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Critical Rotation Frequency for Vortices Configuration in A Harmonically Trapped Boson Atoms

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IN this paper we suggest a convenient quantum statistical approach to calculate the critical rotation frequency for vortex nucleation in a rotating condensate boson atoms. We extende thequantum mecanices approach for achieving the critical rotation rate to enclode the effect of finite temperature. Dependence of the critical rotation on the interatomic interaction, atoms number and temperature is considered. A simple semiclassical approach is suggested. The calculated results showed that the critical rotation rate depending crucially on the above mensioned parameters, its change very rapidlyat very low temperature. Which means that the vortical configuration is achieved for an accurate rotation rate value α, depends on temperature, interatomic interaction and number of condensate atoms. The obtained results provide useful qualitative theoretical results for future Bose Einstein condensation experiments in rotating traps. Motivated by our work, it will be interesting to study the critical rotation for vortex nucleation of rotating condensate boson atoms in optical lattice.

Keywords: Bose-Einstein condensates in periodic potentials, Dynamic properties of condensates, Quantum statistical theory; ground state.

Introduction

Ultracold quantum gases in a state of Bose-Einstein condensates (BECs) [1, 2, 3, 4, 5] provide a good simulator system for which the superfluid behavior can be studied in the weak-coupling regime. One of the prominent features of a superfluid BEC is the way it behaves under rotation. Experiments demonstrated its superfluid nature by the formation of vortex at a rapidly rotating regime [6, 7, 8, 9, 10, 11, 12]. However, rotating BEC have extreme rotation frequency at which the confinement potential is compensated by the centrifugal force . the trapped condensate is therefor unstable at this point. As well as, it has a threshold rotation frequency at which a vortex configuration appears

Many previous theoretical and experimental efforts had been spent to the calculation of the critical frequency needed to create a stable vortex in the rotating frame [13, 14, 15, 16, 17, 18, 19]. These studies show that the critical frequency of vortex nucleation turns out to be smaller than the radial oscillator frequency. Moreover it decreases smoothly with the number of atoms in the condensate. However, most of these theoretical works have been based on the use of the solution of the stationary Gross-Pitaevskii equation for the ground state along the rotation axis z.

The purpose of this work is to discuss the behavior of the critical frequency for vortex nucleation at finite temperature, pointing out the crucial role played by the interactions and the number of atoms. Stringari [20] pointed that the vortex configuration is thermodynamically stable if the trap rotates with frequency Ωlarger than a critical frequency given by,

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$\Omega_c=(F_v-F_0)/(\langle L_z\rangle_v-\langle L_z\rangle_0)$ (1) **(1)**

where F_v, F_0 and $\langle L_z\rangle_v$,$\langle L_z\rangle_0$ are the free energy and the angular momentum of the configuration with and without vortex In the present paper, the free energy and the thermal average of the angular momentum are calculated in terms of the thermodynamic potential when the system is in full thermal equilibrium in the rotating frame.Motivated by the careful study of harmonically confined Bose gas in a rotating trap [21-25], the thermodynamic potential calculated in our previous developed semiclassical approximation [26] can be used to calculate the above mentioned parameters. Usefulness of our approach is in its simplicity and generality: there is no need to know the specific nature of the microscopic details of the superfluid BEC, only knowledge of the system Hamiltonian's and its spectrum are sufficient.

The outcome results showed that the critical rotation rate has a monotonically increasing nature until it reaches to a maximum value and decreases rapidly with the increase of the rotation rate. These maximum value depends strongly on the number of atoms and interatomic interaction parameter. Increasing the interatomic interaction increases α_c while the increases of the number of condensate atoms decreases α_c . the accurate rotation frequency to nucleate ranging from 0.6 ~0.8 $\omega_{\perp}$ for the ENS group trap parameters [10, 11].

This paper is organized as follows. In the first section, we describe the physics of the single-boson atom model. In section two, the critical frequency of vortex nucleation is given. Discussion and conclusion are given in the last section.

