Egypt. J. Phy. **Vol. 49,** pp. 101-107 (2021)

Egyptian Journal of Physics
http://ejfs.journals.ekb.eg/

Calculation of Charmonium Mass Spectrum within the Fourier Grid Hamiltonian Method

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FOURIER Grid Hamiltonian method in the frame work of the nonrelativistic model is a way to study the charmonium states ($c\bar{c}$) based on microscopic effective quark interactions. The newly resulting spectrum doesn't need any renormalization to fit the experimental and recent published theoretical spectra.Furthermore, theoretical calculations are in good agreements with existing experimental data.

Keywords: Fourier Grid Hamiltonian; The nonrelativistic model; Charmonium states.

Introduction

This work is devoted to study the masses spectra of some heavy charmonium states (Fig. 1). The quantum numbers and basic properties of most of the states in the charmoniumfamily can be described as a simple picture of a nonrelativistic quark-antiquark pair $c\bar{c}$. There are several theoretical ways to calculate charmonium spectra; one of these methods is the Fourier Grid Hamiltonian (FGH) method. The heavy meson is said to be a charmonium if it is built up of charm quark (c) and anticharm quark ($\bar{c}$).

Experimentally, the heavy meson properties are used for clearing up of static and dynamic quark properties. Heavy meson spectroscopy is studied experimentally according to the particle data group (PDG)[1] as a comparable platform to the effective theoretical models. Over the years, there are so many articles studied the charmonium spectroscopy [1-10]. One of the most successful models used is the nonrelativistic model. Applying this model allows us to perform the calculation of the spectra based on the nonrelativistic Schrödinger equation, in which the effective potential of quark interaction U(r) is selected phenomenologically. Before the charmonium ($c\bar{c}$) discovery, it was suggested that if a heavy charm quark is existent, a ($c\bar{c}$) mesonstate should be found according to the nonrelativistic model. It is a difficult task to give good quantitative description to the charmonium or bottomonium spectrum based on the QED theory (although it gave a good description of charmonium and bottomonium in the first approximation of Balmer spectrum in QED), so it is advised to use the interplay of perturbative and nonperturbative aspects of QCD [11].

Phenomenologically, the system of quark-antiquark can be studied successfully via some specific models that produce theoretical results comparable to the experimental values.

The motivation of this work is to investigate the reliability of the FGH method to the extracted masses spectra of charmonium systems in the framework of the nonrelativistic quark model. A detailed description of the main formalism to the FGH method [12,13] is reviewed. A brief Survey of the framework used in the current analysis is outlined. Results and discussions are provided.

*Corresponding author: atef@usa.com
DOI :10.21608/ejphysics.2020.42912.1054
Received : 17/9/2020; accepted : 17/11/2020

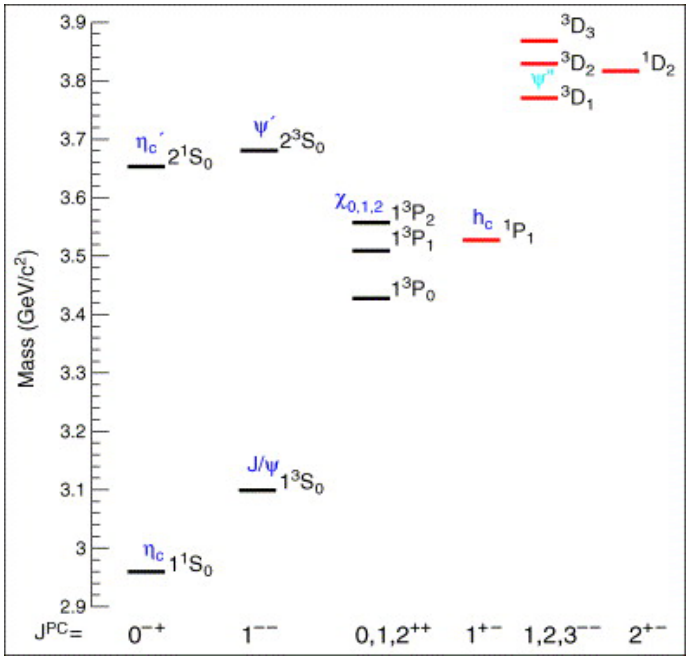

Fig.1. Charmonium spectra.

