrightarrow Y; f(x) = x$ which is the identity function. Then, $\{2\} \in GLPO^*(y, \nu_{12})$ but, $\{2\} \notin GLPO^*(X, \tau_{12})$. Hence, f is not $GLPO^*-$ irresolute but it is $LPO-$continuous.*

On account of Theorem 4.3 and Examples 4.4, 4.5 and 4.6 we have the following corollary.

Corollary 4.2. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \nu_1, \nu_2)$ be a mapping. Then, we have the following diagram:*

$$
\begin{array}{c}
GLPO^* - irrosolute \\
\nearrow \swarrow \qquad \downarrow \nnwarrow \qquad \searrow \nnwarrow \\
\\
\overset{\nwarrow}{} \qquad\qquad \overset{\nwarrow}{} \\
LPO - cts \qquad GLPO^* - cts \qquad GLPO - cts \\
\longrightarrow \qquad\quad \longrightarrow \\
\nwarrow \searrow \qquad\qquad\qquad \nearrow \\
GLPO^{**} - cts
\end{array}
$$

Remark 4.4.

1. *The composition of two GLPO-irresolute [resp., GLPO*-irresolute, GLPO**-irresolute] functions is clearly GLPO-irresolute [resp., GLPO*-irresolute, GLPO**-irresolute] function.*

2. *The composition of $g \circ f$ GLPO-continuous [resp., GLPO*-continuous, GLPO**-continuous] function f and P*-continuous function g is clearly GLPO-continuous [resp., GLPO*-continuous, GLPO**-continuous] function.*

References

[1] N. Levine, Generalized closed set in topology, *Rend. Circ. Mat. Palermo.* 19 (2) (1970), 89-96.

[2] J.C.Kelly, Bitopological spaces, *Proc Lond Math Soc.* 13 (1963), 71-79.

[3] A. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, On supra topological space, *Indian J. Pure Appl. Math.* 14 (1983), 502-510.

[4] A. Kandil and S. A. El-Sheikh, On bitopological spaces, first Intern.conf. *on Mathematics and Statistics, Assuit University* 3 (1990), 73-96.

[5] M. C. Datta, Projective bitopological spaces, *Journal of the Australian Math.* Soc. 13 (1972), 327-334.

[6] B.P. Dvalishvili, Bitopological spaces theory, relation with generalized algebraic structure and applications, *North Holland* (2005).

[7] S.A.EL-Sheikh, Dimension theory of bitopological spaces, Master Thesis, Ain Shams University, Cairo, Egypt(1987).

[8] S.A.EL-Sheikh, A new approach to fuzzy bitopologicl space, *INS*, 137 (2001), 283-301.

[9] M. K. Singal and A. R. Singal, Some more seperation axioms in bitopological spaces, *Ann. soc. Sci. Bruelles* 84 (1970), 207-230.

[10] M. Ganster and I.L. Reilly, Locally closed sets and LC-continuous functions, J. Math and math Sci. 12 (3) (1989), 417-424.

[11] K. Balachandran, P. Sundram and H.Maki, Generalized locally closed set and GLC- continuous function. *Indian J. Pure Appl. Math.* 27 (3)(1996), 235-244.

[12] S.A.EL-Sheikh, Some bitopological properties via grills, *International Mathematical Forum.* 13 (7) (2014), 335-353.

[13] M. E. Abd-El-Monsef and E. F Lashine, Some operation on supratopological spaces, *Dirasat,* XIII (7) (1986), 43-50.

[14] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and E. A. Shalaby, Generalized closed sets in bitopological spaces, South Asian J. Math. 6(2) (2016), 72-81.