Physics for single-boson atom model

Let us consider a system of N non-interacting boson atoms trapped in a cylindrically symmetric harmonic potential ,$V(r)=\frac{1}{2}M(\omega_\perp^2(x^2+y^2)+\omega_z^2z^2)$

with $\omega_x=\omega_y\equiv\omega_\perp$, ω_z are the x,y,z oscillator frequency respectively and M is the atomic mass. The trap can be either static or rotating. The time-independent Hamiltonian of our system is obtained by going into the rotating frame which is related to static Hamiltonian in the lab frame H by $H_{rot}=H_0-\Omega.L$

$$=((p-M\Omega\times r)^2+p_z^2)/2M+M/2(\omega_\perp^2-\Omega^2)r^2+M/2\,\omega_z^2z^2 \quad (2)$$

where $p^2=p_x^2+p_y^2, r^2=x^2+y^2, \omega_\perp=\omega_x=\omega_y$ and $L_z=(xp_y-yp_x)$ is the angular momentum in the z-direction. In most of the experiments on atomic BEC trap geometry described by the Hamiltonian (2) is used. Using cylindrical polar co-ordinates (r, θ, z), one finds that the Hamiltonian H is separable into a pair of harmonic oscillator. Namely an isotropic oscillator in the x −y plane $H_\perp$ and another is one dimensional oscillator along the z -axis, H. The eigenfunctions for them can be written as a product of the eigenfunctions of H and of Hz.

$$H_\perp=\frac{\hbar^2}{2M}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{L_z^2}{r^2}\right]+\frac{1}{2}M\omega_\perp^2r^2 (3)$$

with energyeigen values given in terms of two quantum numbers{n,m}as

$$E_{m,n}=(2n+|m|+1)\hbar\omega_\perp (4)$$

where n=0,1,2,....is the quantum number for energy and m=0,±1,±2,....is angular momentum quantumn umber As well as for the axial Hamiltonian Hz

$$H_z=\frac{p_z^2}{2M}+\frac{1}{2}M\omega_z^2z^2 \ (5)$$

the energy eigenvalue is given by

$$E_{n_z}=\left(n_z+\frac{1}{2}\right)\hbar\omega_z \ (6)$$

If the trap is rotated about the z-axis, the corresponding Hamiltonian is given by

$$H_{rot}=H_\perp+H_z-\Omega L_z \ (7)$$

with eigenfunctions

$$\Phi_{n,m,n_z}=\Phi_{n,m}\Phi_{n_z}(8)$$

and energy-eigenvalues given by [5]

$$E_{n,m,nz}=(2n+|m|-\alpha m)\hbar\omega_\perp+\left(n+\frac{1}{2}\right)\hbar\omega_z(9)$$

where $\alpha=\Omega/\omega_\perp$ is the rotation rate. In calculating Eq.(9) the energy eigenvalues of the angular momentum Lz is calculated. Since [L , Lz] = 0, the eigenstates of the angular momentum operatoris the same eigenstates of L2 with eigenvalues ℏm

$$L_z\Phi_{n,m}=m\hbar\Phi_{n,m}(10)$$

3. Critical frequency of vortex nucleation with interaction effect

The purpose of this section is to calculate the critical frequency for vortex nucleation at finite temperature, pointing out the crucial role played by the interatomic interactions, temperature and number of particles. We suggest a new approach to show that the critical frequency of vortex nucleation in a trapped condensate can be found from statistical mechanics perspective.

The quantum mechanical average of the angular momentum can be determined using the Hellmann-Feynman theorem [27, 28, 29]. The Hellmann-Feynman theorem governs the linear changes in the energy with respect to a parameter of the Hamiltonian,

$$\langle \Phi_{n,m,n_z}(\lambda)|dH/d\lambda|\Phi_{n,m,n_z}(\lambda)\rangle = \frac{dE_{n,m,n_z}}{d\lambda}$$

The parameter λ can be any quantity that appears in the Hamiltonian of the system. Substitution from Eqs.(7) and (9) into (11) for $\lambda \equiv \Omega$, the average angular momentum is given by, $\langle L_z\rangle = m\hbar$