Theoretical Models and Model Calculations

Three-Dimensional FGH Method

In this subsection, the main features of the FGH method are reviewed. It correlates between the potential energy at N grid points and the kinetic energy in the momentum space via forward and reverse Fourier transforms between the coordinate and the momentum space [14,15]. The N × N symmetric matrix H, obtained by discretization, has elements in the form of cosine sums. The task of calculating the bound state eigenvalue and eigenfunctions is thereby transformed to the task of finding eigenvalues and eigenvectors of the matrix H. The eigenvalue equation for a stationary state is given by

$$[\hat{T}+\hat{V}]\left|\psi\right\rangle = E\left|\psi\right\rangle \qquad (1)$$

where, $\hat{T}$ is the kinetic energy which depends only on the square of the relative momentum *P* between the particles, V is a local interaction which depends on the relative distance, and E is the eigenenergy of the stationary state. This equation is a nonrelativistic Schrodinger equation if

$$\hat{T} = m_1 + m_2 + (\widehat{P^2}/2\mu) \qquad (2)$$

where, m_1 and m_2 are the masses of the particles and μ is the reduced mass of the system in configuration space, Eq. (1) can be written as

$$\left[\left\langle r\left|\hat{T}\right|\acute{r}\right\rangle + \left\langle r\left|\hat{V}\right|\acute{r}\right\rangle\right]\left\langle\acute{r}|\Psi\right\rangle d\acute{r} = E\left\langle r|\Psi\right\rangle\Big] \qquad (3)$$

In this concern, we only consider the case of a local central potential

$$\left\langle \mathbf{r}\left|\hat{V}\right|\mathbf{r}'\right\rangle = V(r)\delta(\mathbf{r}-\mathbf{r}') \text{ with } r=|\mathbf{r}| \qquad (4)$$

It is then useful to decompose the wave function into its central and orbital parts

$$\left\langle \mathbf{r}|\psi\right\rangle = R_l(r)\,\mathrm{Y}_{lm}(\hat{r}) \text{where } r=\mathbf{r}/r \qquad (5)$$

To compute the nonlocal representation of the kinetic energy operator, we introduce the basis states $\{\left|K\boldsymbol{\lambda}\right\rangle\}$, which are eigenstates of the operator P^2. They are characterized by good orbital quantum numbers (λ,ν), obey the relation

$$\hat{T}(P^2)\left|k\lambda\nu\right\rangle = T(k^2)\left|k\lambda\nu\right\rangle \qquad (6)$$

and satisfy the orthogonality relation

$$\left\langle\acute{k}\acute{\lambda}\acute{\upsilon}|k\lambda\upsilon\right\rangle = \delta(\acute{k}-k)\delta_{\acute{\lambda}\lambda}\delta_{\acute{\upsilon}\upsilon} \qquad (7)$$

The representation of these states in the configuration space is given by

$$\left\langle r\left|k\lambda\nu\right\rangle = \sqrt{(2k^2/\pi)}\,j_\lambda(kr)Y_{\lambda\nu}(\hat{r}) \qquad (8)$$

where the function $j_l(kr)$ is the spherical Bessel

function. Using the completeness relation of basis states $\{|k\lambda\nu\rangle\}$ and Eq. (7), we can find

$$\langle r|\hat{T}|\acute{r}\rangle = \int_0^\infty dk\,(2k^2/\pi)T(k^2)\sum_{\lambda=0}^{\infty}\sum_{\upsilon=-\lambda}^{\lambda} j_\lambda(kr)j_\lambda(k\acute{r})Y_{\lambda\upsilon}(\hat{r})Y^*_{\lambda\upsilon}(\hat{\acute{r}}) \qquad (9)$$

By introducing the regularized function $u_1(r) = r\,R_r($, Eq. (3) can be written as in ref. [10]

$$(2/\pi)r\int_0^\infty dr'r'u_l(r')\int_0^\infty dk(k^2)T(k^2)$$
$$j_l(kr)\,j_l(kr') + \mathrm{V}(r)u_l(r) = Eu_l(r) \qquad (10)$$

This equation is considered as the basis of the three-dimensional Fourier grid Hamiltonian method.