The effective thermal average of the angular momentum for N atoms can be parametrized from Bose-Einstein distribution [26], $N=\sum_{n,m,n_z} n_{n,m,n_z} = \sum_{n,m,n_z} \frac{Ze^{-\beta E_{n,m,n_z}}}{1-Ze^{-\beta E_{n,m,n_z}}}$

$$\langle L_z\rangle = \sum_{j=1}^{\infty} Z^j \sum_{n,n_z=0}^{\infty} \sum_{m=-\infty}^{\infty} m\hbar e^{-j\beta E_{n,m,n_z}} \quad (13)$$

With $z=e^{\beta(\mu_0-E_0)}$ is the effective fugacity and μ_0 is the chemical potential.

Result in (13) is consistent with the formulas predicted from the quantum mechanics consideration for a set of particles i = 1, N occupying these states, for which the total angular momentum is given by [5],

$$L=\sum_{i=1}^{N} m_i \hbar \quad (14)$$

where mi is the angular momentum of particle i. This can be seen clearly if we set T = 0 in Bose-Einstein distribution, the occupation number nnx;ny;nz vanish for every energy level except the level with energy equal to zero (ground state energy). The energy of the ground state is given by (μ - E0,0,0) which is vanishes due to BEC criterion. Thus, at T = 0 Eq (13) gives

However, Eq.(13) can be parametrized in terms of the thermodynamic q −potential [26, 31-34], $\langle L_z\rangle = \frac{k_B T}{N}\frac{1}{\omega_\perp}\frac{\partial}{\partial\alpha}\sum_{n,n_z=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{Z^j}{j}e^{-j\beta E_{n,m,n_z}}$

$$=-\frac{k_B T}{N}\frac{1}{\omega_\perp}\frac{\partial}{\partial\alpha}\sum_{n,n_z=0}^{\infty}\sum_{m=-\infty}^{\infty}\ln\left(1-Ze^{-\beta E_{n,m,n_z}}\right)$$

where $(1-y) = -\sum_{j=1}^{\infty}\frac{y^j}{j}$ is used here.

In our previous work, the thermodynamic potential [26, 30] is given by, $q(z,T)=q_0(z,T)-\sum_{n,m,n_z}^{\infty}\ln\left(1-ze^{-\beta E_{n,m,n_z}}\right)$

$$= q_0(Z,T) + \frac{1}{(1-\alpha^2)}\left(\frac{k_B T}{\hbar\omega_g}\right)^3\left[g_4(z)+\frac{\mu_0}{k_B T}g_3(z)\right] \quad (17)$$

with $\alpha = \frac{\Omega}{\omega_\perp}$ is the rotation rate,

$\omega_g = (\omega_\perp^2\omega_z)^{1/3}$ is the geometrical average of the oscillator frequencies for the confining potential and $\mu_0/(k_B T)=\eta[(1-\alpha^2)(1-T^3)]^{2/5}/T$

is the chemical potential with η , first introduced by Stringari et al. [1, 16], determined by the ratio between the chemical potential at T = 0 value calculated in Thomas-Fermi approximation and the transition temperature for the non-interacting particles in the same trap, i.e. $\eta=\mu_0(N,T=0)/(K_B T_0)$, $T=T/T_0$ is the reduced temperature with $N=N_0+z\left(\frac{\partial q(z,T)}{\partial z}\right)_T$

$$= N_0 + \frac{1}{(1-\alpha^2)}\left(\frac{k_B T}{\hbar\omega_g}\right)^3\left[g_3(Z)+\frac{\mu_0}{k_B T}g_2(Z)\right] \quad (18)$$

thus

$$\frac{N_0}{N} = 1-\mathcal{T}^3-\eta[(1-\alpha^2)(1-\mathcal{T}^3)]^{\frac{2}{5}}\frac{\zeta(2)}{\zeta(3)}\mathcal{T}^2 \quad (19)$$

With ζ is the Riemann zeta function. Substitute from Eq.(17) into Eq.(16) gives,

$$\langle L_z\rangle_v = \hbar\left(\frac{\omega_z}{\omega_\perp}\right)^{\frac{1}{3}}\mathcal{B}\left[\frac{2\alpha}{(1-\alpha^2)}\left[\frac{\zeta(4)}{\zeta(3)}\mathcal{T}^4 + +\eta A\mathcal{T}^3\right] - \frac{4}{5}\frac{\alpha}{(1-\alpha^2)}\eta A\mathcal{T}^3\right] \quad (20)$$

Where $A = [(1-\alpha^2)(1-\mathcal{T}^3)]^{\frac{2}{5}}, \mathcal{B} = \left(\frac{(1-\alpha^2)N}{\zeta(3)}\right)^{\frac{1}{3}}$

The free energy is given,

$$F = E - TS \quad (21)$$

where S is the entropy of the system [35, 36]

$$S = k_B\left[-q(z,T) + \frac{E}{k_B T} - N\frac{\mu_0}{k_B T}\right] \quad (22)$$

and

$$E = k_B T^2\left(\frac{\partial q(z,T)}{\partial T}\right)_Z \quad (23)$$

is the total energy. In terms of the T the normalized free energy per particle is given by, F_v=(k_B T)[q(Z,T)/N-μ_0/(k_B T)]

$$= \hbar\omega_g\left[(k_B T)\frac{q_0(Z,T)}{N} - \mathcal{B}\left[\frac{\zeta(4)}{\zeta(3)}\mathcal{T}^4 + \eta\mathcal{A}\mathcal{T}^3\right] - \frac{\mathcal{A}\eta}{\mathcal{T}}\right] \quad (24)$$

As well as F0 is given by

$$F_0 = \hbar\omega_g\left[(k_B T)\frac{q_0(z,T)}{N} - \mathcal{B}_0\left[\frac{\zeta(4)}{\zeta(3)}\mathcal{T}^4 + \eta\mathcal{A}_0\mathcal{T}^3\right] - \frac{\mathcal{A}_0\eta}{\mathcal{T}}\right] \quad (25)$$

Now, it is straightforward to discuss the behavior of the frequency of vortex nucleation at finite temperature. Substitute from Eqs. (20) and (24) in Eq.(1) leads to,

$$\alpha_c = \frac{(\mathcal{B}-\mathcal{B}_0)\frac{\zeta(4)}{\zeta(3)}\mathcal{T}^4 + (\mathcal{B}_0 A_0 - \mathcal{B}A)\eta\mathcal{T}^3 + (A_0 - A)\frac{\eta}{\mathcal{T}}}{\mathcal{B}\left[\frac{2\alpha}{(1-\alpha^2)}\left[\frac{\zeta(4)}{\zeta(3)}\mathcal{T}^4 + \eta A\mathcal{T}^3\right] - \frac{4}{5}\frac{\alpha}{(1-\alpha^2)}\eta A\mathcal{T}^3\right]} \quad (26)$$

Results and Discussion

The calculated results from Eq.(26) for the rate of critical rotation are represented graphically in Fig. 1 as a function of the normalized temperature for different interaction parameters.

This figure shows that the behavior of the critical rotation rate agrees with the experimental results for the creation of vortex by adiabatic increase of the rotation frequency [11]. The critical rotation for nucleating vortex decreases suddenly with the increase of the normalized temperature T. However, in order to avoid this scenario, superfluid helium scenario [37], Stringari [20] pointed that at very low temperature a safer procedure to generate vortexes is first to rotate the gas at higher temperature and then to cool it via evaporation cooling. At very low temperature, superfluid helium [37] exhibits high barrier for the nucleation of vortexes. The figure also reveals that nucleating vortex with the non-interacting system is a hard task.

In order to extracte the most interesting vortical region from our approach we have to use Eq.(26).