Discretization

The continuous variable r will be replaced by a grid of discrete values r_i defined by

$$r_i = i\Delta \text{ With } i = 0,1,2,\ldots\ldots\ldots\ldots,N \qquad (11)$$

where Δ is the uniform spacing between the grid points. Regularity at the origin imposes $u_1(r_0 = 0) =$. For bound states, we have $lim_{r\to\infty}\, u_1(r) =$. Consequently, we choose to set $u_1(r_N = N\Delta) =$. This last condition is not necessary, but it does not spoil the accuracy of solutions. The normalization condition of the radial wave functioncan be written as

$$\int_0^\infty dr[u_1(r)]^2 = 1 \qquad (12)$$

The discretization of this integral on the grid is given by

$$\Delta\sum_{i=1}^{N-1}[u_l(r_i)]^2 = 1 \qquad (13)$$

The grid spacing Δ in the configuration space determines the grid spacing Δk in the momentum space. The maximum value of r considered to be $r_N = N\Delta$, the wave function works on a sphere of diameter $2r_N$ in the configuration space. This length determines the longest wavelength $\lambda_{\max}$ and, therefore, the smallest frequency Δk which appears in the k -space is

$$\Delta k = (2\pi/\lambda_{\max}) = (\pi/N\Delta) \qquad (14)$$

Now, we have a grid in configuration space and a corresponding grid in momentum space as

$$k_s = s\Delta k = (s\pi/N\Delta)$$
$$\text{with } s = 0,1,\ldots\ldots\ldots,N \qquad (15)$$

For $V_i = V(r_i)$, the discretization procedure replaces the continuous Eq. (10) by an eigenvalue matrix problem

$$\sum_{j=1}^{N-1} H_{ij}\phi_j^n = e_n\phi_j^n \text{ for } i = 1,2,\ldots\ldots,N-1 \qquad (16)$$

where,

$$H_{ij} = (2\pi^2/N^3)ij\sum_{s=1}^{N} S^2T((\pi s/N\Delta)^2)$$
$$j_l(\pi si/N)j_l(\pi sj/N) + V_i\delta_{ij} \qquad (17)$$

The (N-1) eigenvalue $e_n$at Eq. (16) corresponds approximately to the first (N-1) eigenvalues of Eq. (10). In case of potential possesses continuum spectrum, only eigenvalues below the dissociation energy are considered. Other eigenvalues which forma discrete spectrum of positive energies, are spurious and correspond to standing wave solutions satisfying $u(r) =$ at $r =$ and $r = l$. The eigenvector ϕ produces approximately the values of the radial part for$n$th solution of Eq. (10) at the grid points.

Brief survey of the potential model

The standard practice of describing charmoniumis adopted via the nonrelativistic kinematics by using the FGH method. The nonrelativistic (NR) potential [16-25] is the best way to study the heavy meson spectra because of the smallness of the relativistic effects. However, it is suitable to insert the spin-orbit and spin-spin interaction to the potential. The spin-orbit interaction and tensor interaction are considered in the vacuum suggestion inside the heavy meson. It is recommended to add the tensor and spin-orbit terms into the considered potential to reflect the complete vision of the vacuumidea, the potential then can be given as Ref. [16]

$$V(r) = -(4\alpha_s / 3r) + br + (32\pi\alpha_s / 9m_c^2)(\sigma/\sqrt{\pi})^3 e^{-\sigma^2 r^2} S_c.S_{\bar{c}} + (l(l+1)/2\mu r^2) + (1/\mathrm{m}_c^2)[((\alpha_s / r^3) - (b/2r))\vec{L}.\vec{S}] + (\alpha_s / r^3)\mathrm{T}_s \quad (18)$$

where, the spin-orbit operator is diagonal ina $|J, L, S\rangle$ basis, with the matrix elements

$$\langle \vec{L}.\vec{S} \rangle = [J(J+1) - L(L+1) + S(S+1)]/2 \quad (19)$$

The tensor operator T_s has non-vanishing diagonal matrix elements only at L >0 spin-triplet states, which are

$$T_s = \begin{cases} -L/(6(2L+3)), J = L+1 \\ 1/6, \mathrm{J} = \mathrm{L} \\ -(L+1)/(6(2L-1)), J = L-1 \end{cases} \quad (20)$$

Now, the potential model will be applied to get the spectra of charmonium ($c\bar{c}$) bound states with wave functions determined by the radial Schrödinger equation. The spectroscopic results of the charmonium and open charm systems are presented and discussed on the next section.