Analytically, the accurate critical rotation rate is given by the root of the equation

$\frac{d\alpha_c}{d\alpha} = 0,$ i.e.

$$\frac{d}{d\alpha}\left[\frac{(\mathcal{B}-\mathcal{B}_0)\frac{\zeta(4)}{\zeta(3)}\mathcal{T}^4 + (\mathcal{B}_0\mathcal{A}_0 - \mathcal{B}\mathcal{A})\eta\mathcal{T}^3 + (\mathcal{A}_0 - \mathcal{A})\frac{\eta}{\mathcal{T}}}{\mathcal{B}\left[\frac{2\alpha}{(1-\alpha^2)}\left[\frac{\zeta(4)}{\zeta(3)}\mathcal{T}^4 + \eta\mathcal{A}\mathcal{T}^3\right] - \frac{4}{5}\frac{\alpha}{(1-\alpha^2)}\eta\mathcal{A}\mathcal{T}^3\right]}\right] = 0 \quad (27)$$

Figure 2 is devoted to illustrate the variation of αc with the trap rotation rate for diffferent interaction parameter η . This figure shows that the critical rotation rate has a monotonically in- creasing nature until it reaches to a maximum value and then decreases rapidly with the increase of the rotation rate. Increasing the interaction parameters leads to an increase at the value of the critical frequency specially at its maximum. Increasing the critical rotation rate with the interaction parameter is understandable in the sense that the in situ area of interacting system is greater than the one of the non-interacting due to the repulsive forces between atoms. Finally it is clear that for α = 1.0, the confinement potential is compensated by the centrifugal force and the critical frequency to achieve vortical configuration is vanishes.

Phase diagram for vortices nucleation region for a harmonically rotating trap is illustrated in Fig. 3, using Eq.(19). In this figure the gray region shows the region of the absent of the condensate. The area under the surface represents the condensate region with BEC without vortex (white region) and BEC plus vortex (red region).

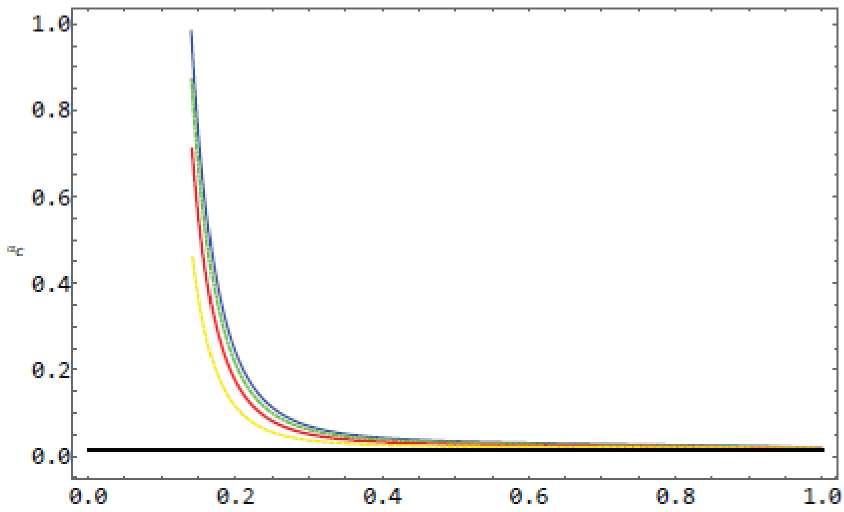

Figure 1: Critical rotation rate versus the reduced temperature T for different values of interaction parameter η = 0.0 (black line), 0.1 (yellow line),0.2 (red line), 0.3 (green line) and 0.4 (blue line) at α= 0.1 and and N = 2.5 × 105 . The trap parameters of the ENS group [10, 11] are usedω_⊥= 175.0 × 2π Hz and ω_z== 10.3 × 2 π Hz(trap parameters).