Numerical Results and Discussion

Over the developing era of particle physics, charmonium spectroscopy has played an important role in understanding the quark-antiquark interaction in the framework of quantum chromodynamics (QCD) in particle physics. The charmonium system provides a perfect platform for studying the strong force to improve the reliability of calculation for the charmonium spectroscopy. In this context, we restrict the mass region of charmonium spectrum to be below 4.5*GeV*. The charmonium parameters were obtained by fitting the potential model in Eq. (20) to the known charmonium spectrum. The resulted values are $\mathrm{m_c} = \mathrm{m_{\bar{c}}} = 1.4619\ (GeV)$, $\alpha_s = 0.4942$, b= 1.1446 $(GeV)^2$ and σ= 1.1412 (*GeV*). These parameters are readjusted to reproduce the overall experimental results of the charmonium particles. The masses of S, P and D-wave states are tabulated in Table 1 and are in good agreement with experimental data as well as other theoretical model predictions [26]. These results are satisfactorily reflecting their experimental data. The mass spectra of charmoniummes ons are also shown graphically in Fig. (2), (3) and (4). The obtained results reflect well the experimental measurements, especially for ground states. One can see obviously both the mass trajectories exhibit good behaviors and they are roughly consistent with each other. However, any increase on the mass will led to decrease on the mass shift and vice versa.

Conclusion

In this work, the charmonium spectra predicted in the mass region below 4.5 GeV, for S, P and D-wave levels are studied. The FGH method is employed in the framework of the nonrelativistic quark model. Based on the obtained results, one can observe that the FGH method produces the charmonium states in high accuracy up to the level which can be compared to experimental results, as listed in Table 1 and shown in Fig. (2), (3) and (4). The considered method will be helpful to clarify the nature of the newly discovered mesons. Due to the lack of experimental spectra for charmonium, as light deviation on theoretical results is found. New measurements are recommended to measure charmonium spectra to provide more precise data for the future tests of different charmonium states. Furthermore, data at wide ranges of mass regions and energy levels of charmed mesons are also required in order to perform a systematic study.

TABLE 1. Chamonium spectrain units of GeV. Calculated masses are listed in the third column in comparison to the results of Ref. [26] listed in the fourth column andexperimentalmasses listed in the fifth column.

StateName	Theoretical masses in GeV	Ref. [26]	Experimental masses in GeV [1]
$c\bar{c}1^1S_0\eta_C(1S)$	3.0303	2.981	2.983± 0.5
$c\bar{c}2^1S_0\eta_C(2S)$	3.6511	3.625	3.637± 1.2
$c\bar{c}3^1S_0\eta_C(3S)$	4.037	4.032	….
$c\bar{c}4^1S_0\eta_C(4S)$	4.4662	4.368	…..
$c\bar{c}\ \ 1^3S_1J/\Psi(1S)$	3.0922	3.089	3.096± 0.006
$c\bar{c}2^3S_1\Psi(2S)$	3.6686	3.672	3.686± 0.025
$c\bar{c}\ \ 3^3S_1\Psi(3S)$	4.0693	4.060	4.039± 0.001
$c\bar{c}\ \ 4^3S_1\Psi(4S)$	4.4059	4.386	4.421±0.004
$c\bar{c}1^3P_1\ \ \chi_{c1}(1P)$	3.5825	3.505	3.510±0.05
$c\bar{c}2^3P_1\ \ \chi_{c1}(2P)$	4.062	3.923	….
$c\bar{c}1^3P_2\ \ \chi_{c2}(1P)$	3.5909	3.556	3.556±0.07
$c\bar{c}2^3P_2\ \ \chi_{c2}(2P)$	4.0725	3.970	….
$c\bar{c}1^1P_1\ \ h_C(1P)$	3.5907	3.524	3.525±0.11
$c\bar{c}2^1P_1\ \ h_C(2P)$	4.0712	3.941	….
$c\bar{c}1^3D_2\ \ \Psi_2(1D)$	4.0247	3.800	….
$c\bar{c}2^3D_2\ \ \Psi_2(2D)$	4.1507	4.156	…..
$c\bar{c}1^3D_3\ \ \Psi_3(1D)$	4.0168	3.806	….
$c\bar{c}2^3D_3\ \ \Psi_3(2D)$	4.1451	4.164	…..

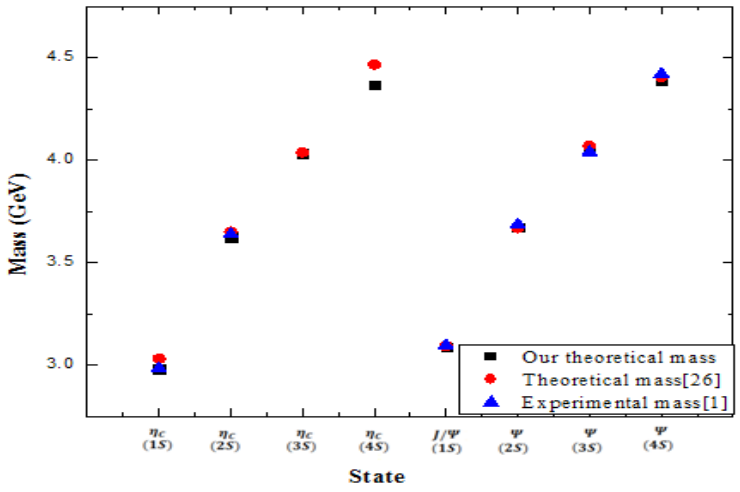

Fig. 2. The trajectories of our theoretical masses, other theoretical masses ref. [26] and experimental masses varying with the S-wave levels.

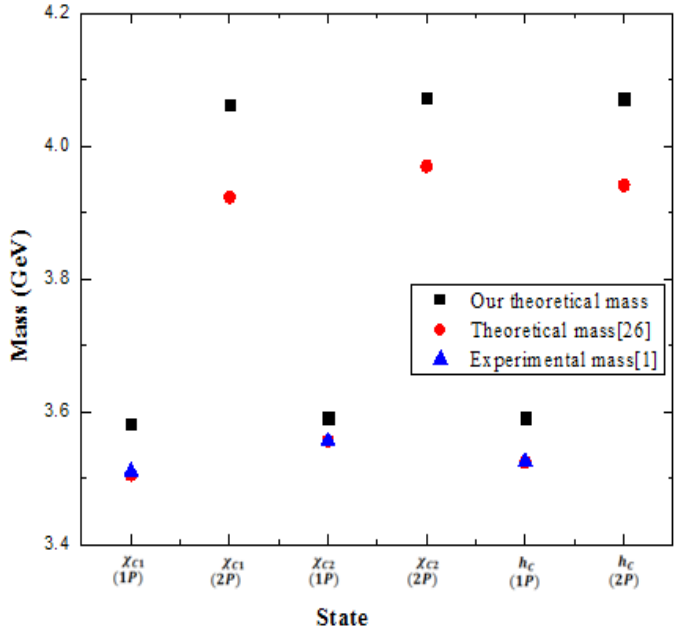

Fig. 3. The trajectories of our theoretical masses, other theoretical masses ref. [26] and experimental masses varying with the P-wave levels.

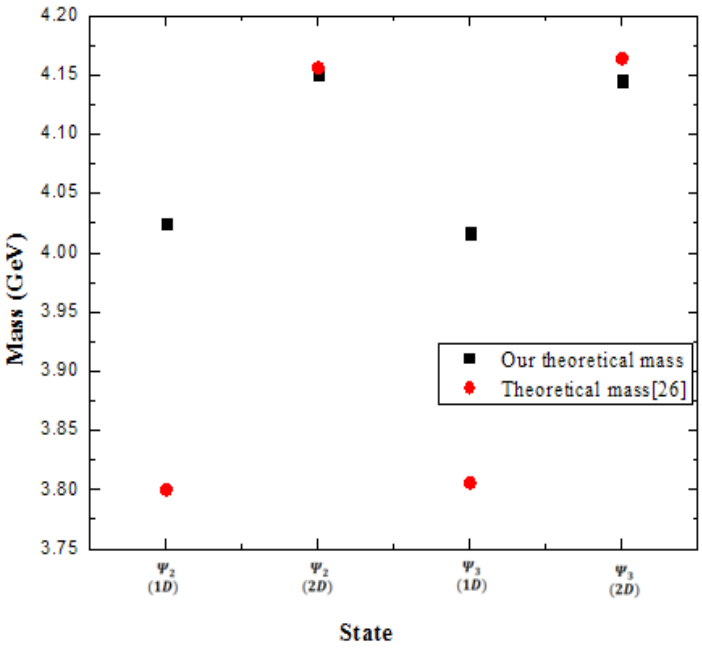

Fig. 4. The trajectories ofour theoretical massesand theoretical masses ref. [26] varying with the D-wave levels.

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