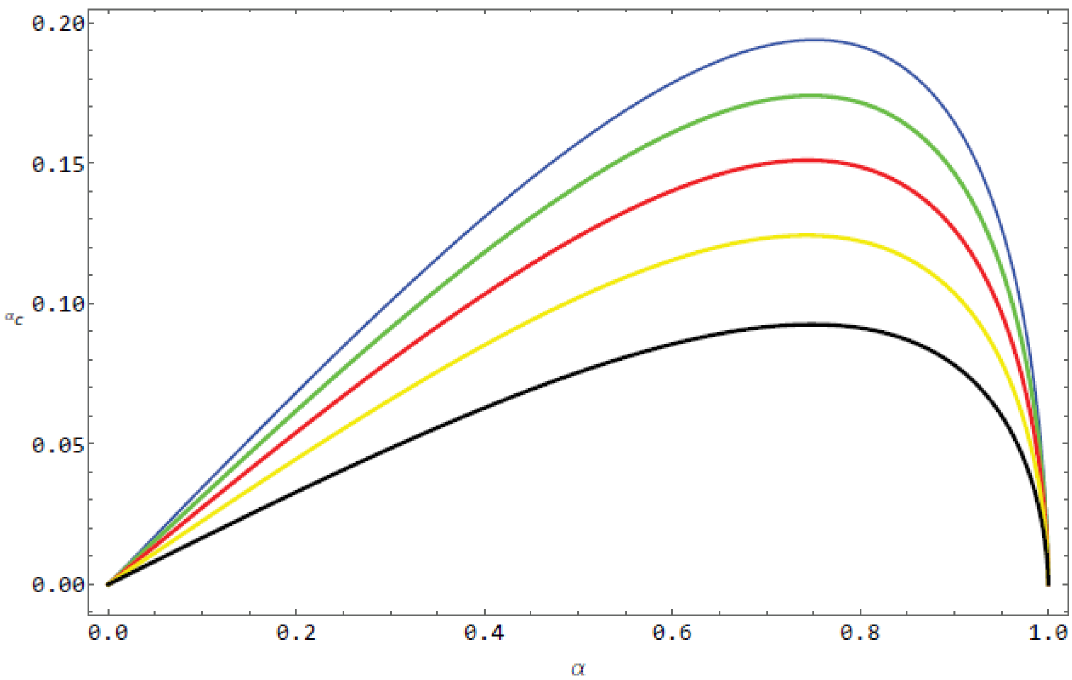

Figure 2: Critical rotation rate versus the rotation rate α for different values of interaction parameter η = 0.0 (black line), 0.1 (yellow line),0.2 (red line), 0.3 (green line) and 0.4 (blue line) at T = 0.5 and N = 2.5 × 105 . The trap parameters have the same values of figure(1).

Effect of the number of particles in the critical rotation rate is illustrated graphically in Fig. 4.This figure shows that for the interacting system the critical rotation rate turns out to be smaller than the oscillator frequency for the confining harmonic potential and to decrease smoothly with the number of atoms. However, increasing the number of atoms leads to directly increases the condensate atoms number, the nucleation of the vortex may become harder if the number of atoms in the condensate is large.

Conclusion

In this paper, we have calculated the critical rotation rate of vortex nucleation for rotating trapped condensate boson atoms. The trap parameters of the ENS group experiments are used [10]. In conclusion, our results show that the accurate region for nucleating vortex should be observable at temperatures not dramatically smaller than the BEC transition temperature. The critical rotation rate for this system depends significantly on the interatomic interaction parameter and the number of particles at finite temperature.

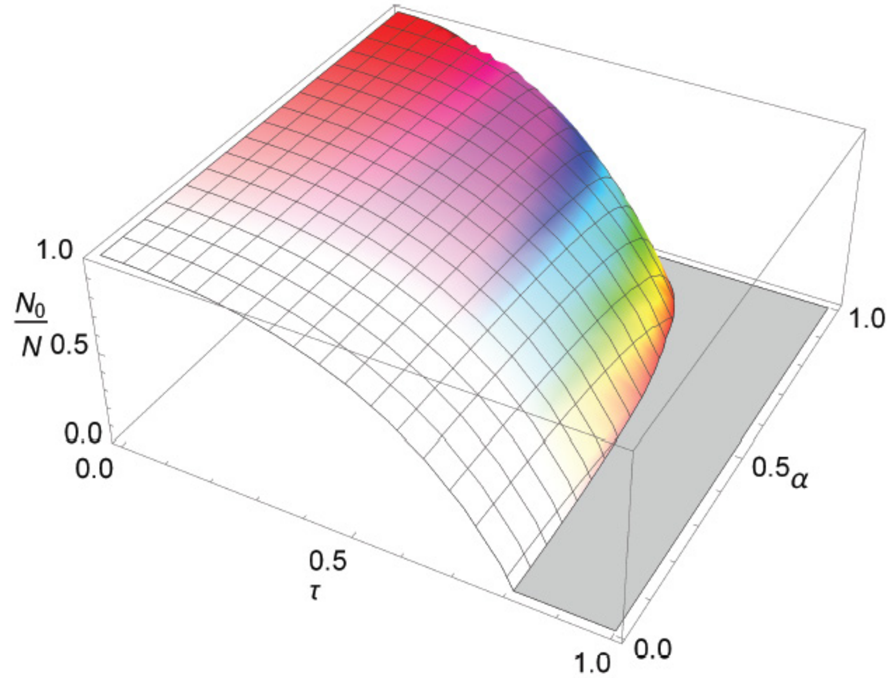

Figure 3: Phase diagram for vortices in a harmonically trapped Bose gas. The red region represents the region for N = 2.5 × 105 and the interatomic interaction η = 0.3. The trap parameters have the same values of figure(1).

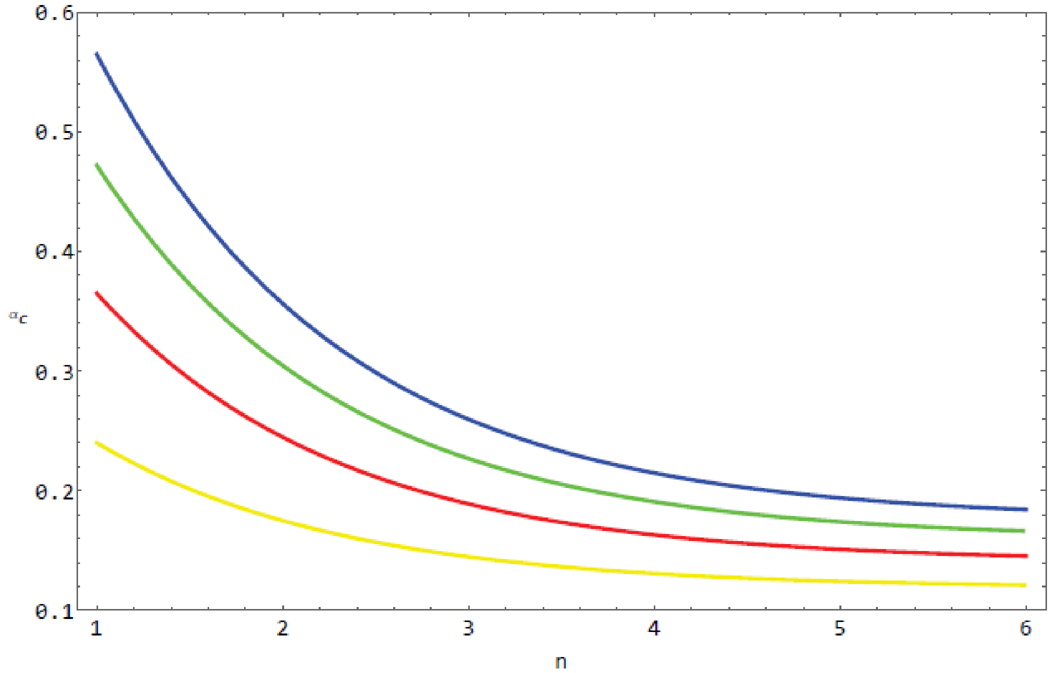

Figure 4: Critical rotation rate versus the number of particles N = 2.5 × 10n for di□erent values of interaction parameter η = 0.1 (yellow line),0.2 (red line), 0.3 (green line) and 0.4 (blue line) at T = 0.5 and N = 2.5 × 105 . The trap parameters have the same values of figure(1).

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